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The influence of Computeralgebra Systems in the Function concept

The computer - a child of mathematical thinking - has changed and will change mathematics in the future fundamentally. The first phase of the use of computers in mathematics was characterized by the growing importance of numerical methods. Mathematics tried to learn and to use the language of computers. In the next phase - the phase of computeralgebra systems (CAS) - computers tried to learn the mathematical language of the students. And now we are observing a third phase: computers are starting to change the language of mathematics by offering new language elements.

Mathematics is a language and like other languages it has its own grammar, syntax, vocabulary, word order, synonyms, conventions, a.s.o. [Esty, 1997]. This language is both a means of communication and an instrument of thought. The purpose of this lecture is to describe the influence of the tool of a computeralgebra system - especially the TI-92 - on mathematical thinking and working. The experience and the examples come from the Austrian CAS-projects. In the last project nearly 2000 pupils took part in 70 experimental classes ranging from the 7th to the 11th grade.

1. Some didactical prerequisites

1.1 The way into mathematics supported by CAS

One possible model to describe the path of the learners „into“ mathematics is a spiral [Buchberger, 1993]

The spiral begins with observations, data material or problems, the solution of which can be found in the development of algorithms or in the creation of new concepts.

Through analysing, experimenting or generally through heuristic strategies, assumptions are found, sentences formed and initial ideas of proof sought.

By proving and substantiating, in other words, by exactifying, one enters into the next stage of the spiral: Theorems and sentences which can now be assumed to be correct.

Thus, supported by acquired knowledge, one proceeds to develop those algorithms or programs which are necessary for problem solving. Testing of and consolidating the developed algorithms by practicing is an integral part of this stage.

The actual part of the spiral ends with the next step. The insight and strategies are now used to solve the initial problems or related problems.

When new problems evolve and new additional knowledge is necessary or new algorithms need to be developed, then the stages of the spiral are repeated once again.

When experiencing a loop in the Creativity Spiral one can distinguish three important fields of activity in the learning process, whereby it is not always possible to draw a sharp line among them:

Phase 1: The heuristic, experimental phase

Developing conjectures, forming hypotheses, devising proving and problem solving strategies, developing naive, elementary conceptions.

Phase 2: The exactifying phase

Corroborating assumptions, proving hypotheses, programing (including testing) exactifying concepts.

Phase 3: The application phase

Solving problems by applying the concepts and algorithms developed in phases 1 and 2: modeling, operating and interpreting.

The order is not important and especially phases 2 and 3 may be reversed or experienced simultaneously.

Within this spiral the learner moves more and more deeply into mathematics using the experience he has gained in his previous contact with the creativity spiral. Naturally it is sometimes necessary in classroom teaching to deviate from this path which is typical for developing mathematics as a science.

One possible deviation caused by the computer is dangerous: Theoretically, CAS, if used as a Black Box, would offer the possibility of doing mathematics without mastering algorithms. The pupil could form assusmptions in the heuristic phase, skip over the theoretical corroboration of algorithms and the practicing of calculating skills and then using the CAS as a Black Box immediately turn to the applications.

In my presentation I would like to point out the chances available with the use of CAS although we must not forget the dangers.

Firstly, no competence in solving problems, in building models or in justifying the correctness of the model can exist without a certain knowledge of the applied algorithm and without basic calculating skills. Secondly, it is our educational duty to place assumptions gained through intuition on a firm foundation by corroboration in the exactifying phase and not only to rely on the competence of tools like CAS.

1.2 The Module Principle

One method of describing the changes in teaching and learning mathematics caused by CAS is to formulate new didactical principles.

Using modules is not new for the learners. Every formula used by the pupils can be seen as a module e.g. Hero's formula for the area of a triangle or the use of the cosine rule in trigonometry.

The computer, and especially CAS, opens a new dimension of modular thinking and working. The programming features of the CAS allow the students to create modules which can be used, on the one hand, in the white box phase as didactical tools and, on the other hand, as black boxes for problem solving.

While the modules of traditional math education mostly are the starting point for calculations, the CAS-modules often also do the calculations.

We can call the modules
"knowledge-units"
- in which knowledge is compressed and
- in which operations can be recalled as a whole package.

Creating modules means building a cognitive scheme, condensing cognitive experience. Using modules causes cognitive relief and a reduction of complexity at the same time operations and complex knowledge can be activated as a unit.

This compressed knowledge can also cause new sorts of mathematical objects or new elements of mathematical language and, last but not least, a re-organisation of the mathematical activity.

2. The CAS as a medium for prototypes

General concepts become cognitively available through concrete representatives or in our words through prototypes (e.g. if you use the common concept „table“ you are thinking about concrete prototypes which you have experienced).

The computer allows us a greater variety of prototypes of a concept and also offers some which were not available before.

Typical for this thesis is the **concept of functions**. The pupil will find access to this concept not through a clear cut abstract definition but rather through a supply of suitable prototypes which draw the pupil´s attention to the vital characteristic of the concept. In this process the important activity is the establishment of relationship among the individual prototypes. It is in this way that the learner can comprehend that the individual prototype is simply one of many possibilities of appearance of the concept of function. Not until after this process does it make any sense to verbalize or formally define the concept „function.“

Observing traditional mathematics education you can find the following **prototypes of the fundamental concept of functions**:

- **word formulas**
- **symbolic prototypes like terms, parametric equations, polar equations**
- **graphs**
- **tables**

The computer also offers new prototypes e.g.
- **recursive models**
- **programs**

The fundamental idea of functions plays a central role in mathematics education and many of the examples of final exams deal with the function concept.

Some goals dealing with the function concept:

- find a certain prototype of a function

(find the term of a function)
- given is a prototype of a function, find another suitable one

(given is the term, find the graph)
- investigate certain attributes of a function

(find certain points like extremums or points of inflection)
- investigate the influence of parameters in the solution

(investigate the influence of the angle and the velocity in the path of a kicked stone)

- change a given prototype

(factorize the term of a function for finding the zeros)

- use certain prototypes of functions for problem solving

(use the recursive model to investigate a certain financial problem)

The following examples will show that CAS make it much easier to find a certain prototype of a function. CAS also change the possibilities of using a certain prototype. Using a term of a function, factorizing is done by the CAS as a black box. The graphic mode of the TI-92 allows the pupil to calculate certain results, like the extremums, derivatives, integrals a.s.o. CAS do not only calculate a table, it is also possible to calculate within the table.

2.1 The use of the graphic prototype

Example 2.1: Optimisation without differential calculus (9th grade)
At the angles of a rectangular where l = 10 dm and w = 8 dm congruent squares are cut off to get a cuboid with maximum volume. Determine the length of the square.

One can find this exercise in many mathematic school books dealing with differential calculus. Pupils often solve this problem automatically by looking for the zeros of the first derivation. Some examine the maximum using the second derivation. Because of calculating they forget that the main goal is optimisation.

CAS allow the pupil to deal with this example in precalculus (e.g. in 9th grade) by exploring the graph of the function. Using the *Trace-mode* of the TI-92 pupils can move along the graph, looking for a suitable definition area, looking for the zeros, the maximums and minimums and interpreting certain points according to the application problem (figure 2.1). By zooming they could get a higher exactness of the solution (figure2.2). It was fascinating to listen to the pupil´s discussion about how sensible exactness is

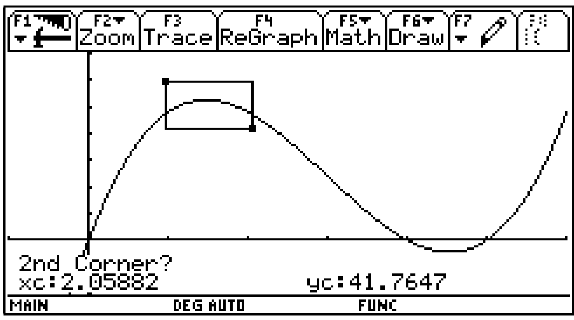


figure 2.1

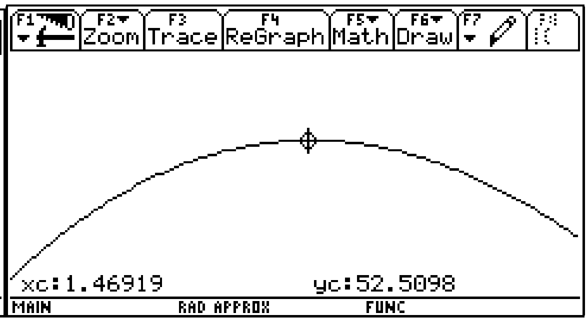


figure 2.2

Experimenting by visualizing and interpreting are the main activities of the learners and not operating.

Example2.2: The fundamental idea of linearisation; differentiable functions. (calculus, 11th grade)

If one tries to teach linearisation of a differentiable function by starting with an abstract concept which can be found in nearly every calculus book, only a few students will be in the position to grasp the fundamental idea. The first step should be finding conjectures by visualisation in a heuristic phase. In the graphic window students can walk along the graph using the *Trace-mode* (figure 2.3). By zooming the CAS can be used like a „function-microscope“ (figure2.4). By repeated enlarging the graph looks like a straight line (figure 2.5).

This observation should lead to the conjecture that a certain group of curves can be replaced by a straight line in a small domain around a given point - in the end this is the fundamental idea of linearisation. A next step could be using the *Trace-mode* to look for two points of this supposed straight line to find the equation of the line. In the exactifying phase which should follow, students should try to find the „best“ of all the lines with these attributes - the tangent. Preparing the exactifying phase in such a heuristic phase with conjectures, more students would follow the exactifying steps.

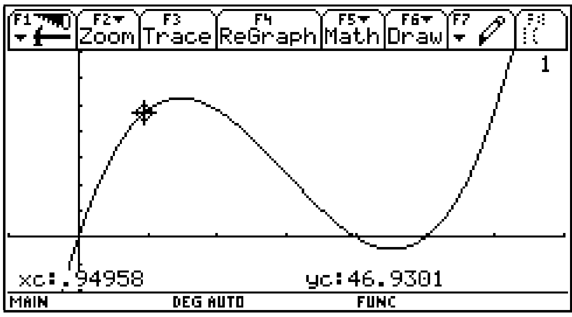


figure 2.3

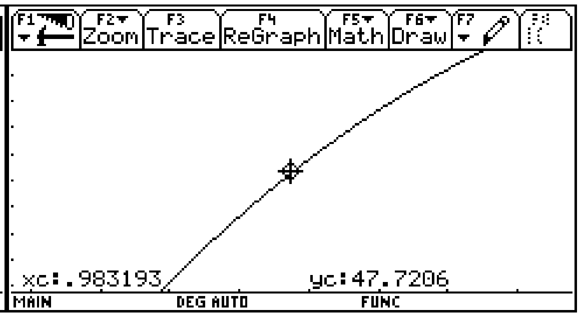


figure 2.4

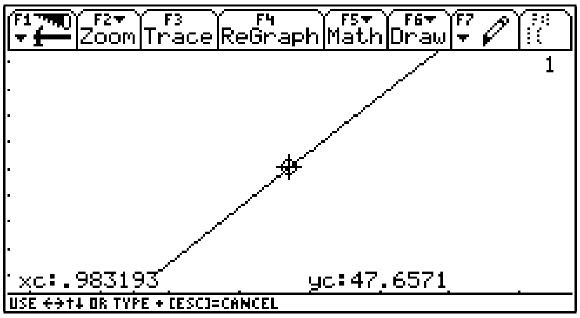


figure 2.5

It is also necessary for the pupils to recognize that the idea of linearisation does not work with every function. One example of this is to examine the absolute value of a given function in the neighborhood of a zero (figure2.6). In spite of repeated zooming it is impossible to find a straight line with such a property (figure2.7). This function is continous but not differentiable.

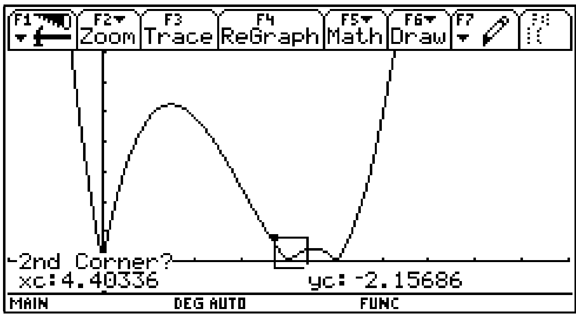


figure 2.6

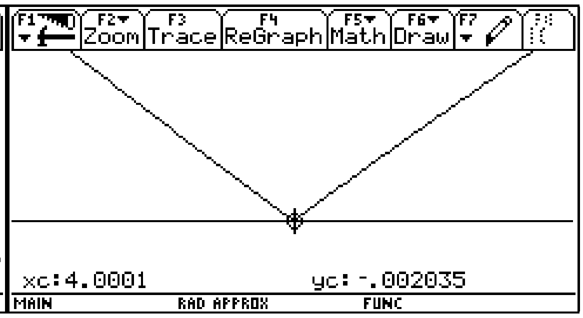


figure 2.7

2.2 The use of the parametric prototype

Example 2.3: Exploring the idea of the parameter [Wheeler,1998]
The USS Arlington and the USS Heights are sailing on the Atlantic Ocean: Their progress is being monitored by radar tracking equipment. As they come onto the observer´s rectangular screen, the USS Arlington is at a point 900 mm from the bottom left corner of the screen along the lower edge. The USS Heights is at a point 100 mm above the lower left corner along the left edge. One minute later the positions have changed as follows: (a) the USS Arlington has moved to a location on the screen that is 3 mm west and 2 mm north of the previous location and (b) the USS Heights has moved 4 mm east and 1 mm north.
Will the two ships collide if they maintain their speeds and remain on their respective course? If so, when? If not, how close do they actually come to each other?
Note: 10 mm ÷ 1 km

This example combined with interesting results of classroom observations starting in 7^{th} grade was presented by Judy Wheeler (jwheele@remc11.k12.mi.us) at the T^3-conference in Nashville in March 98. The concept which was shown in this lecture is a very good example for the creativity spiral, for the way to the analytic geometry. Students should explore the idea of the parametric equation.

I would like to describe another point of view. Our 9^{th} grade students had the following theoretical prerequisites: Vector calculation, solving systems of equations by hand and with the aid of the TI-92, the parametric equation of a line, intersection of lines given by parametric equations.

The first attempt of the pupils using their theoretical knowledge was to find the parametric equations of the two lines (arl(t) and hei(s)). Since they had to find the point of intersection they remembered that they should use different names for the parameters of the two functions (figure 2.8). With the features of the TI-92 Plus they solved the system of equations arl(t) = hei(s) (figure2.9). After having the coordinates of the point of intersection their first reaction was: A point of intersection exists and therefore the ships will collide.

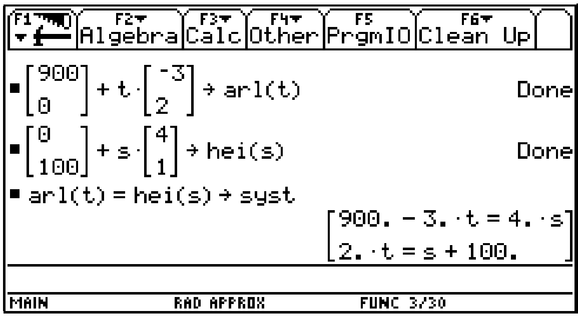


figure 2.8

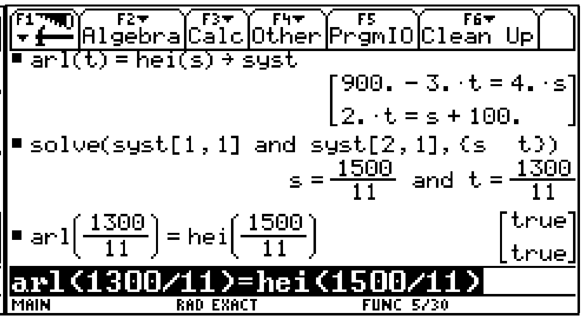


figure 2.9

But there were also doubts about this interpretation of the mathematical result: If the ships´ paths cross and they are both traveling on the surface of the water, that does not necessarily mean that they will collide.

Now a discussion about the role of the parameter started.
Some conclusions:

- The parameter represents the time.
- The parameters of the point of intersection are not equal with respect to the equations of the two lines which intersection means the two ships are not at this point at the same time.
- If we want to compare the positions of the two ships at the same time, we have to use the same variable t in both parametric equations: t=0 means the starting position, t=1 the position of the ships after a minute a.s.o.

The *Parametric-mode* of the TI-92 offers a special chance to visualise these conclusions and to come to a better understanding of the role of the parameter (figure 2.10)After looking for an appropriate viewing window (figure2.11) and changing to the graph-window the calculator is in the simulation mode, and the students can watch the ships moving along their paths. The parameter *tstep* allows the students to steer the velocity of the movement on the screen.

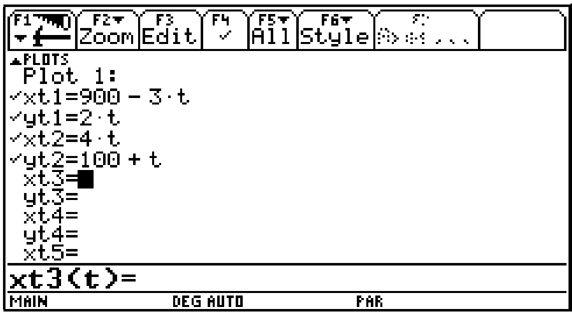


figure 2.10

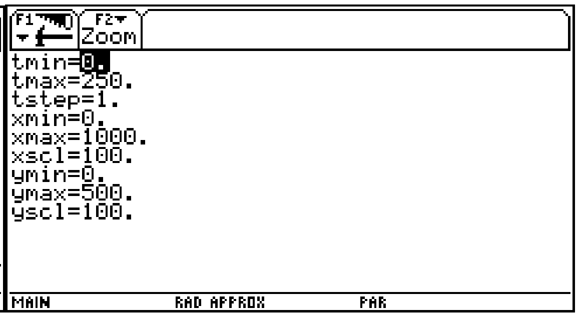


figure 2.11

With the *Enter-key* they can stop the movement at any time and compare the positions at this time (figure 2.12 and 2.13). It is easily seen that the USS Arlington passes the point of intersection before the Height arrives.

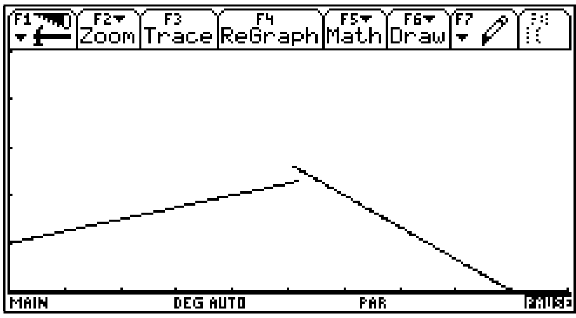


figure 2.12

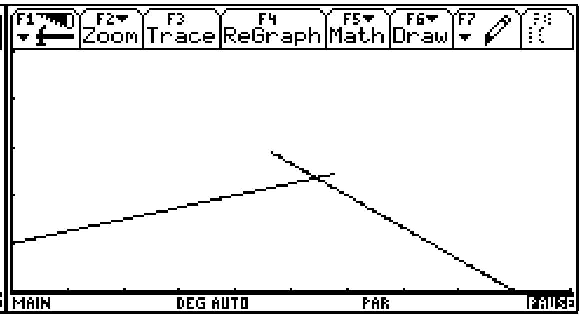


figure 2.13

Another point of view is to use tables (figure 2.14 and 2.15). The suitable section of the table can be found by experimenting in the graph-window using the *Trace-mode*.

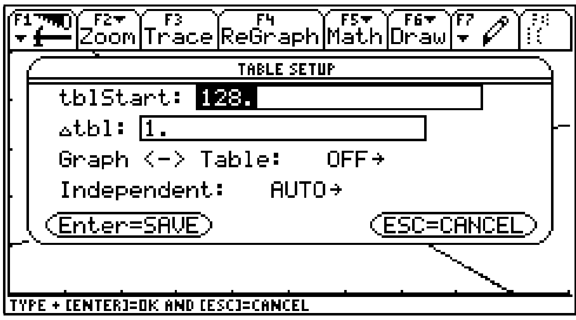
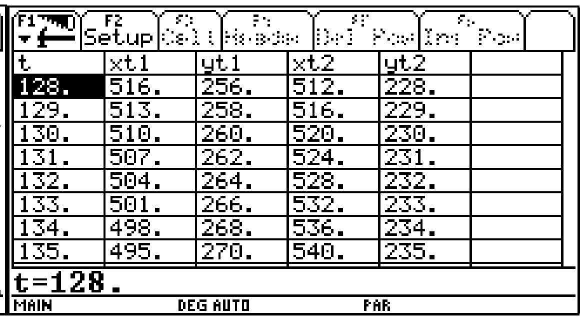


figure 2.15



t	xt1	yt1	xt2	yt2
128.	516.	256.	512.	228.
129.	513.	258.	516.	229.
130.	510.	260.	520.	230.
131.	507.	262.	524.	231.
132.	504.	264.	528.	232.
133.	501.	266.	532.	233.
134.	498.	268.	536.	234.
135.	495.	270.	540.	235.

t=128.

figure 2.14

The TI-92 also allows the learner to find the distance of the ships at any time in an experimental way:The F5 Math toolbar menu offers the tool *Distance*. The calculator asks for the first and the second point. The first point can be found when following the path of the USS Arlington (in the *Trace-mode*). Then the pupils have to jump to the path of the other ship Height to define the second point. Now the TI92 calculates the distance (figure 2.16).

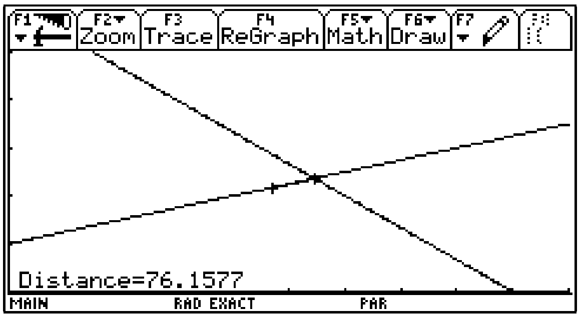
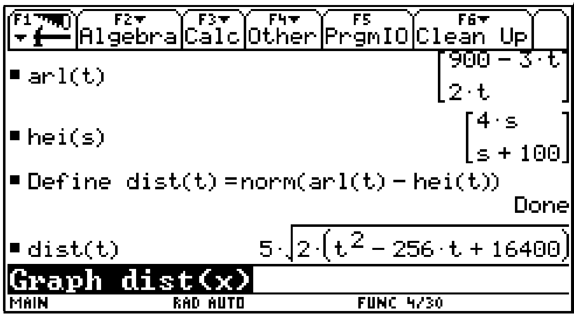


figure 2.16

In traditional mathematics education students and also teachers normally are content to have found one way of solving a problem. Several areas of mathematics are taught side by side often with no connection. A great advantage of such a powerful tool like the TI-92 is that it is possible to compare various ways of solving a problem. This is a chance for the students to discover important connections between several mathematical areas, to prove the results and to come to a better understanding, not only of the mathematical theory and the matematical model but also of the solution of the applied problem.

Such a new representation of the „ship-problem“ is to return to the *home-screen* and to use the prerequisites in vector calculation and in the area „functions“ (graph, zeros, maximums of functions a.s.o.). The first step is to define a function dist(t) (figure 2.17). This is the length of the vector arl(t)-hei(t). After finding a suitable viewing window using the *Trace-mode* the pupils can find the minimum of the function in an experimental way and they can also suppose that no zero exists. Using the F5 Math toolbar menu the TI-92 as a black box also calculates the minimum point (figure 2.18)



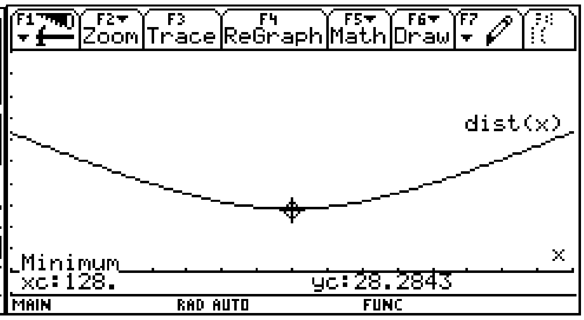


figure 2.17 figure 2.18

In the calculus course (in our country in 11th grade) this black box can be made white. That does not mean that the students, after having explored the idea of derivations, should do the calculations themselves. They have to find a model to decide on the appropriate algorithm, in other words they have to do the planning, the calculation is done by the CAS (figure 2.19). The answer is: There is no point with distance 0, that means no collision point, and the smallest distance is reached after 128 minutes.

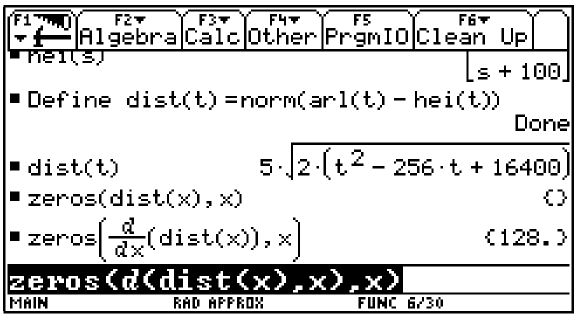


figure 2.19

2.3 The use of the recursive prototype

Example 2.4: A financial problem
One takes out a loan of \$ 100.000,- and pays in yearly instalments of \$ 15.000,-. The rate of interest is 9%. After how many years has he paid off his debts?

In traditional mathematics education such problems could be solved for the first time in 10th grade, because the students need geometric series and calculating skills with logarithms. The computer offers a new model, the recursive model. From that pupils now work such problems in 7th grade

The first step is finding a word formula, which describes what happens every year:
Interest is charged on the principal, the instalment is deducted.

Translated into the language of mathematics:
$K_{new} = K_{old}*(1+p/100) - R$

The TI-92 offers a special way to come to a better understanding of a recursive (or better an iterative) process: The activities of storing and recalling make the pupils conscious of the two important steps of a recursive process: Processing the function and feedback (K_{new} Y K_{old}) (figure 2.20). Looking at the list of values the quality of an exponential growth becomes much clearer than by calculating with logarithms. The typical problem of paying in instalments can be recognized: During the first phase the loan is nearly equal because the greatest part of the instalment is used for the interest (figure 2.21). The experimental solution is obtained by repeated using of the enter-key until the first negative value appears (figure 2.22). The variable n shows the number of the years.

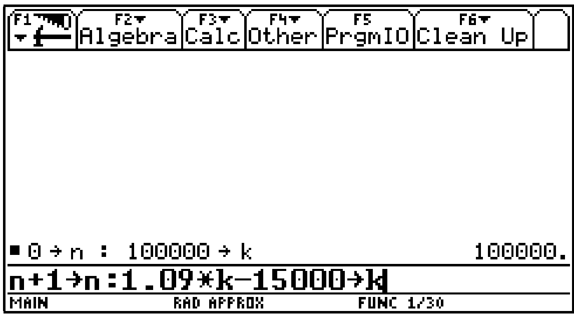


figure 2.20

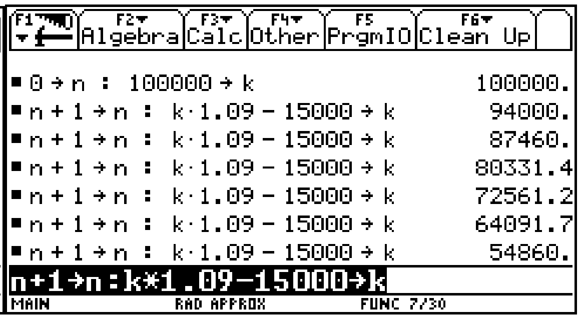


Figure 2.21

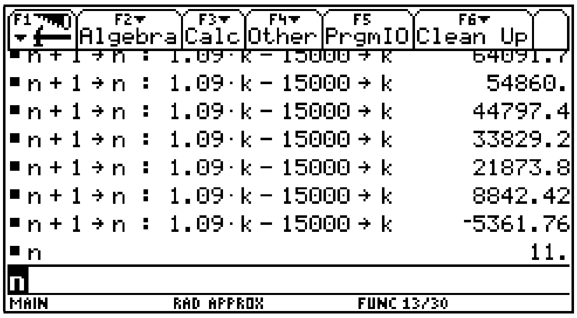


figure 2.22

In this learning phase the pupils should explore the fundamental idea of a recursive process by experimenting and working step by step. We call such a phase White Box Phase, a phase of cognitive learning. In the next learning phase - the Black Box Phase - students can use the Sequence Mode of the TI-92 as a black box (figure 2.23). They can easily experiment with several rates of interest and installments. Simulating is done by the CAS. The students have to find a suitable model and to interpret the results, either the table or the graph (figures 2.24 and 2.25).

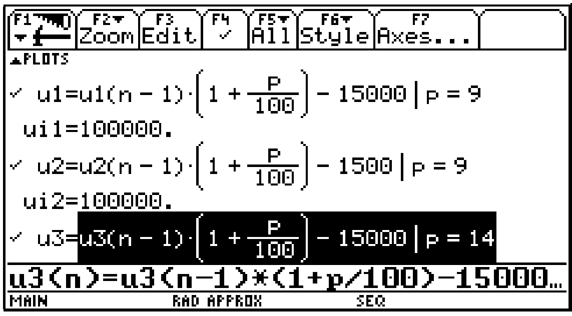
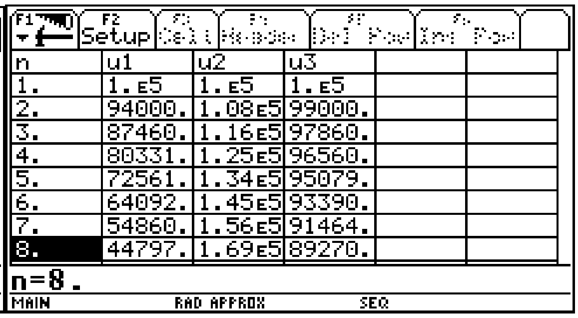


figure 2.23



n	u1	u2	u3
1.	1.E5	1.E5	1.E5
2.	94000.	1.08E5	99000.
3.	87460.	1.16E5	97860.
4.	80331.	1.25E5	96560.
5.	72561.	1.34E5	95079.
6.	64092.	1.45E5	93390.
7.	54860.	1.56E5	91464.
8.	44797.	1.69E5	89270.

figure 2.24

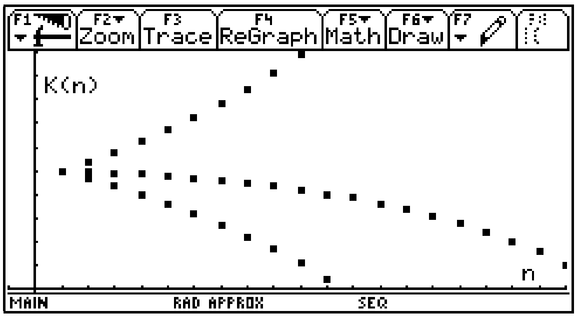


figure 2.25

3. The Window Shuttle Method

It is not enough to make various prototypes of a general concept available, the establishment of the relationship among the individual prototypes leads the pupil´s attention from the specificity of a singular prototype to the superior general conceprt

In order to recognize the „prototypical“ as an invariable characteristic, the prototype often has to be changed.

In traditional mathematics education prototypes mostly are available in a serial way. A typical example is the discussion of curves: One prototype, the term is given. The students have to find the graph by calculating the zeros, the extremums, the inflection points and they have to determine a table of values.

The main importance of the computer is that the learner can use several prototypes parallely. The given term allows the pupil to draw the graph directly and the table is the result of activating one key of the tool. The real learning process consists of shuttling between several prototypes and investigating the influence of changes of one prototype in the others. Therefore we call this didactical concept the **Window Shuttle Method**.

The steps of the learning process according to the window shuttle technique and the role of CAS in this process:

- The pupil activates various adequate prototypes for the problem or the concept in different windows of CAS, for example a symbolic prototype in the algebra window and a graphic prototype in the graphic window.
- The pupil now works with the individual prototypes, whereby the advantages of CAS such as interactivity, easy manipulation and repetition can be applied.
- The multiple window technique enables the learner to work simultaneously with various prototypes. In continuous interaction between the algebra and graphic window and the table, the effect of algebraic operations on the graphs or on the table values can be examined or ideas for activities in the algebra window can result from examining graphs or tables. Furthermore the consequences of the alteration of individual parameters in the algebra window, can be examined directly in other windows.

A concept or a solution of a problem develops by shuttling back and forth between the various forms of representation, meaning between different windows in CAS.

The observation of the students involved in our CAS project strengthens the thesis that the tool CAS does not only support cognition, it becomes part of cognition.

Example 3.1: First experience with the function concept in the 7th grade: Direct - indirect proportion

This example is part of an investigation, called „observation window“ in the Austrian CAS project [Klinger, 1997]. The Goal was to observe the pupils´ behavior: The learners should choose a prototype of a function suitable for a given problem and they should discover and use strategies for the proof of a definite functional relation.

The initial problem for indirect proportions was rather simple:
The distance between Vienna and Insbruck is 500 km. Calculate the driving time for several mean velocities.

According to the goals of this investigation - pupils should activeley discover new concepts and strategies - it was necessary to give precise instructions:

- Calculate the time for the velocities in the given table. What happens if the velocity is two times, three times, ten times, k-times greater than before?

- Find a formula in the y-Editor. Using this formula find the values of a table. Check the correctness of the values of the table given by the teacher.
- Calculate the product of the velocity and the appropriate time, at first in the Home Screen and then in the Data/Matrix Editor. Select 7 values of the given table. What is noticeable?
- Find the graph of the table values in the Graphic Window. Walk along the graph, using the Trace Mode and check the values of the teacher´s table.

The first goal was to observe which prototype the pupils will prefer. The following were available (figure 3.1 to 3.5):

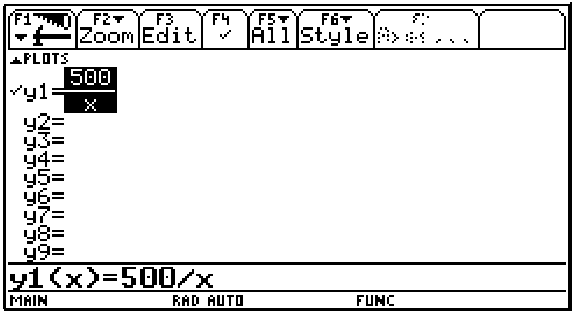
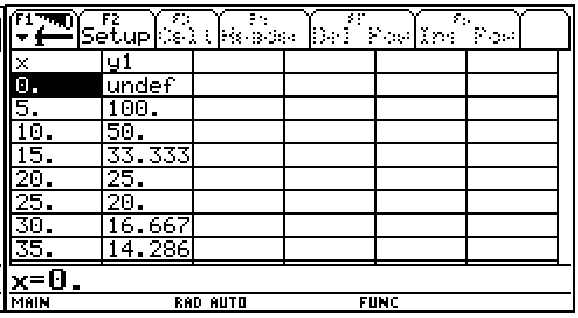
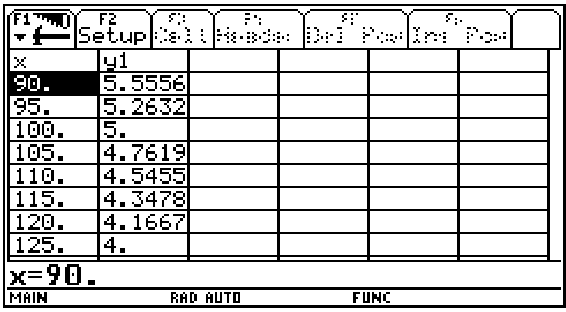


figure 3.1



x	y1
0.	undef
5.	100.
10.	50.
15.	33.333
20.	25.
25.	20.
30.	16.667
35.	14.286

figure 3.2



x	y1
90.	5.5556
95.	5.2632
100.	5.
105.	4.7619
110.	4.5455
115.	4.3478
120.	4.1667
125.	4.

figure 3.3

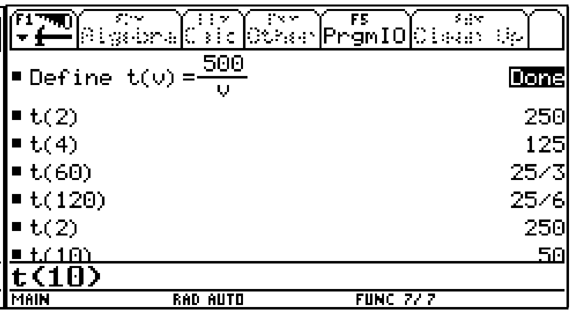


figure 3.4

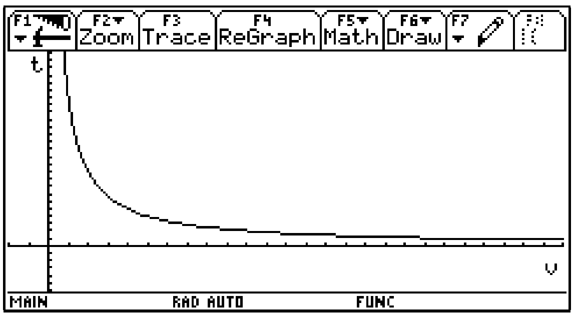


figure 3.5

The formula was found in the y-Editor (figure 3.1) Shuttling to the table the TI-92 calculates the table values using this formula. Pupils can find a suitable section of the table (figure 3.2 and 3.3). A surprise was the definition of the function in the Home screen and the use of this prototype in the testing phase. Such a strategy is unusual in 7th grade.
Just as new is the frequent choice of the graphic prototype which, with the help of the CAS, is now available very easily by shuttling from the table to the graphic window. Shuttling back is possible by using the Trace mode which allows the learner to observe the coordinates of the respective points
The second goal was to discover and to select proof-methods for indirect proportions

The following strategies were used:

Strategy 1: Examine in the Table editor in several cases:

- two fold corresponds to one half
- five fold corresponds to one fifth
- n-fold corresponds to one n^{th}

Strategy 2: Proof with the formula either in the Home screen or in the y-Editor by using the „with- operator" or calculating function values of the defined function (figure 3.6)

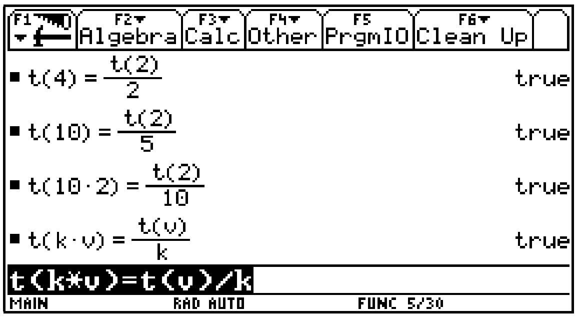
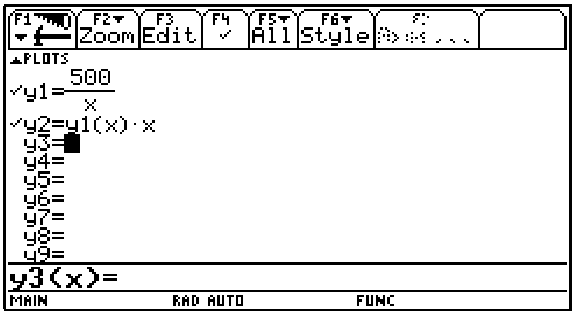


figure 3.6

Strategy 3: Proving the following rule in the Home screen or in the y-Editor (included the control of the table) or in the Data/Matrix Editor (figures 3.7, 3.8, 3.9):

- The product of the argument and the function value is constant



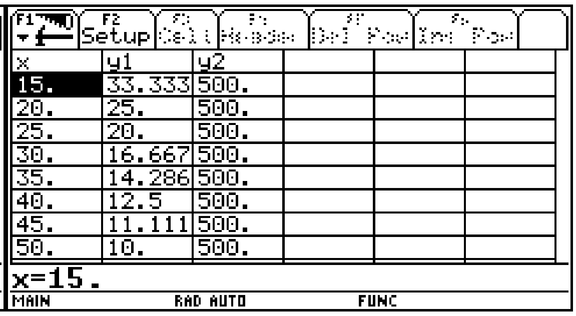


figure 3.7 figure 3.8

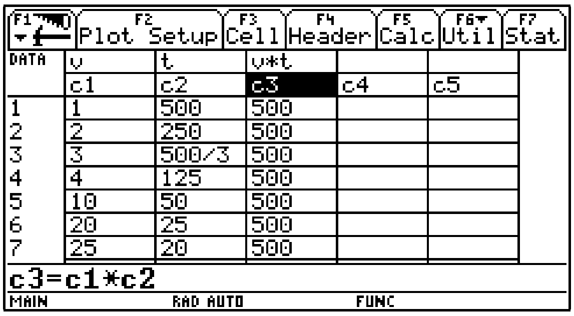


figure 3.9

Strategy 4: Drawing the graph in a suitable intervall in the graphic window.
The „typical curve" is called hyperbola

The third goal: These strategies should also enable the learners to decide in certain examples that neither a direct nor an indirect proportion exists.

The pupils had to examine the following problem:
The force of gravitation of the earth with respect to the distance.
The pupils got a table of values in the Data/Matrix Editor. Observing the graph (figure 3.10) could cause the supposition: It looks like a hyperbola - it is an indirect proportion. But shutteling to the Data/Matrix Editor and using strategy 3, pupils found out: The product of argument and function value is not constant (figure 3.11)

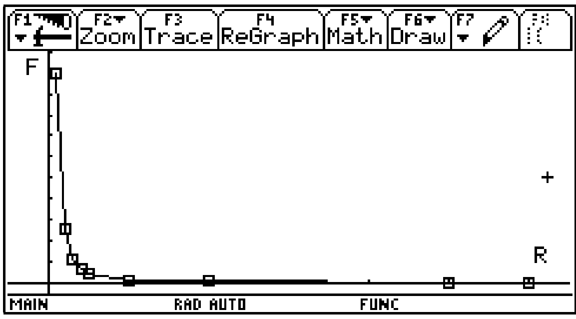


figure 3.10

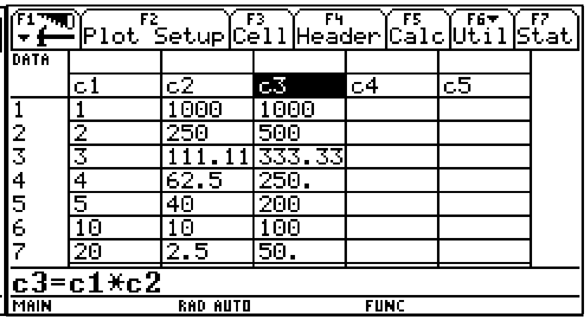


figure 3.11

Some results of pupils´ behavior:
Pupils use the possibility of having several prototypes of the function parallely at their disposal. Shutteling between several prototypes becomes a common practice and allows then to use the advantages of certain prototypes.
Several pupils develop preferences to several prototypes. In traditional math education, the table often was the only prototype which was at their disposal. I did not expect that pupils of the 7th grade will also use the graph and the defined function (see figure 3.6), the last one is prefered by more gifted children.
It is not only easier now to get tables, the opportunity to calculate with whole rows is the main importance of function prototypes in the Data/Matrix Editor.
The testing strategies strengthen the decision competence according to the type of the function.

Example 3.2: The third Kepler rule
Circulating time of planets with respect to the distance from the sun [Schmidt, 1997]

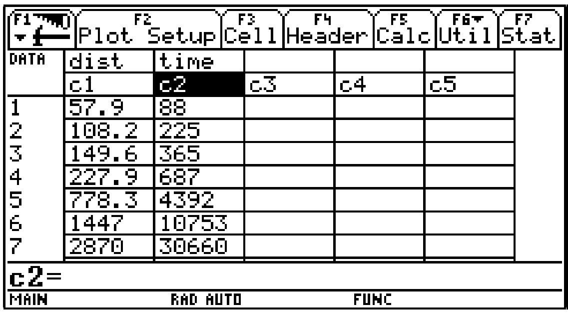
Goals of this example:
- Describing of real phenomena with functions.
- Starting with measured data, the students should discover the rule by using several prototypes of functions.
- The features of CAS and the Window Shuttle Method should open up new possibilies for an experimental and pupil-oriented learning process

The students got a table with observation data:

Planet	Distance (in mil. Km)	Circulation time in days
Merkur	57.9	88
Venus	108.2	225
Erde	149.6	365
Mars	227.9	687
Jupiter	778.3	4392
Saturn	1447.0	10753
Uranus	2870.0	30660
Neptun	4497	60150
Pluto	5907	90670

figure 3.12

The first step was to enter the data of the table into the Data/Matrix Editor of the TI-92 (figure 3.12). To come to suppositions about the sort of functions it is better to „shuttle“ to the graphic window (figure 3.13). Zooming strengthened the assumption: It could be a power function of the type $y = a.x^c$.



	dist	time			
	c1	c2	c3	c4	c5
1	57.9	88			
2	108.2	225			
3	149.6	365			
4	227.9	687			
5	778.3	4392			
6	1447	10753			
7	2870	30660			

figure 3.12

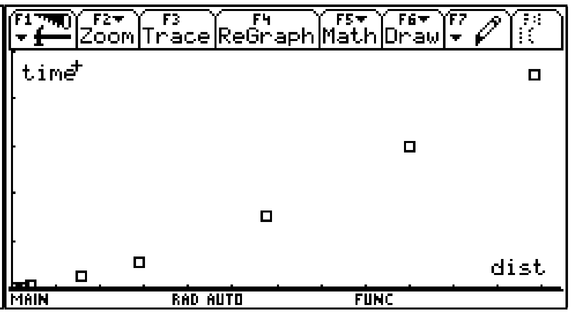


figure 3.13

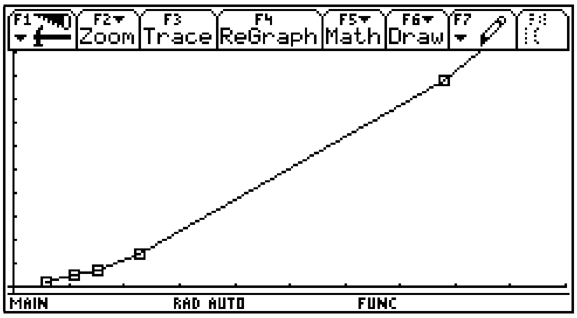
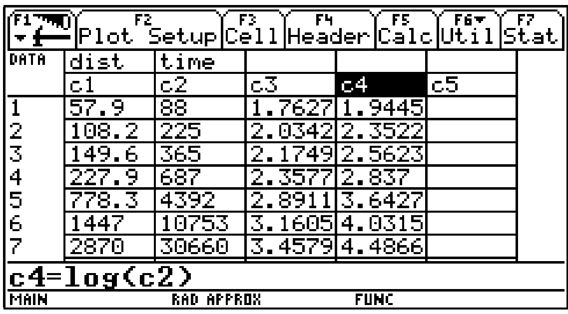


figure 3.14

Because the pupils had experience with the idea of linear regression they decided a „log-log-plot“ with axes log(x) and log(y) which allowed them to use their knowledge:

$$\log(y) = \log(a.x^c)$$
$$\log(y) = c.\log(x) + \log(a)$$

Defining new rows c3 = log(c1) and c4 = log(c2) the calculation was done by the TI-92 (figure 3.15).
If the suppositon is correct the points should be situated on a straight line (figure 3.16)



	dist	time			
	c1	c2	c3	c4	c5
1	57.9	88	1.7627	1.9445	
2	108.2	225	2.0342	2.3522	
3	149.6	365	2.1749	2.5623	
4	227.9	687	2.3577	2.837	
5	778.3	4392	2.8911	3.6427	
6	1447	10753	3.1605	4.0315	
7	2870	30660	3.4579	4.4866	

figure 3.15

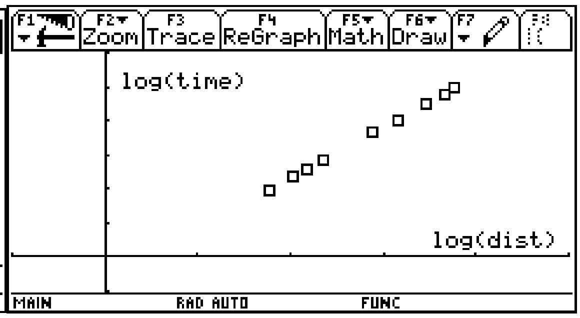


figure 3.16

By using the TI-92 as a black box the pupils could now find the equation of the linear regression (figure 3.17 and 3.18). The correlation coefficient is very good (near 1). Shuttling to the graphic window enabled the visualisation of this result (figure 3.19)

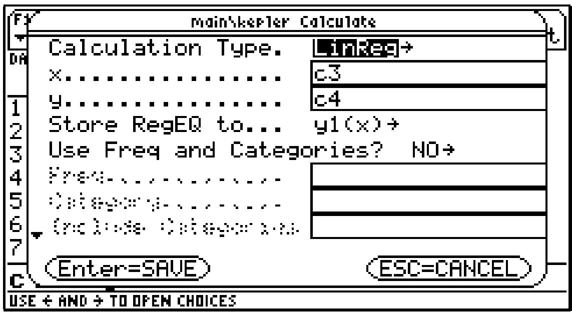


figure 3.17

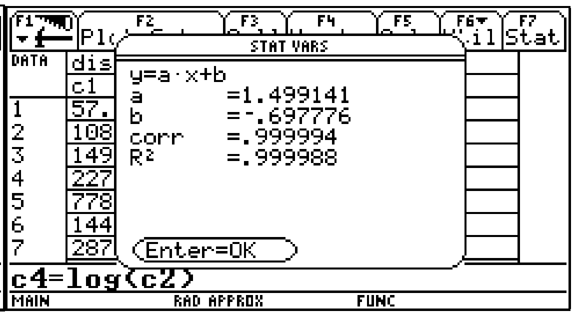


figure 3.18

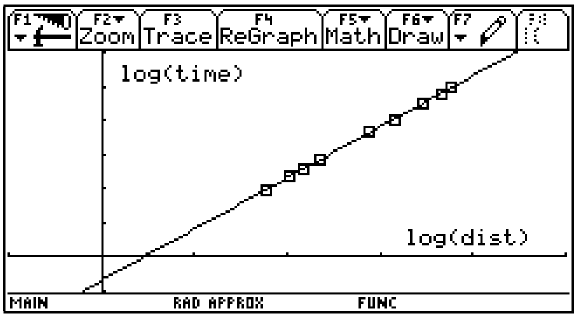


figure 3.19

Now the assumption was confirmed: The circulation time of the planets with respect to the distance from the sun is a power function of the type $y = a.x^c$.

To find the parameters a and c the pupils first had to jump to the y-Editor where the equation of the linear regression was stored. Remembering the equation log(y) = c.log(x) + log(a) they found out that the slope of the line is c and the y-intercept of the line is log(a). Thus they calculated the equation of the desired power function (figure 3.20). After squaring the equation the 3rd Kepler Rule can be seen:
„The squares of the circulation times of the planets and the cubes of the radius are proportional"
Shuttling to the y-Editor and the grahic window and drawing the power function the result can be visualized (figure 3.21 and 3.22)

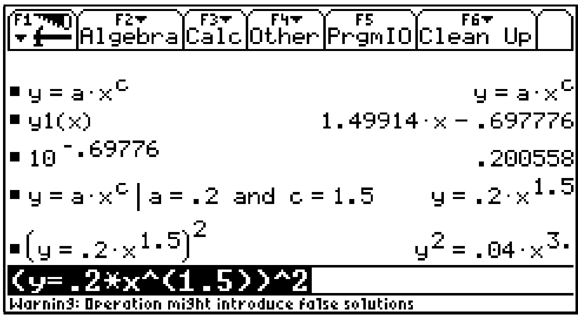


figure 3.20

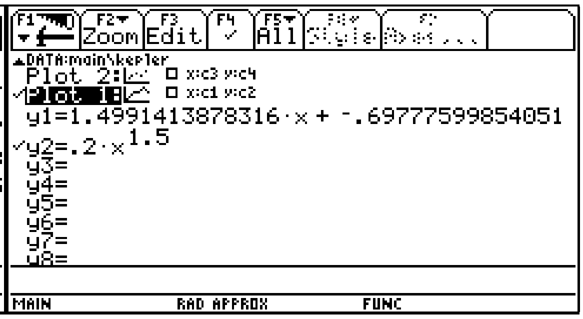


figure 3.21

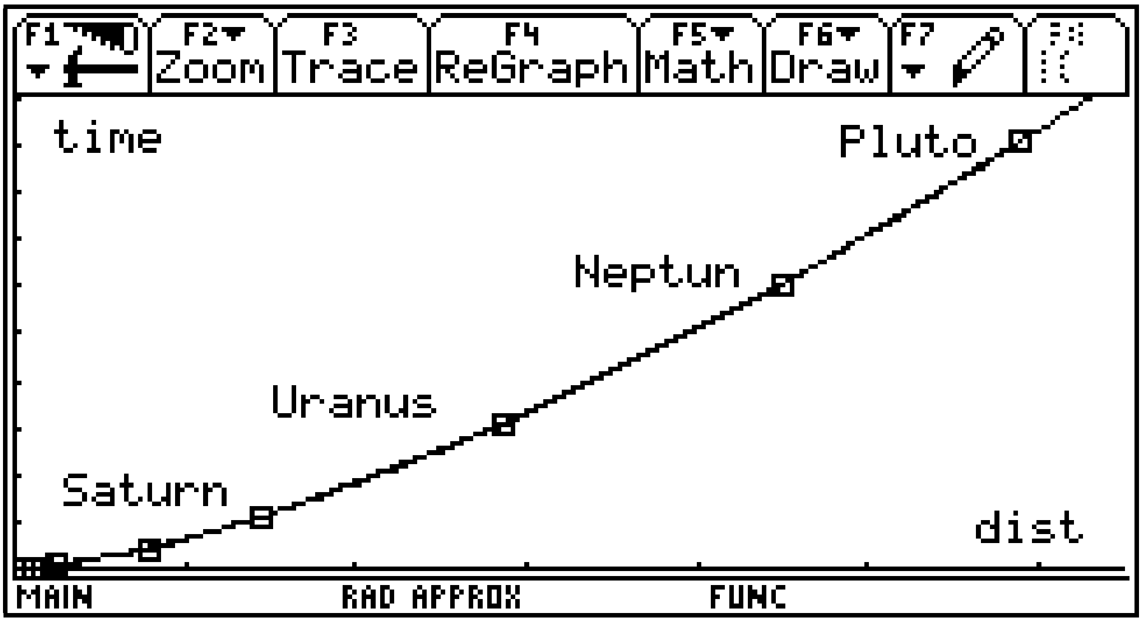


figure 3.22

4. Functions as modules

Students are more often using the possibility of CAS to store modules for calculating drawing a.s.o as functions and after that to use these functions in the problem solving process. This feature of CAS could be a lecture topic on ist own. I will speak about another aspect of using modules:

Modules created by the teachers

Two reasons can be observed for teachers creating modules:

- **Modules** which can be **used as didactical tools** in the white box phase to support the discovery of new mathematical areas.
- **Modules used as black boxes for operating** in the problem solving process.The advantage of these tools is that the students can concentrate their activity in modelling and interpreting, but in my opinion the sense and the yield regarding the fundamental goals of mathematics education depends on the question: Should the black boxes, or at least some of them, become white for the learners?

Example 4.1: The program package "VECTOR-CALCULATIONS"
by Thomas Himmelbauer [Himmelbauer, 1997]

This program package consists of a large number of programs and functions (nearly 140) which make vector calculations possible in $\breve{E}^2$ and $\breve{E}^3$ and allow students to draw points, line segments, lines, circles and planes. Activating the package, a separate toolbar appears which lets students display menus for selecting functions or programs.

Some exercises

Exercise 1: Find the distance of two skew lines

Step 1:
Using the function ***gepktrtg*** students get the parametric equations of the straight lines. The arguments of the function are two vectors, the position vector of a point and a direction vector. The equations are stored in the variables ***ger1*** and ***ger2***

Step 2:
Evaluating the function ***abswinge*** the result is the distance of the two skew lines. The arguments of the function are the names of the equations of the two lines. (figure 4.1)

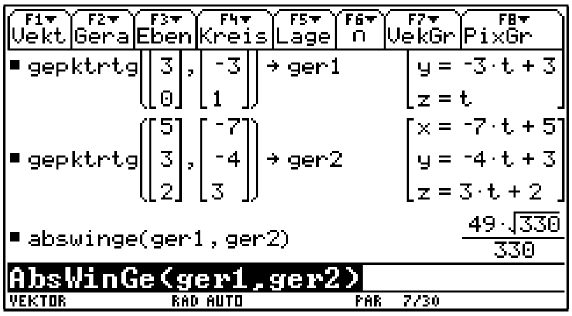


figure 4.1

Exercise 2: Find out the position of planes

Step 1:
Store the equations of the 3 planes in the variables eb1, eb2 and eb3
Step 2:
Evaluate the function ***drebenen,*** the result is a quadruple. The first element gives the answer: "*A point of intersection exists*", the others are the 3 coordinates of the point 8figure 4.2)
A possible answer of another example could be: "*Plane 1 and plane 2 are identical, plane 3 is intersecting plane 1 and plane 2.*"

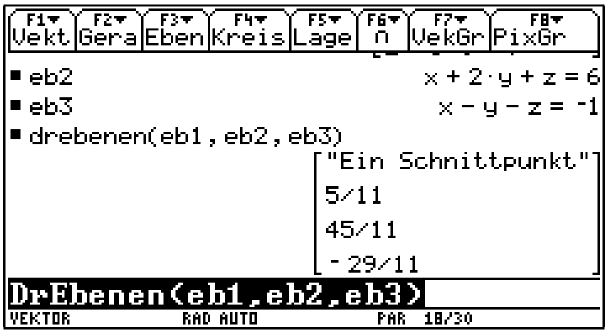


figure 4.2

Summary:
The teacher who created this powerful package of functions and programs uses it in a sensible way: In a white box phase students have to learn the theory and to construct some of these modules themselves and after this phase they can use thes modules for problem solving as black boxes.

But if there is no white box phase before or after the learning phases in which students have to find strategies and algorithms to solve such problems of the analytic geometry the idea of those black box modules is very contrary to the fundamental goals of mathematics education.

I agree with the following definition of mathematics:

Mathematics is the technique of solving problems by reasoning

Simply finding suitable black box functions by searching for suitable words in the menues and inserting given values is not doing mathematics.

Some problems we observed:

- It is very difficult to remember the meaning of more than 70 function and program names. The names are short cuts of meanings of the functions, e.g the function name "abswinge" is a short cut of "**Abs**tand zweier **win**dschiefer **Ge**rader" (in English: "distance of two skew lines"). Pupils are able to remember short cuts of modules which they constructed themselves but not the meaning of so many words of these menues.
- The user interface is unusual. The decision of using a comma or a semicolon or using a quadruple or pairs of values is often not very clear. The consequences are error messages and the correct input version is often only found by trial and error. The interpretation of the results sometimes is not easy.
- The greatest problem is that students who only use this powerful system of modules as a large black box simply learn words and how to use these words but they do not learn problem solving by reasoning.

5. Final summary

Especially the experiments in those classes where CAS is available in every working situation, in other words, not only in the classroom but also at home and in the exam situation, the hypothesis was confirmed that cognition must be seen as a functional system which encompasses both man and the tool as well as material and social context. Accordingly, the tool „computer" is not only an amplifyer of human performance, it can also change thinking qualitatively and can generate new ability.

We especially observed this significance of the tool „Computeralgebra System" in the field of functions.

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