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The influence of technology in several roles of mathematics

Abstract:
Studying the history of mathematics you can recognize that tools have always fundamentally influenced the development of mathematics. The computer, a child of mathematical thinking, has changed the several roles of mathematics as well as the ways of teaching and learning mathematics.

In my lecture I will formulate my thesis concerning the influence of technology by using three roles which mathematics can have:

- *Mathematics – a "two phase" process: the abstract phase and the concrete phase*
- *Mathematics – a language*
- *Mathematics – a thinking technology*

Using these three roles of mathematics I will give concrete examples for some of the changes caused by the use of technology:

- *A more pupil centred, experimental way of learning with a shift of emphasis from operating to modelling and interpreting.*
- *Technology supports both phases of mathematical activity – the abstract and the concrete phase.*
- *The use of technology allows the students to create new language elements.*
- *The use of technology not only supports cognition – it becomes part of cognition.*

Content

1. A short journey in the history of mathematics

A driving force behind the advancement of civilisation has always been the desire to overcome problems of daily life with the help of technical aids.

Watching the history of mathematics you can recognize that calculation tools had always influenced the development of mathematics.

Long before our current algorithms for written calculation were invented mathematicians used calculation boards called abacus. The name “abacus” can be derived from the Greek word “abakion” which means “round board”. One of the first calculation boards coming from the 4th century before Christ was found on the isle of Salamis. The first written document about calculating with the abacus comes from Gerbert who became pope in 999 and was called pope Silvester II. The first picture shows this evolution: The first step was the use of the calculation board, the second one calculating by using written numbers [Kaiser; Nöbauer 1984; pp 29-43].

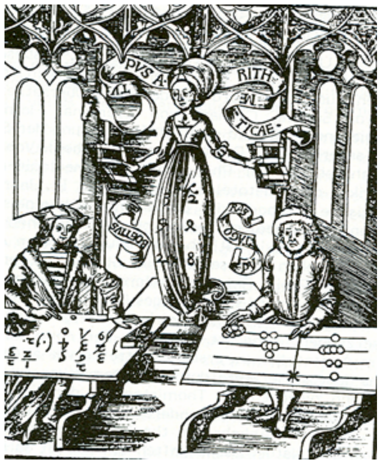


Fig 1.1

The invention of the first calculation machines: The first machine was built by Wilhelm Schickhardts, professor for biblical languages at the University of Thüringen in 1623. A sketch of the machine was found in a letter to Johann Kepler (Fig. 1.2). The machine could realize the arithmetical operations automatically. A reconstruction of the machine can be seen at the University of Linz.

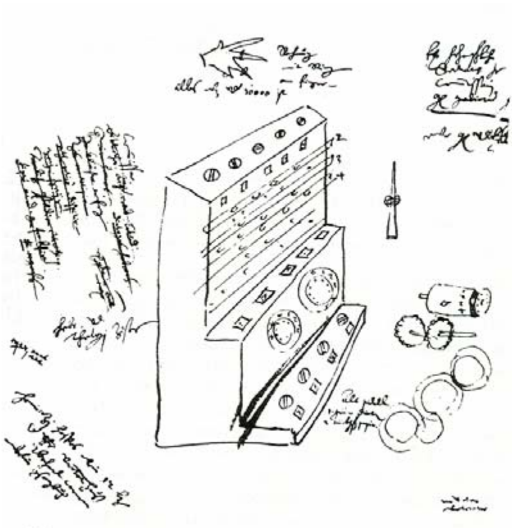


Fig. 1.2

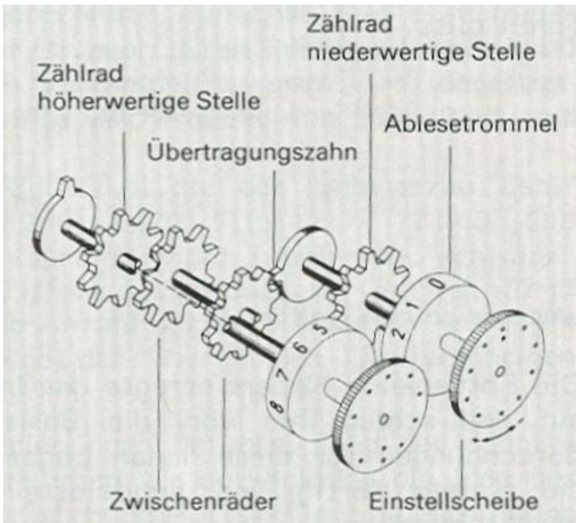


Fig. 1.3

The central component was the decadal “counting wheel” (Fig. 1.3)

A fundamental change in the roles of mathematics and a prerequisite for the growing use of mathematical methods in new areas was the invention of computers. Konrad Zuse in Germany and Howard Aiken in the US built the first prototypes in 1941 (Fig. 1.4) and 1943.

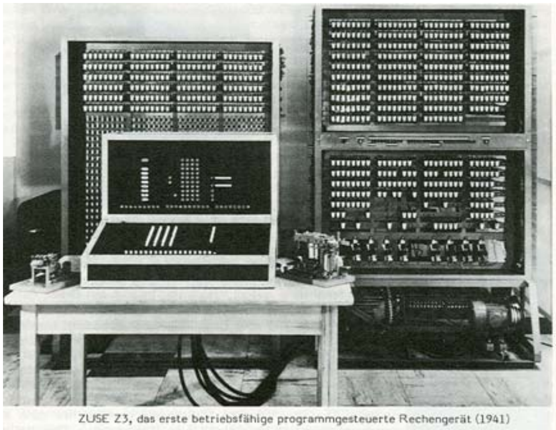


Fig. 1.4

The computer – a child of mathematical thinking – has influenced the role of mathematics and especially mathematics teaching more than all the other tools in previous history.

The topic of my lecture can only be to give some examples of changes caused by the computer.

2. Roles of mathematics and consequences for mathematics education

This lecture is a tribute to my most important teacher Bruno Buchberger. He is not only an excellent mathematics scientist, he developed the theoretical basis for the computer algebra and the emphasis of his recent work is “artificial reasoning”. More interesting for my work is the second side of Buchbergers research: By watching the way of a mathematician into the world of mathematics and by watching the student´s learning process he also formulated didactical concepts and principles.

These didactical ideas confirm the thesis of Jean Piaget that the genesis of knowledge in the sciences and in the individual follows the same mechanisms [Wittmann, 1981, S.59]. This recognition leads to an extremely dynamic model for the learning of mathematics.

The task of our organisation ACDCA is to observe the students in technology supported classes and to develop and to test new ways of teaching and learning in a technology supported learning environment.

Some of Buchbergers thesis which he presented at the last ACDCA conference “visit-me 2002” in Vienna [Buchberger, 2002]:

- The goal of mathematics is automation.
- The goal of mathematics is to trivialize mathematics.
- The goal of mathematics is explanation (= making things of high dimension plain = making complicated things simple).
- Mathematics is didactics.
- The process of trivialization is completely non-trivial.
- Think nontrivially once and act trivially infinitely often.

These thesis – especially the sentence. “mathematics is didactics” can be used as guidelines for developing didactical principles, as well as for developing a curriculum. They are also fundamental ideas for our ways of using technology in mathematics education.

Another approach to the discussion of several roles of mathematics comes from Roland Fischer, Professor at the University of Klagenfurt, the main topics of his recent research are "Human beings and Mathematics " and "Society and Mathematics". The topic of his last lecture in Vienna was "**The significance and usability value of mathematics**" [Fischer, 2004]. For the sociologist H. Tenbruck, whom Fischer quoted in his article, the progress of science is a process of trivialising. He distinguishes between the significance and the usability value of a science, especially of mathematics. At the beginning of a mathematical process the significance is rather high and there is normally no usability value. The process of trivializing causes a decreasing significance (or value of knowledge as Fischer says) and an increasing usability value.

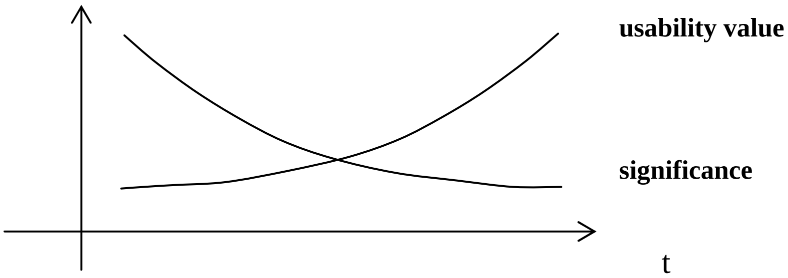


Fig. 2.1

A characteristic of mathematics is that the group of people who can recognize the significance or better the value of knowledge of mathematics is rather small.

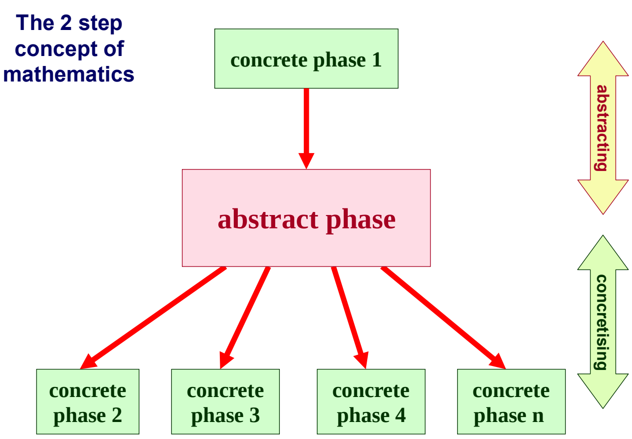
My point of view differs from Fischer and Tenbruck in two points:

1. I don´t agree that during the process of trivializing the significance or the value of knowledge is decreasing. What is decreasing during this process of trivializing is the degree of consciousness of the value of knowledge. When driving a car it is not important to think about how the engine works – the degree of consciousness is low but not the significance.

2. The development of mathematics must be seen as a process happening in different phases. A possible model is a "two phase model" consisting of:

- the phase of abstracting and
- the phase of concretising

The "two phase concept" of mathematics



The phase of abstracting:
Starting with a concrete problem a new theory, a new algorithm, new concepts have to be found. For this purpose it is necessary to detach from the concrete problem, to develop theories and algorithms in the abstract phase.

The phase of concretising:
It is exactly this process of detaching, of abstracting which opens the opportunity of a many-sided use of this mathematical area.

Figure 2.2

At the beginning in the first concrete phase the value of knowledge (or the value of recognition) as well as the value of usability are rather low.

Not until the abstract phase the value of knowledge does increase and reaches its maximum at the end of this phase. At this point the mathematical actors should also have the highest degree of consciousness. Learners may recognize that for the process of trivializing a non trivial mathematics is necessary. The value of usability remains low.

The abstract phase is the starting point for the second – important – phase of concretising. We can say: **The power of mathematics is the power of concretising**

In this phase of concretising not the significance is decreasing for me but the degree of consciousness of the mathematics which is behind the algorithms used to solve concrete problems. The significance is not decreasing rather it is represented in the high value of usability in the concrete phase.

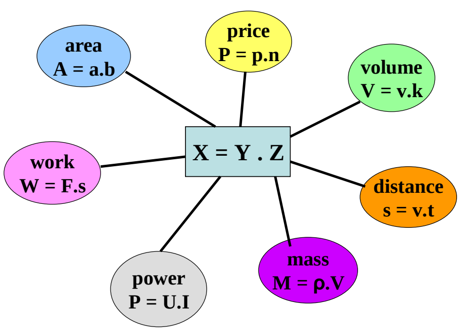
Two examples:

Example 2.1: The derivative of a function

- Concrete problem 1: Slope of the secant line ⇔ slope of the tangent
- Abstract phase: We have to detach from the tangent problem and to concentrate on the first derivative as the limit of the quotient of differences as $\triangle$x approaches zero. I am always disappointed if I ask a student: "*What is the first derivative of a function?*" and he answers "*it is the slope of the tangent*". Such a student did not experience the abstract phase of the mathematical learning process.
- This abstract phase opens the many sided use of the differential calculus

Example 2.2: Integral calculus

- Concrete problem 1: The area between a curve and the x-axis
- Abstract phase: It is better to detach from the concept of area and to concentrate on limit of sums, to concentrate on the fundamental theorem of calculus, on the concept of measure a.s.o. Again:
- This abstract phase opens the many sided use of the integral calculus



The picture tries to show the power of the integral. Several problems are related because of the common mathematical model. Therefore I always say to my students: The education in calculus starts in the fifth grade when students discover the area of a rectangle.

figure 2.3

The influence of technology in this 2-phase model of mathematics

The twentieth century is characterized by an enormous growth of the usability value: Mathematics has permeated nearly all our ways of living and working. Mathematising of many sciences like Psychology, National Economy, Biology, Chemistry a.s.o. shows this growing usability value. The computer plays a central role in this process. Computer supported mathematics allows us to develop more and more precise models of reality. On the other hand by using the mathematical logic we can build these technological tools and by using the theory of formal languages we are able to develop programming languages which are the prerequisite for more and better models.

2.1 The influence of technology in the phase of abstracting:

Thesis 2.1:

Technology like coputeralgebra systems (CAS) help to increase the value of knowledge and the degree of consciousness of the learners.

Example 2.3: Calculate the definite integral $\int_a^b x^2\,dx$ using the definition of the definite integral e.g. use the idea of „midsums“.

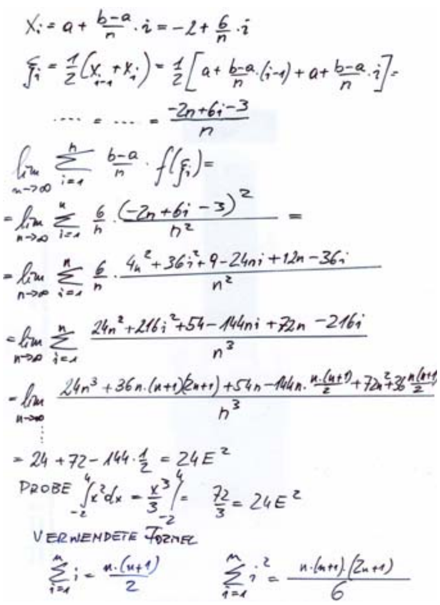


figure 2.4

In the traditional math education we often observe, especially in calculus, the lack of the abstract phase and the unreflective usage of Black Boxes. All students know, for example, that the indefinite integral of x^5 is $x^6/6$ but only a few have some knowledge about Riemann sums. One explanation is that only a few of these Riemann sums can be calculated by the students themselves. It is difficult to simplify the expressions and very difficult to calculate the necessary limits

Using a CAS opens new possibilties: Modelling - that means finding the formula of the sum, - is done by the students, operating is done by the computer. But we hope that the students before experience a White Box phase dealing with limits of sums. Therefore we created an important didactical principle , the White Box/ Black Box Principle as a guideline for computer supported teaching and learning

After dividing the intervall [a,b] into n equal parts and calculating the mid points $\xi(i)$ the function values $f(\xi\,(i))$ have to be computed (figure 2.5). Thus far the the operation could also be done by hand. But the sum normally cannot be calculated by a high school student as you can see in figure 2.4. Anyway this complex calculation is not so important for the understanding of the integral concept. Using the TI-92 the sum is available very quickly and also the limit of the sum (figure 2.6 and 2.7). The expected result is $b^3/3 - a^3/3$.

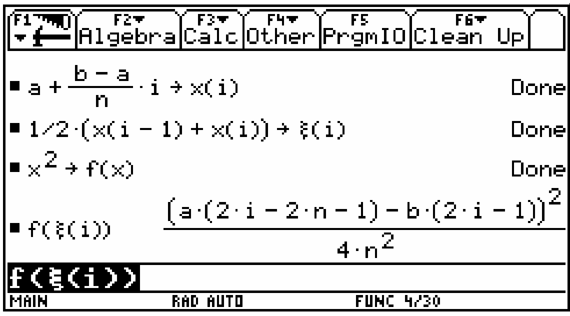


figure 2.5

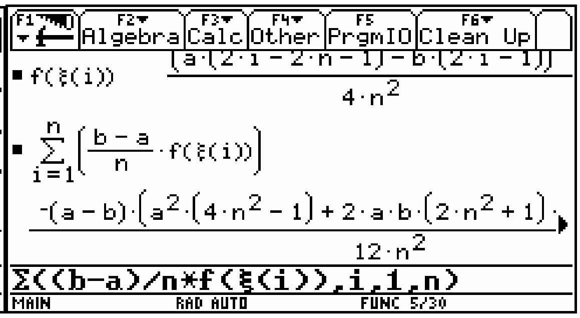


figure 2.6

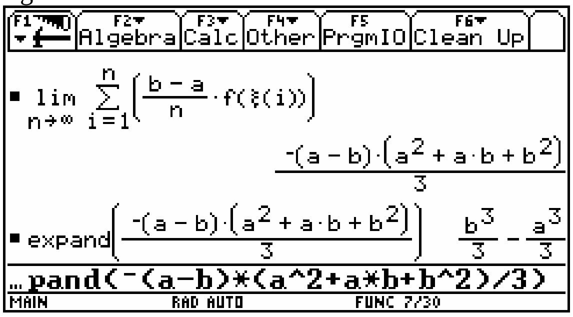


figure 2.7

2.2 The influence of technology in the phase of concretising

In an Austrian daily newspaper the author of a commentary, entitled „Environment instead of math“, demanded that math lessons be abolished and more important topics, such as environmental protection, take their place in the curriculum. There is one reproach in the article that we should take seriously: The author writes: „*I experienced math as time wasted in senseless mental gymnastics*“.

Thesis 2.2:

> The computer caused an enormous growth of the usability value of mathematics. The powerful calculation competence, the possibility of simulating, building complex models by programming, visualizing a.s.o. open a large number of new concrete applications. This possibility has also changed the technology supported math education. We can offer to the students a more application oriented, a more interesting, a more meaningful mathmematics.

The time of the lecture is too short to give enough examples for the technology supported concrete phase and I think a lot of the lectures of this conference will deal with these possibilities. One possibility is to visit the homepage of ACDCA: www.acdca.ac.at. Therefore I will concentrate on some reasons for the influence of the computer in the concrete phase:

Some reasons:

- More complex and also new models are available.
- Complex operations could be done by the CAS.
- Several representations of results lighten the interpretation of the given problem.
- Mathematics could become a „service subject“ for other subjects and thus support cross curriculum teaching.

Dangers caused by the use of technology

But the use of technology also causes a lot of dangers, one of which I will express in my

Thesis 2.3:
Mathematics is not only mathematising

This dangerous definition "Mathematics ⇔ Mathematising" I heard in a lecture concerning the role of constructivism in mathematics education and I also observe this interpretation in classrooms where technology is used.
Theoretically, CAS, if used as a Black Box, would offer the possibility of doing mathematics without mastering algorithms. The pupil could form assumptions in the heuristic phase, skip over the abstract phase, over the corroboration of algorithms and the practicing of calculating skills and then using the CAS as a Black Box immediately turn to the applications.

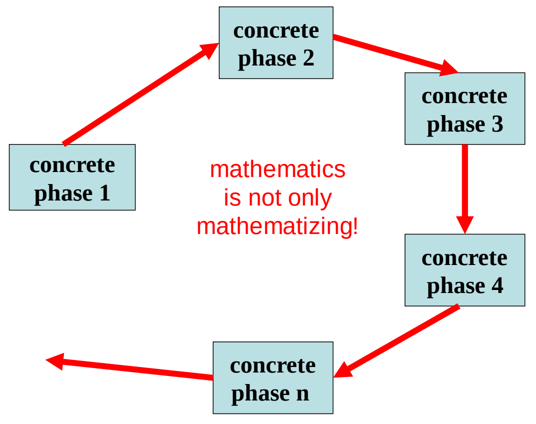


figure 2.8

If mathematising means this way of doing mathematics, forgetting the abstract phase, forgetting the necessity of developing and mastering algorithms, then in my opinion it is a dangerous definition of mathematics.

This way of mathematics education disregards the NCTM-principles concerning "Learning the Basics" and also the standard "Reasoning and Proof" where you can read: "*Students should recognize reasoning and proof as fundamental aspects of mathematics*"

3. Mathematics as a language

In a German curriculum can be read

Students should learn three sorts of languages:

- *the mother tongue*
- *foreign languages and*
- *the language of mathematics*

Observing the evolution of natural sciences in the twentieth century we can acknowledge that

“The book of nature is written in the language of mathematics”

Mathematics is a language and like other languages it has its own grammar, syntax, vocabulary, word order, synonyms, conventions, a.s.o. [Esty, 1997]. This language is both a means of communication and an instrument of thought.

One main goal of the learning of mathematics is to have the students assimilate the basic concepts and language skills which are fundamental to mathematics. Mathematical language skills include the abilities to read with comprehension, to express mathematical thoughts clearly, to reason logically, to recognize and employ common patterns of mathematical thought. *[Esty, W., 1997, preface]*

Unique among languages is its ability to provide precise expressions for nearly every thought or concept that can be formulated in its terms. The power of the modern mathematical language may be seen in the following two examples: On the one hand the original formulation of theorems of ancient Greek mathematicians and on the other hand their equivalents in modern math language: [http:www.cut-the-knot.org/language/index.shtml]

Ancient mathematical language closer to the native language	**Modern language of mathematics**
If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments. (Euclid, Elements, II.4, 300B.C.)	$(a+b)^2 = a^2 + b^2 + 2.a.b$
The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle. (Archimedes, On the Sphere and the Cylinder, 220B.C.)	$A = r.2\pi r/2 = r^2\pi$

figure 3.1

These examples and our investigations in the classroom corroborate again J. Piagets Thesis that the genesis of knowledge in the sciences and in the individual follows the same mechanisms. Also students during their way into mathematics acquire more and more new language elements and necessary rules for using the language of mathematics for problem solving.

On the other hand it is a bad didactical mistake starting too early with ten year old children using the pure modern mathematical language. When I visited math lessons and looked at the pupils´ exercise books my first question to the teacher was: where can I find the native language.

The evolution of the language of mathematics follows as a translation process form the native language to the mathematical language and back.

Teachers support this process by using tools coming from language lessons e.g. using a “vocabulary”

English	Mathematik
“so you get”	**=**
“3-times of”	**• 3**
“p% von”	**• p/100**
“increase by p percent”	**• (1+p/100)**

figure 3.2

If we maintain that the main role of mathematics is problem solving, consisting of the activities modelling – operating – interpreting, then **a main goal of mathematics learning is the translation process from a problem formulated in the native language to a mathematical model written in the language of mathematics.**

The translation process from the student´s language into the language of mathematics mostly takes place in three stages:

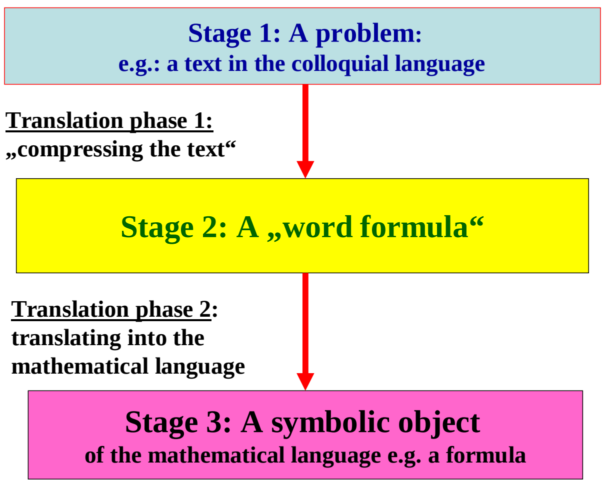


figure 3.3

The more difficult step for the students is the first one: Compressing the text - finding the main linguistic parts which have to be translated – finding a so called “word formula” still expressed in the colloquial language but suitable for a direct translation.

The influence of technology in the language of mathematics:

Although being a child of mathematical thinking, computers of the first period needed their own language and it was difficult to use them in mathematics lessons. Besides they were only able to use numerical methods, a fact that also changed the science landscape: numerical mathematics was dominating.

The translation process in this area at the end of the 70ies and the beginning of the 80ies last century was more complicated when using a computer: Two translation acts were necessary: The translation from the colloquial language into the language of mathematics and afterwards the translation into the language of the computer. This additional translation process and the need of an additional language was the reason that computers were rarely used in mathematics education.

A decisive step in the evolution of technology was the development of computer algebra systems (CAS). I started my first classroom experiment with a CAS in 1986 using a HP28C calculator in a 10^{th} grade. It was a fascinating experiment and the starting point of my research activity in this field but my students had still also to learn the language of the calculator

With the further development of powerful software systems like Derive, computers more and more learned to understand the language of mathematics: This was the key to a widespread use of technology in schools: Computers understood the language which the students used in their exercise books and they were not only able to use the algorithms which the students needed, they offered a lot of new algorithms and new sorts of presentations:

3.1 The translation process

Thesis 3.1:

Technology supports the translation process from the native language into the language of mathematics

- Technology, especially CAS allows the students to transform the word formula directly into a symbolic object of the mathematical language by defining variables, terms or functions or writing programs.
- The CAS allows a greater variety of prototypes of a formula and also offers some which were not available before. While in traditional mathematics education often only one prototype is available and used, now the CAS offers several prototypes parallely.
- The CAS offers and allows a greater variety of testing strategies, in this case testing if the formula is suitable for the problem and mathematically correct.

Example 3.1: A financial problem

One takes out a loan of K=$ 100.000,- and pays in yearly instalments of R=$ 15.000,-. The rate of interest is p=9%. After how many years has he paid off his debts?

In traditional mathematics education such problems could be solved for the first time in 10^{th} grade, because the students need geometric series and calculating skills with logarithms. The computer offers a new model, the recursive model. From that pupils now work such problems in 7^{th} grade

The first step is finding a word formula, which describes what happens every year:

Interest is charged on the principal K, the instalment R is deducted.

Translated into the language of mathematics:

$K_{new} = K_{old}*(1+p/100) - R$

After finding a formula in the phase of modelling, operating is the next activity. Using a CAS calculating takes on a new meaning. The Voyage (TI 92, TI 89, …) offers a special way to come to a better understanding of a recursive (or better an iterative) process: The activities of storing and recalling make the pupils conscious of the two important steps of a recursive process: Processing the function and feedback (K_{new} => K_{old}) (figure 3.4 entry line). Looking at the list of values the quality of an exponential growth becomes much clearer than by calculating with logarithms. The typical problem of paying in instalments can be recognized: During the first phase the loan is nearly equal because the greatest part of the instalment is used for the interest (figure 3.5). The experimental solution is obtained by repeated usage of the enter-key until the first negative value appears (figure 3.6). The variable n shows the number of the years.

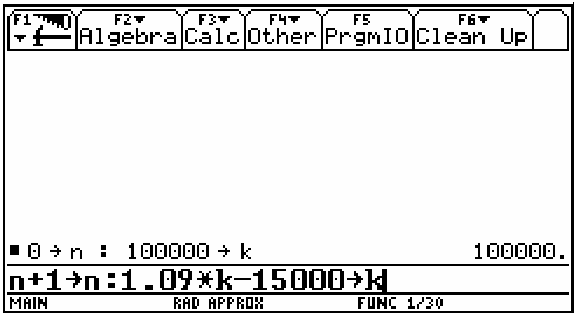
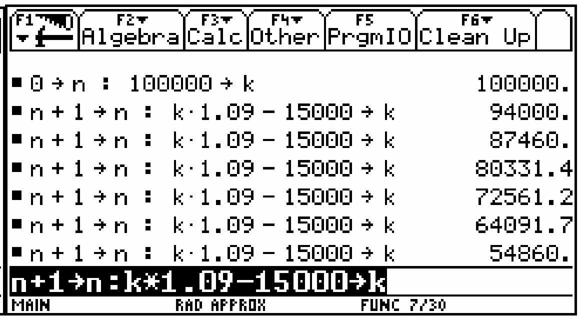


figure 3.4



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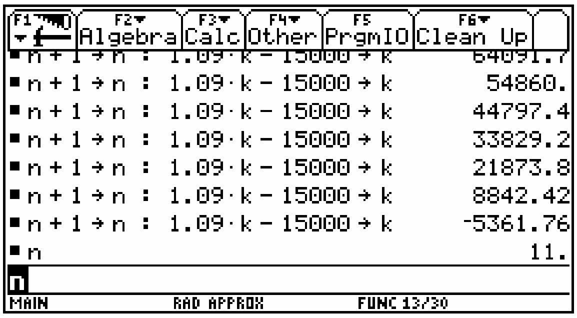
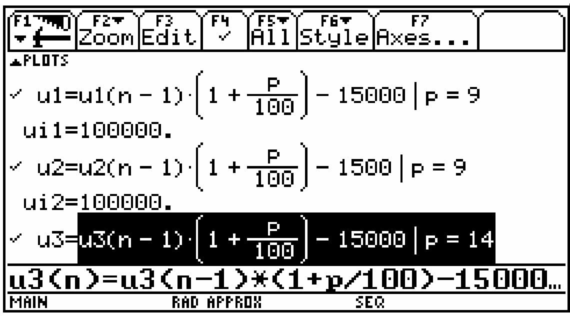


figure 3.6

In this learning phase the pupils should explore the fundamental idea of a recursive process by experimenting and working step by step. We call such a phase White Box Phase, a phase of cognitive learning.

By using the Sequence Mode in the next learning phase - the Black Box Phase – a new prototype of the formula is available (figure 3.7). The learners can easily experiment with several rates of interest and installments. Simulating is done by the CAS. The students have to find a suitable model and to interpret the results, either the table or the graph (figure 3.8 and 3.9).



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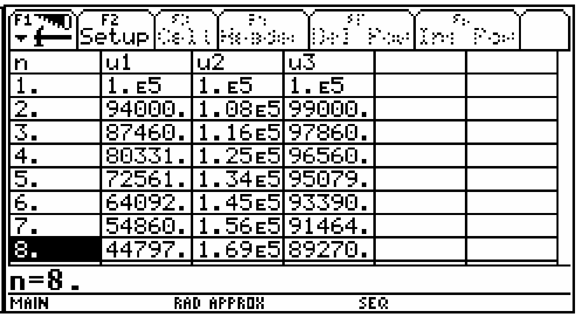

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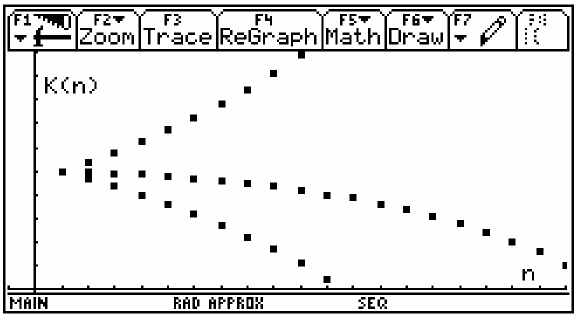


figure 3.9

Thesis 3.2:

Technology offers new language elements and allows the students to create their own new language elements

3.2 New language elements offered by the technology:

Starting a CAS and looking at the menu shows us the great variety of new language elements which the tool offers to the user: The possibilities extend from ***expand*** and ***factor*** in the Algebra menu to language elements for solving differential equations.

The development of the mathematical language of the learners by gaining these new language elements means a chance and also a danger:

The new elements are dangerous if they are only black boxes, if students have only learned to press certain buttons like Skinners` animals.

Although not every language element offered by the computer can become a white box for the learner the possibility of the technology to expand the mathematical language is a chance if students acquire the important and fundamental language elements by exploring and developing the algorithms which stand behind these commands in a white box phase of learning before using them as black boxes in a problem solving process.

Some didactical comments to the "White Box/ Black Box Principal" which describes the corresponding learning activity:

The White Box Phase: the phase of cognitive learning

The student should be led to a mathematical concept, an algorithm or to a mathematical theory. The skills developed in this phase should be carried out by hand, in other words, also trained without the use of the computer. Basic skills should be automatized by practicing.

The computer should only be used as a Black Box for those contents which the student has explored in earlier White Box phases or as a didactic tool to support the investigation of a new White Box.

Possible activities in the White Box Phase:

- Formulating a problem: finding a conjecture; developing a concept; development of an algorithm; proving.

- Calculating numerous examples without CAS; experimental learning with the support of the computer; CAS supported usage of Black Boxes which were explored in earlier White Boxes
- Discussion of the solutions, the limitations and the possibilities of generalizing; independent development of modules which can be later used in the Black Box Phase as Black Boxes.

The Black Box Phase: the phase of knowledgeable application

The student should now be in the position to use the algorithms and concepts developed in the White Box Phase to solve practical problems or to use these in further White Box Phases. The computer is used as a Black Box to process the actual algorithms . The student must decide what to do, explain his decisions, but need not carry out the calculations himself.

New language elements created by the students:

By storing, defining functions or writing programs students can develop their own new language elements. I will go into this topic in the chapter 4.3 “modular thinking and working” in more detail.

Now I will give only one example: One weak point of the TI-89 and TI-92 or Voyage promotes and strengthens the use of modules as new language elements: the small screen. More complex expressions cannot be totally seen on the screen and therefore operations with such expressions become confusing. So the structure of such operations is more comprehensible and clearer if students use the name of the expressions instead of the expressions themselves.

Example 3.2: Discovering algorithms like substitution method, Gauss algorithm a.s.o.

Step 1: In traditional math education students have to work ***„in the equations“***. They have to do the calculating themselves:

(I) 3.x - 2.y = 12 |+2.y
<u>(II) 7.x + 2.y = 8</u>
(I) 3.x = 12 + 2.y |:3
<u>(II) 7.x + 2.y = 8</u>
(I) x = (12 + 2.y)/3
<u>(II) 7.(12 + 2.y)/3 + 2.y = 8 |.3</u>
(II) 84 + 14.y + 6.y = 24 |-84
(II) 20.y = -60 |:20
(II) y = -3
a.s.o.

Step 2: Using CAS students can work ***„with the equations“***. Calculating, substituting and using algorithms to solve the single linear equations, which they learned in former White Box phases, is now done by the computer as a Black Box. Students have to decide on the operations, the CAS have to do them (figure 3.9).

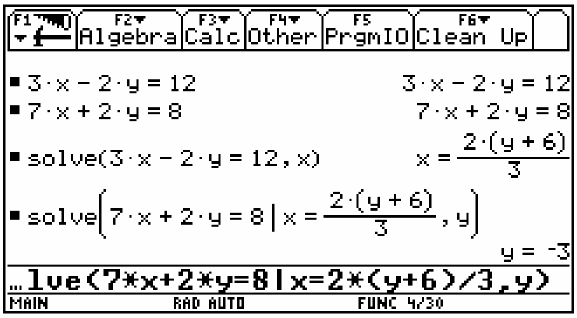


figure 3.10

Linear systems of equations in ninth grade

Step 3: In our project where we observe students who are familiar with the CAS we found out that the operations are not carried out with the equations but ***„with the names of the equations“*** which the students had defined as new elements of the mathematical language.

Using the idea of the Gauss algorithm to find out the Cramer rule.

Phase 1: Students have to speak about strategies to remove one variable in their colloquial language. E.g.: *„We have to multiply the first equation with a_{22} and the second equation with $-a_{12}$ and have to add the two equations. After that we have to solve the new equation for the variable x_1 and so on.“*

Phase 2: Using the CAS these two steps can be connected closely. After storing the first equation in the variable ***equ1*** and the second equation in the variable ***equ2*** the activities expressed in the word formula of phase 1 can now be carried out by using the CAS.

At first for a better understanding of the several calculating steps it would be better to separate the particular steps:

Word formula	Language of Mathematics
Multiply the first equation with a_{22} and the second equation with $-a_{12}$ and add the two equations.	***a_{22}.equ1 + (-a_{12}).equ2***
Store the new equation with respect to the variable x_1 in the variable equ2n	***a_{22}.equ1 + (-a_{12}).equ2 => equ2n***
solve the new equation for the variable x_1	***solve(equ2n, x_1)***

Students having more experience with using CAS and especially the more talented students more and more prefer to translate the word formula of phase 1 into one abstract expression:

Multiply the first equation with a_{22} and the second equation with $-a_{12}$ and add the two equations. Now solve the new equation for the variable x_1	***solve(a_{22}.equ1 - a_{12}.equ2, x_1)***

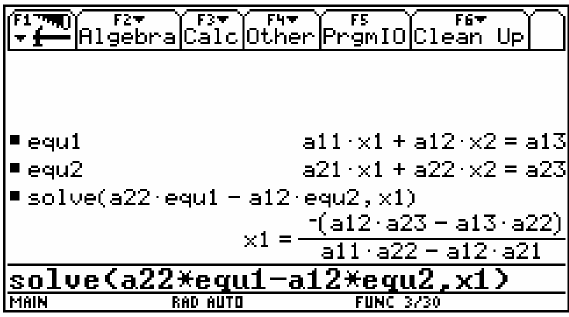


figure 3.11

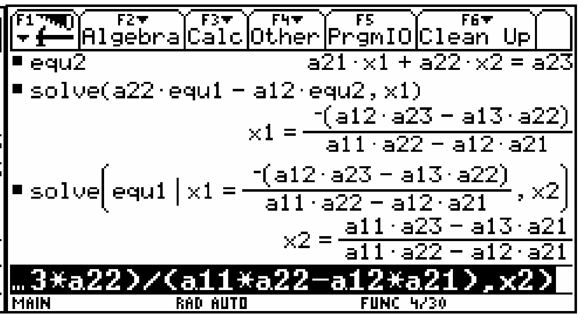


figure 3.12

This result shows a new quality of mathematical thinking caused by CAS:

Thesis 3.3:

The tool of CAS do not only support cognition, they become part of cognition.

4. Mathematics – a thinking technology

One of my students whom I met ten years later – she studied medicine – said:
"I have forgotten all I have learned in your math lessons, I am not able to calculate with vectors, to solve differential equations a.s.o. – but the way of argumenting, the way of thinking logically, the necessity at first to define a concept or to describe it before using it are competences which are also very useful in my medicine studies."

I answered: I am very content – I did not expect you to be thinking about integrals while you are operating me.

> ***"mathematical thinking technology is the essence of science and the essence of a technology based society" (Buchberger)***

Teachers who read our curriculum start (and end) reading at the contents (Algebra, Analysis, Stochastics …), they forget to read the main part: The educational mission of the subject. There they would find out that

> **the thinking technology which is necessary when doing mathematics, independent from special contents, is the main contribution to the general education in our society.**

We must not say: "Our students have to learn integrals!" – we have to argue: "What thinking technology students are gaining when learning integrals".

Thinking technology can also be acquired in other subjects but mathematics needs special ways of thinking. To describe these ways we can use Buchbergers “creativity spiral” as a model of the learners way into the mathematics [Buchberger, 1992].

The spiral begins with observations, data material or problems, the solution of which can be found in the development of algorithms or in the creation of new concepts.

Through analysing, experimenting or generally through heuristic strategies, assumptions are found, sentences formed and initial ideas of proof are thought.

By proving and substantiating, in other words, by exactifying, one enters into the next stage of the spiral: Theorems and sentences which can now be assumed to be correct.

Thus, supported by acquired knowledge, one proceeds to develop those algorithms or programs which are necessary for problem solving. Testing of and consolidating the developed algorithms by practicing is an integral part of this stage.

The actual part of the spiral ends with the next step. The insight and strategies are now used to solve the initial problems or related problems.

When new problems evolve and new additional knowledge is necessary or new algorithms need to be developed, then the stages of the spiral are repeated once again.

When experiencing a loop in the Creativity Spiral one can distinguish **three important phases of activity** in the learning process [Heugl, a. o., 1996], whereby it is not always possible to draw a sharp line among them:

Phase 1: The heuristic, experimental phase

Developing conjectures, forming hypotheses, devising proving ideas and problem solving strategies, developing naive, elementary conceptions. The characteristic brain activities of this phase are: Plausible, inductive conclusions

Phase 2: The exactifying phase

Corroborating assumptions, proving hypotheses, programming (including testing) exactifying concepts.

- The exactifying phase serves to school exact and critical thinking and supports the ability to formal logical conclusions.
- The pupil should become acquainted with the manner of working in the field of mathematics and become trained in proving.
- Furthermore, the ability of reasoning should be educated.

Phase 3: The application phase

Solving problems by applying the concepts and algorithms developed in phases 1 and 2: modeling, operating and interpreting.

If we consider thinking technology as the central goal of mathematics education it is not enough to offer a few chosen applications but rather to develop a general qualification of application, in other words a reflection on the process of application itself [Fischer, 1985, p85ff]

The influence of technology in the thinking technology

In a memorable lecture at a congress at the University of Klagenfurt in 1991 Prof. Willi Dörfler formulated this thesis concerning the influence of technology in mathematics [Dörfler, 1991]:

- *If we understand cognition as a functional system which encompasses man and tools and the further material and social context, then* ***new tools can change cognition qualitively and generate new competences****. Learning is then not simply the development of existing competences but rather a systematical construction of functional cognitive systems*
- *The computer and computer software must therefore be seen as an* expansion and a strengthening of cognition.
- *There is a shift in activities* **from doing to planning** *and interpreting.*
- *The thought process develops advantageously* **using concrete representations or models of the given problem**. *Good software systems offer a number of graphic and symbolic elements, enabling the user to construct various cognitive models on the screen.*
- ***The computer as a medium of prototypes****: General concepts are made cognitively available by prototypical representation. The computer offers not only a larger variety of prototypes, but rather, and more importantly, those which would not be available without the computer.*
- ***Modulation of thought****: The computer can be used for storage and for processing of many various modules and therefore supports modular thinking and working.*

Some examples for computer supported heuristic strategies

which do not only support cognition. Watching our student´s behaviour we can say these heuristic strategies become part of their cognition.

1. Experimenting

The student looks for assumptions by systematic trial. An assumed model is tested, accepted as usable, improved or discarded. In the next step either the learner operates with the tested model or he develops a better model. Testing means checking whether the model fulfills the given requirements and whether it is applicable for larger given requirements.

2. Proceeding step by step

The CAS offer results directly, often without any provisional results. The pupil should try to follow the results presented by CAS by following the path step by step („many things *the computer can do as a black box, I can do too*“).

3. Visualising

A special quality of mathematics is the possibility of graphic representation of abstract facts. Apart from free hand drawings, it is difficult to develop graphs without using a computer. Finding the most important points and characteristics of functions in order to be able to draw the graph is the main goal of the discussion of curves in analysis. CAS allow us to draw the graph faster and more directly than the data that is supplied by the curve discussion. Visualising is one of the most interesting contributions of CAS towards a better comprehension of abstract problems. In addition, when observing pupils working with CAS we have found that especially in group or partner work, the visual communication is an important prerequisite and support of spoken and written communication.

4. Zooming

The possibility of scaling down or enlarging graphs as well as being able to move the cursor along a graph, are two additional enrichments when visualising.

5. Simulating

The computer enlarges the definition of what is a solution. Until now, a problem was considered to be solved when the pupil could find certain numbers or an algebraic protoype of the solution such as a term or an equation. Now, however, the simulation of a process, (using an iterative or a recursive model) is also accepted as a solution. Without the use of a computer the learner was hardly in the position to solve problems for which only an iterative model could be found.

The following two results of our research should show how cognition is changing by the use of technology:

4.1 Modular Thinking and Working

Using modules is not new for the learners. Every formula used by the pupils can be seen as a module e.g. Hero's formula for the area of a triangle or the use of the cosine rule in trigonometry.

Some of these modules are Black Boxes for the pupils which they can find in a formula book. But I hope that in the teaching process many of the modules at first are derived in a White Box phase and after that they are used in the problem solving process as Black Boxes. Knowing such a module means that the student has a model for his problem but when using this module he has to do the calculation himself.

The computer, and especially CAS, opens a new dimension of modular thinking and working. The modular way of thinking is typical of informatics science and in informatics applications. Watching teachers and students in our research project we also found a change of mathematical thinking and working when creating and using modules.

The programming features of the CAS allow the students to create modules which can be used, on the one hand, in the White Box phase as didactic tools and, on the other hand, as Black Boxes for problem solving.

While the modules of traditional math education mostly are the starting point for calculations, the CAS-modules often also do the calculations. W. Dörfler [Dörfler, 1991, pp71] calls any modules

"knowledge-units"
in which knowledge is compressed and
in which operations can be recalled as a whole package.

Creating modules means building a cognitive scheme, condensing cognitive experience. Using modules causes cognitive relief and a reduction of complexity, operations and complex knowledge can be activated as a unit at the same time.

This compressed knowledge can also cause new sorts of mathematical objects or new elements of mathematical language and, last but not least, a reorganisation of the mathematical activity.

Teaching mathematics in accordance with the Module Principle shows the close connection between mathematics and informatics:

Implementing a module in the field of mathematics means to define, to exactify, to describe constructively, to structure, to use or to develop a suitable algorithm.

In computer science implementing refers to formulating in the language of CAS, applying earlier modules , utilising system resources or programming.

Using tools like CAS this close relationship between mathematics and informatics will not only become more noticeable at the universities but also in schools. Bruno Buchberger characterises this new situation by demanding a new subject called „Mathformatics“.

Using the Module principle the learning process could be divided into **three phases**:

Phase 1: Understanding, structuring of knowledge, modelling.

Before creating a module it is necessary to understand the mathematical subject and to connect the new subject with themes the students have learned before, just as to have experience with the language of the used CAS.

Phase 2: Implementing, testing, documenting.

Implementing a module means using the language of CAS for defining functions, using earlier modules or programming

Testing becomes one of the most important activities.

It is necessary that other learners are also able to understand the module and not only how to use it and therefore good documentation is necessary.

Phase 3: Problem solving by using modules as Black Boxes.

By applying constructed modules the actual reorganisation of mathematical activity can take place.

These "knowledge-units" allow the learners to concentrate their activity on modelling, testing and interpreting. Operating will be done by the module.

Depending on their source we distinguish **three sorts of modules**:

(1) Modules produced by the students

These are the most important ones. Pupils create the modules in a White Box phase, sometimes in single work, mostly in pair work or in groups. The testing of a module will often be done by other groups. After that, often supported by the teacher, the best versions of the module will be offered to all the pupils. In the Black Box phase the modules are used mostly via function call for problem solving.

Typical applications: Complex calculations, formulas or tools which are often used, modules for visualisation or simulation, modules used as didactic instruments. For problem solving it is often useful to connect some smaller modules in order to form a more complex module.

(2) Modules created by the teacher

To support the learning process of the students, it is sometimes useful to offer modules which could be used as didactic tools in the White Box phase or for complex calculations which are not the main goal in the problem solving process. In some of these modules the inner structure can be made accessible for the students, others will stay Black Boxes.

(3) Modules which are made available by the CAS

CAS are usually delivered as a system of modules partly offered in the core of the system and partly being available as a system extension. Some of these modules should become White Boxes for the learners, others will be used as Black Boxes, e.g. not every module for solving differential equations should become a White Box. The new flash-technology opens a new dimension of system extension and change.

Working with modules causes a modular thinking
A special expert in using modules is Eberhard Lehmann from Berlin [Lehmann, 2002]. He uses the module concept in the classroom starting in the 7th grade. He calls this didactical concept "**concept of building stones**". Walking on the spiral into mathematics means developing a pool of building stones which can be used for problem solving.

Goals of a module oriented mathematics education:

- Defining modules
- Analysing modules, using modules for experimental learning
- Developing a pool of modules as a source for modelling, for problem solving
- Using modules as black boxes
- Connecting modules, building new more complex modules by using existing modules as building stones

The results of Lehmanns investigations show that students being familiar with modules use them as new language elements and demonstrate a new quality of mathematical thinking. Important is not only to built a module and afterwards to forget it but to see the opportunity to use the constructed module in several ways.

Examples 4.1: The module "difference quotient"
[Lehmann, 2002, pp24]

To discover the of the differential quotient as a limit of the difference quotient a module which can be used for experimental learning is very helpful.

Step 1: Defining a module „difference quotient"

$$\frac{f(x+h)-f(x)}{h} \rightarrow diffq(x,h)$$

Defining means storing as a function of two variables

Step 2: Using the module „difference quotient" - exploring the graphs for several values of h

$$\textit{Graph } diffq(x,h)\,|\,h=\{-1,-0.5,-0.1,\ 0.1,\ 0.5,\ 1\}$$

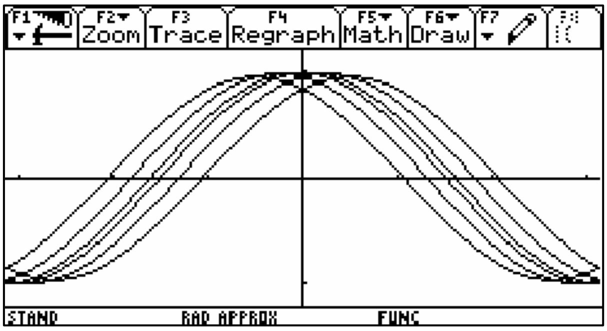


figure 4.1

Step 3: Connecting modules produced by the students with modules offered by the CAS

$\lim_{h\to 0}\left(\boldsymbol{diffq}(x,h)\right)$

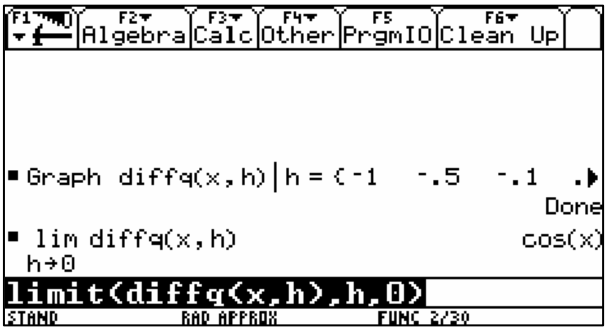


figure 4.2

Example 4.2: Distance of points and curves
[Lehmann, 2002, pp101]

The module "distance" is especially suitable as a building stone for building more complex models by the cross connecting of given modules.

Step 1: Defining a module „distance of two points" P(a,b) and Q(c,d)

$\sqrt{(\mathbf{a\text{-}c})^2+(\mathbf{b\text{-}d})^2}$ => **distance(a,b,c,d)**

Step 2: Using the module „distance" - e.g. calculating the distance of points and curves

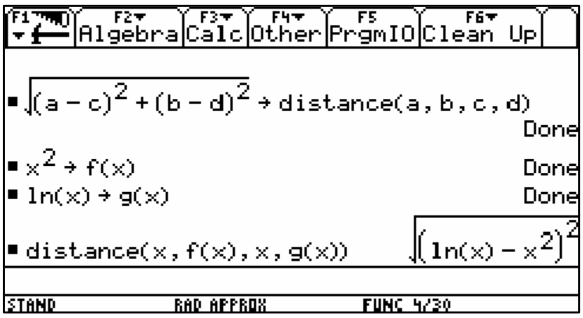


figure 4.3

Step 3: Connecting modules produced by the students with modules offered by the CAS - e.g. looking for the smallest distance of two curves f and g by using the CAS-modules ***„solve"*** and ***„differentiate":***

Find the shortest distance of the functions f(x) = x^2 and g(x) = ln(x)

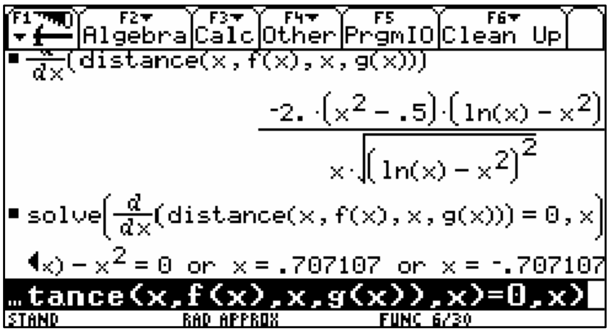


figure 4.4

Step 4: „Shutteling“ to the graphic-window and examine and discuss the results

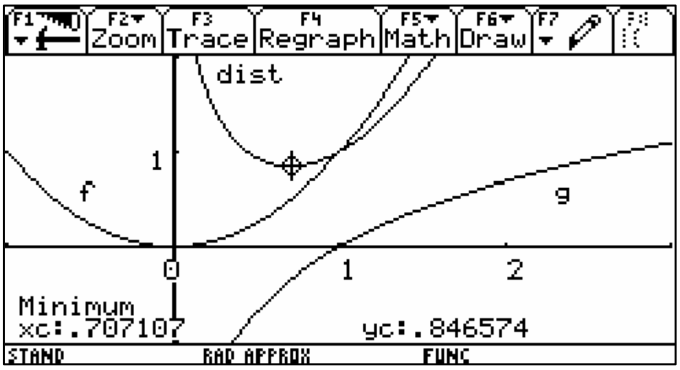


figure 4.5

Using modules – a chance and a danger:

Example 4.4: The program package "VECTOR-CALCULATIONS"
by Thomas Himmelbauer, Vienna [Himmelbauer, 1997]

This program package consists of a large number of programs and functions (more than 140 functions and programs) which makes vector calculations possible in $\mathbb{R}^2$ and $\mathbb{R}^3$ and allows students to draw points, line segments, lines, circles and planes. Activating the package a separate toolbar appears which lets students display menus for selecting functions or programs.

Some examples:

Example 4.4.1: Find the distance of two skew lines

Step 1: Using the function ***gepktrtg*** students get the parametric equations of the straight lines.
The arguments of the function are two vectors, the position vector of a point and a direction vector. The equations are stored in the variables ***ger1*** and ***ger2***

Step 2: Evaluating the function ***abswinge*** the result is the distance of the two skew lines. The arguments of the function are the names of the equations of the two lines (figure 4.?? <nash6>

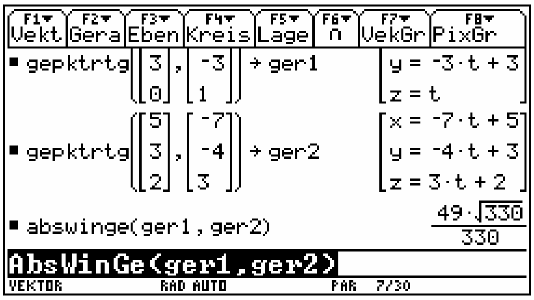


figure 4.6

Example 4.4.2: Find out the position of planes

Step 1: Store the equations of the 3 planes in the variables ***eb1***, ***eb2*** and ***eb3***.

Step 2: Evaluate the function ***drebenen,*** the result is a quadruple. The first element gives the answer: "*A point of intersection exists*", the others are the 3 coordinates of the point (figure 4.7)

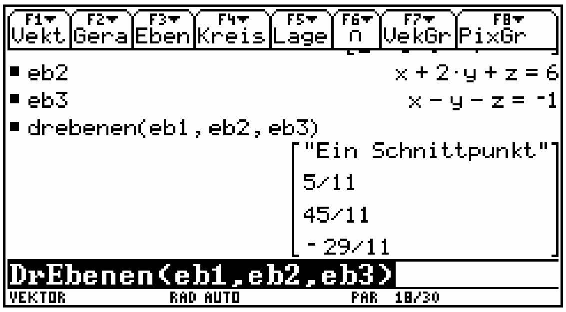


figure 4.7

The chance:

If the powerful package is used in the same way as the teacher who developed this system of modules it is a chance: The first phase is a White Box phase in which students have to develop problem solving strategies and they also have to do the programming for some of the modules themselves.

The danger:

But if there is no White Box phase before or after in which students have to find strategies and algorithms to solve such problems of analytic geometry the idea of those Black Box modules is very contrary to the fundamental goals of mathematics education.

Finding suitable Black Box functions by searching for suitable words in the menues and inserting given values is not doing mathematics.

4.2 The "Window-Shuttle" Strategy

The Computeralgebra System as a medium for prototypes

General concepts become cognitively available through concrete representatives or in our words through prototypes (e.g. if you use the common concept „table“ you are thinking about concrete prototypes which you have experienced).

The computer allows us a greater variety of prototypes of a concept and also offers some which were not available before.

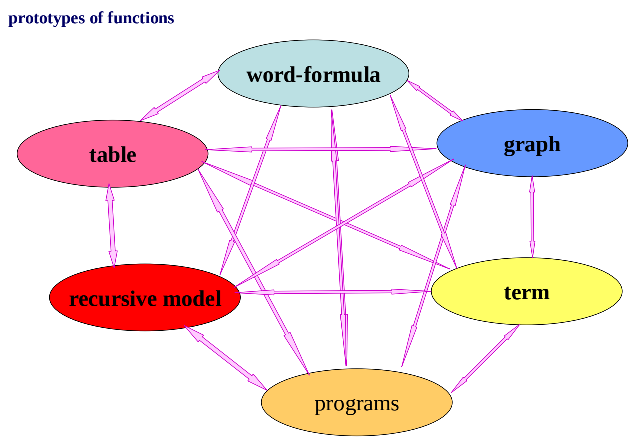
Typical for this thesis is the **concept of functions**. The pupil will find access to this concept not through a clear cut abstract definition but rather through a supply of suitable prototypes which draw the pupil´s attention to the vital characteristic of the concept. In this process the important activity is the establishment of relationship among the individual prototypes. It is in this way that the learner can comprehend that the individual prototype is simply one of many possibilities of appearance of the concept of function. Not until after this process does it make any sense to verbalize or formally define the concept „function.“

Observing traditional mathematics education you can find the following **prototypes of the fundamental concept of functions**:

- **word formulas**
- **symbolic prototypes like terms, parametric equations, polar equations**
- **graphs**
- **tables**

The computer also offers new prototypes e.g.

- **recursive models**
- **programs**



It is not enough to make various prototypes of a general concept available, the establishment of the relationship among the individual prototypes leads the pupil´s attention from the specificity of a singular prototype to the superior general conceprt

In order to recognize the „prototypical“ as an invariable characteristic, the prototype often has to be changed.

Figure 4.8

In traditional mathematics education prototypes mostly are available in a serial way. A typical example is the discussion of curves: One prototype, the term is given. The students have to find the graph by calculating the zeros, the extreme values, the inflection points and they have to determine a table of values.

The main importance of the computer is that **the learner can use several prototypes parallely**. The given term allows the pupil to draw the graph directly and the table is the result of activating one key of the tool. The real learning process consists of **shuttling between several prototypes** and investigating the influence of changes of one prototype in the others. Therefore we call this didactical concept the **Window Shuttle Method**.

The steps of the learning process according to the window shuttle technique and the role of CAS in this process:

- The pupil activates various adequate prototypes for the problem or the concept in different windows of CAS, for example a symbolic prototype in the algebra window and a graphic prototype in the graphic window.
- The pupil now works with the individual prototypes, whereby the advantages of CAS such as interactivity, easy manipulation and repetition can be applied.
- The multiple window technique enables the learner to work simultaneously with various prototypes. In continuous interaction between the algebra and graphic window and the table, the effect of algebraic operations on the graphs or on the table values can be examined or ideas for activities in the algebra window can result from examining graphs or tables. Furthermore the consequences of the alteration of individual parameters in the algebra window, can be examined directly in other windows.

A concept or a solution of a problem develops by shuttling back and forth between the various forms of representation, meaning between different windows in CAS.

The observation of the students involved in our CAS project strengthens the thesis that the tool CAS does not only support cognition, it becomes part of cognition.

Example 4.5: First experience with the function concept in the 7th grade: Direct - indirect proportion

This example is part of an investigation, called „observation window“ in the Austrian CAS project [Klinger, 1997]. The goal was to observe the pupils´ behavior: The learners should choose a prototype of a function suitable for a given problem and they should discover and use strategies for the proof of a definite functional relation.

The **initial problem** for indirect proportions was rather simple:

> The distance between Vienna and Innsbruck is 500 km. Calculate the driving time for several mean velocities.

According to the goals of this investigation - pupils should actively discover new concepts and strategies - it was necessary to give precise instructions:

- Calculate the time for the velocities in the given table. What happens if the velocity is two times, three times, ten times, k-times greater than before?
- Find a formula in the y-Editor. Using this formula find the values of a table. Check the correctness of the values of the table given by the teacher.
- Calculate the product of the velocity and the appropriate time, at first in the Home Screen and then in the Data/Matrix Editor. Select 7 values of the given table. What is noticeable?
- Find the graph of the table values in the Graphic Window. Walk along the graph, using the Trace Mode and check the values of the teacher´s table.

The first goal was to observe which prototype the pupils will prefer. The following were available (figure 4.9 to 4.13):

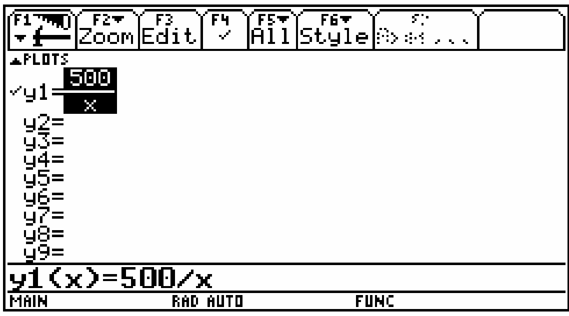


figure 4.9

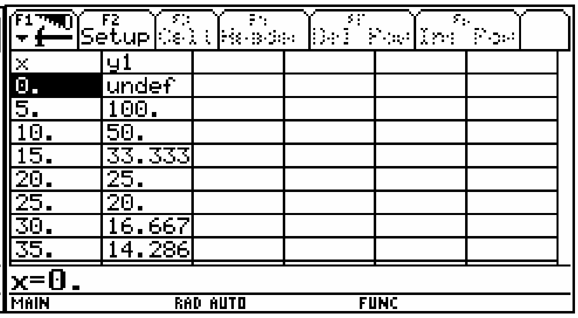


figure 4.10

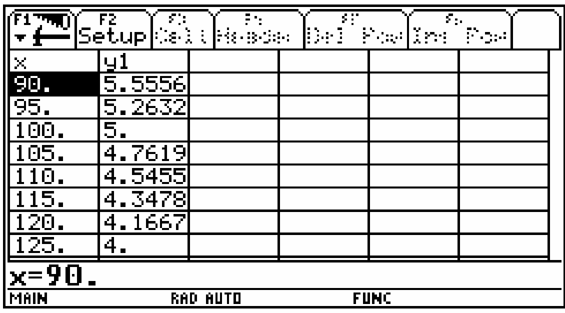


figure 4.11

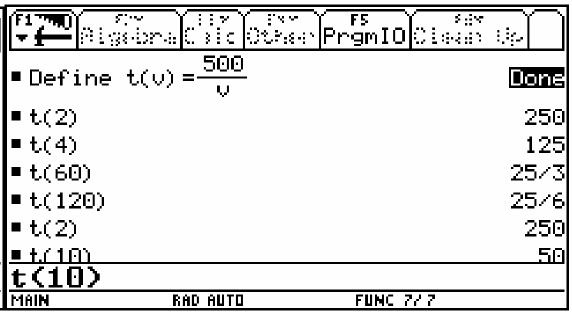


figure 4.12

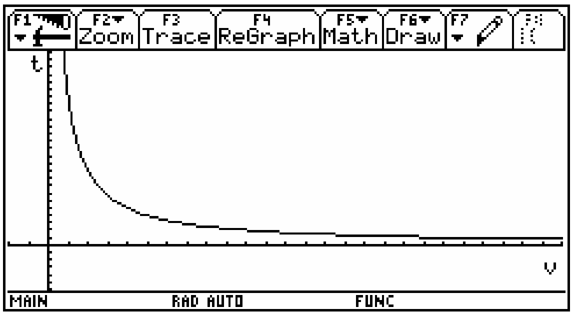


figure 4.13

The formula was found in the y-Editor (figure 4.9) Shuttling to the table the TI-92 calculates the table values using this formula. Pupils can find a suitable section of the table (figure 4.10 and 4.11). A surprise was the definition of the function in the Home screen and the use of this prototype in the testing phase. Such a strategy is unusual in 7th grade.
Just as new is the frequent choice of the graphic prototype which, with the help of the CAS, is now available very easily by shuttling from the table to the graphic window. Shuttling back is possible by using the Trace mode which allows the learner to observe the coordinates of the respective points

The second goal was to discover and to select proof-methods for indirect proportions

The following strategies were used:

Strategy 1: Examine in the Table editor in several cases:

- two fold corresponds to one half
- five fold corresponds to one fifth
- n-fold corresponds to one nth

Strategy 2: Proof with the formula either in the Home screen or in the y-Editor by using the „with- operator“ or calculating function values of the defined function (figure 4.14)

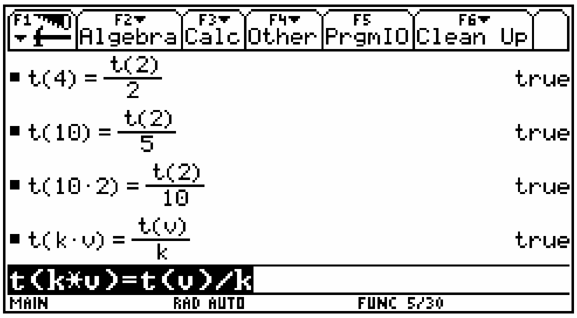


figure 4.14

Strategy 3: Proving the following rule in the Home screen or in the y-Editor (included the control of the table) or in the Data/Matrix Editor (figures 4.15 to 4.17):

- The product of the argument and the function value is constant

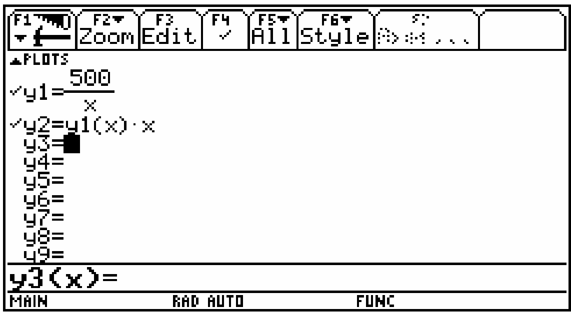


figure 4.15

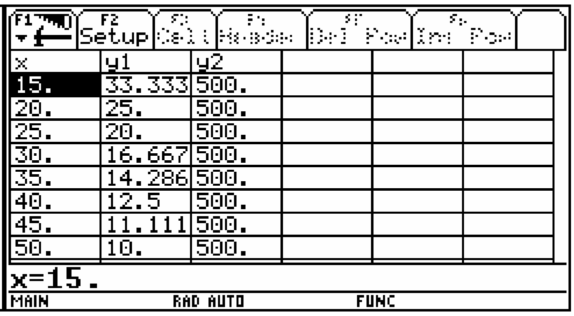


figure 4.16

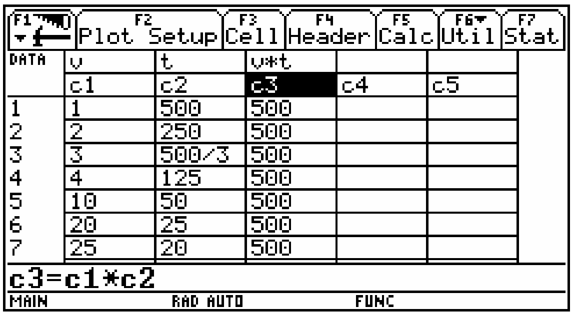


figure 4.17

Strategy 4: Drawing the graph in a suitable intervall in the graphic window.
The „typical curve“ is called hyperbola

The third goal: These strategies should also enable the learners to decide in certain examples that neither a direct nor an indirect proportion exists.

The pupils had to examine the following problem:
The force of gravitation of the earth with respect to the distance.

The pupils got a table of values in the Data/Matrix Editor. Observing the graph (figure 4.18) could cause the supposition: It looks like a hyperbola - it is an indirect proportion. But shutteling to the Data/Matrix Editor and using strategy 3, pupils found out: The product of argument and function value is not constant (figure 4.19)

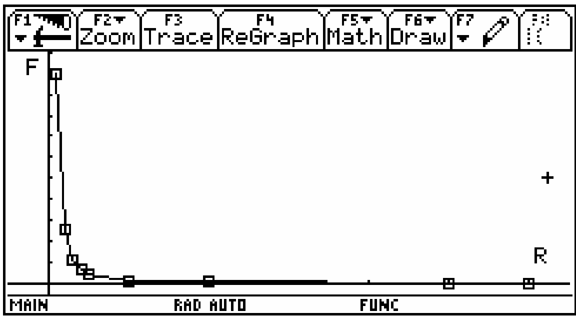


figure 4.18

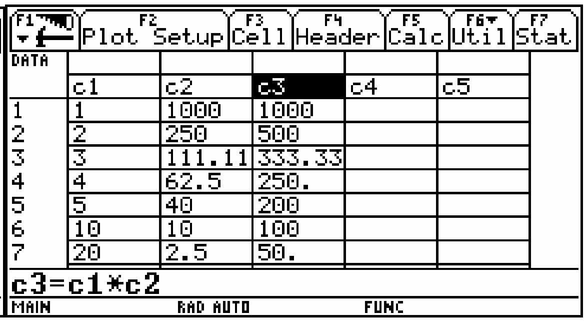


figure 4.19

Some results of pupils´ behavior:
Pupils use the possibility of having several prototypes of the function parallely at their disposal. Shutteling between several prototypes becomes a common practice and allows then to use the advantages of certain prototypes.
Several pupils develop preferences to several prototypes. In traditional math education, the table often was the only prototype which was at their disposal. I did not expect that pupils of the 7th grade will also use the graph and the defined function (see figure 3.6), the last one is prefered by more gifted children.
It is not only easier now to get tables, the opportunity to calculate with whole rows is the main importance of function prototypes in the Data/Matrix Editor.
The testing strategies strengthen the decision competence according to the type of the function.

Example 4.6: The third Kepler rule
Circulating time of planets with respect to the distance from the sun [Schmidt, 1997]

Goals of this example:
- Describing of real phenomena with functions.
- Starting with measured data, the students should discover the rule by using several prototypes of functions.
- The features of CAS and the Window Shuttle Method should open up new possibilies for an experimental and pupil-oriented learning process

The students got a table with observation data:

Planet	Distance (in mil. Km	Circulation time in da
Merkur	57.9	88
Venus	108.2	225
Erde	149.6	365
Mars	227.9	687
Jupiter	778.3	4392
Saturn	1447.0	10753
Uranus	2870.0	30660
Neptun	4497	60150
Pluto	5907	90670

figure 4.20

The first step was to enter the data of the table into the Data/Matrix Editor of the TI-92 (figure 4.21). To come to suppositions about the sort of functions it is better to „shuttle“ to the graphic window (figure 4.22). Zooming strengthened the assumption: It could be a power function of the type $y = a.x^c$.

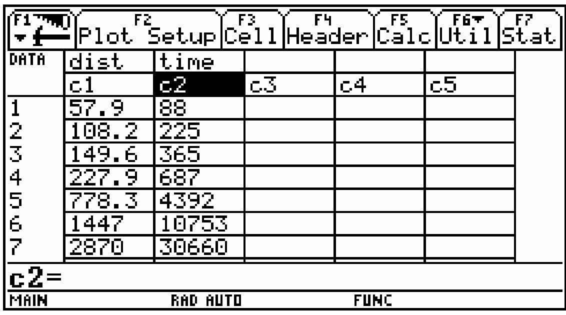


figure 4.21

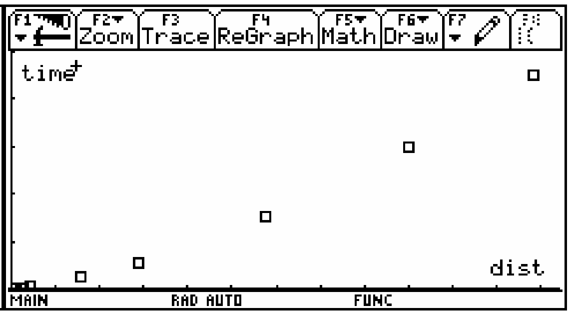


figure 4.22

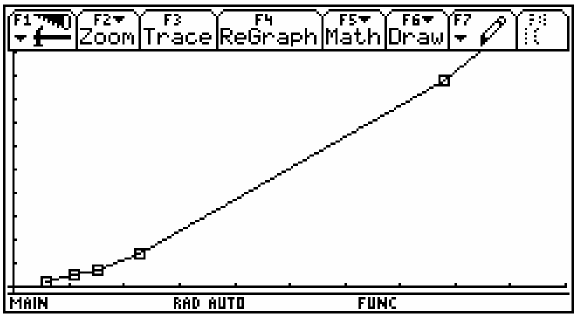


figure 4.23

Because the pupils had experience with the idea of linear regression they decided a „log-log-plot“ with axes log(x) and log(y) which allowed them to use their knowledge:

$$\log(y) = \log(a \cdot x^{c})$$
$$\log(y) = c \cdot \log(x) + \log(a)$$

Defining new rows c3 = log(c1) and c4 = log(c2) the calculation was done by the TI-92 (figure 4.24). If the suppositon is correct the points should be situated on a straight line (figure 4.25)

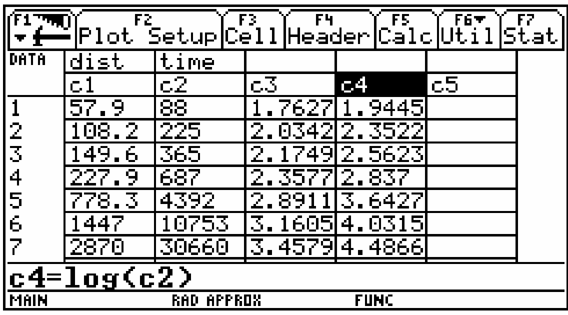


figure 4.24

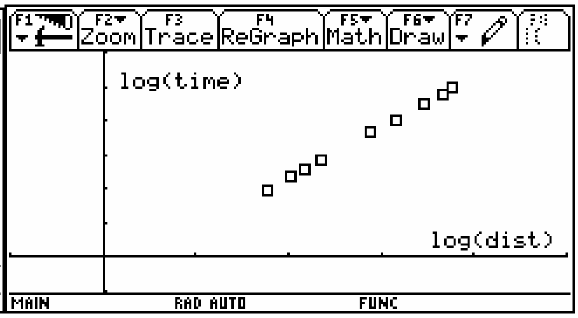


figure 4.25

By using the TI-92 as a black box the pupils could now find the equation of the linear regression (figure 4.26 and 4.27). The correlation coefficient is very good (near 1). Shuttling to the graphic window enabled the visualisation of this result (figure 4.28)

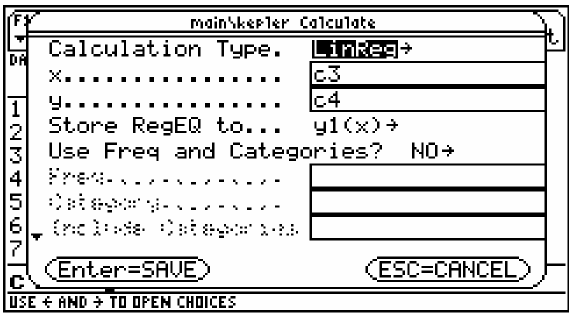


figure 4.26

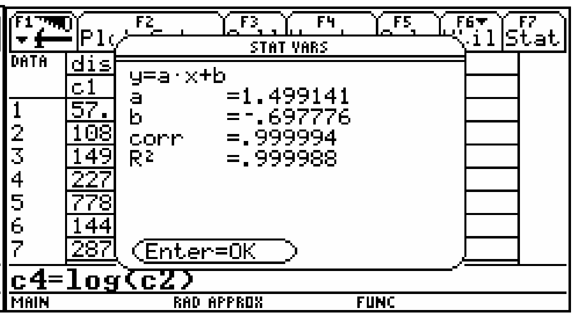


figure 4.27

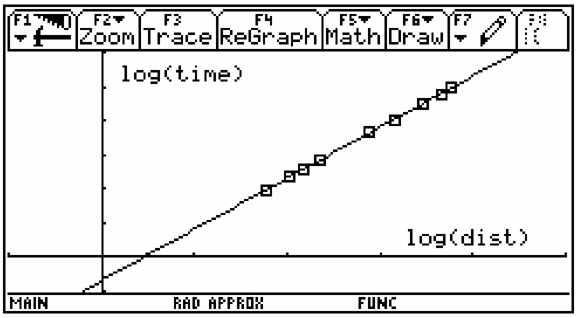


figure 4.28

Now the assumption was confirmed: The circulation time of the planets with respect to the distance from the sun is a power function of the type y = a.x^c.

To find the parameters a and c the pupils first had to jump to the y-Editor where the equation of the linear regression was stored. Remembering the equation log(y) = c.log(x) + log(a) they found out that the slope of the line is c and the y-intercept of the line is log(a). Thus they calculated the equation of the desired power function (figure 4.29). After squaring the equation the 3rd Kepler Rule can be seen:

„The squares of the circulation times of the planets and the cubes of the radius are proportional"

Shuttling to the y-Editor and the grahic window and drawing the power function the result can be visualized (figure 4.30 and 4.31)

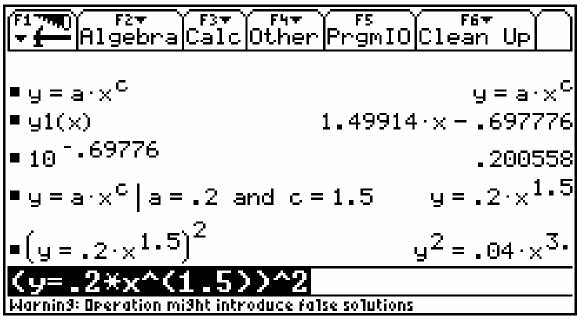


figure 4.29

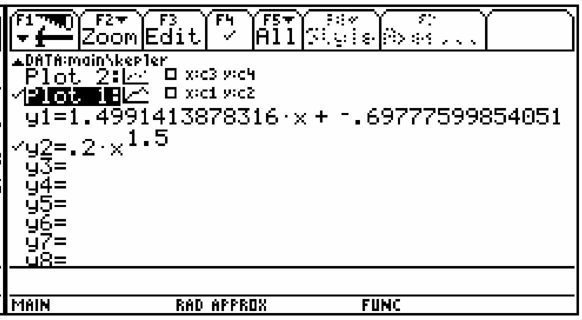


figure 4.30

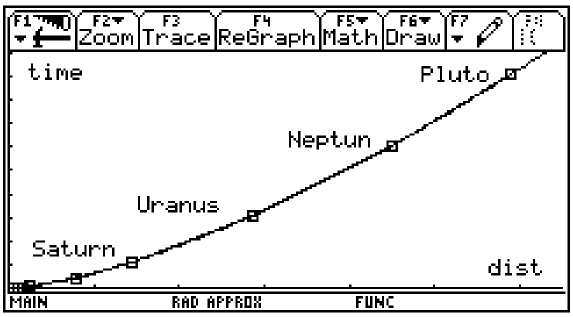


figure 4.31

Summary:

Now I will come to the end, compressing my thoughts about the influence of technology in several roles of mathematics, looking for a summary or a “word formula” of my lecture:

- Technology supports the value of knowledge in the abstract phase as well as the value of usability in the concrete phase of the mathematical process
- Technology changes the language of mathematics
- Technology causes a new quality of mathematical thinking

These changes of mathematical thinking and of the language of mathematics must also be considered in mathematics education.

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Figures

Figure 1.1:
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Figure 1.2:
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Figure 1.3:

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