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Technology – Standards - Assessment

The influence of the use of Technology in Standards and Assessment

Abstract

After a short survey of the results of our research projects in Austria concerning the use of technology in mathematics education I will try to compare ***some concepts of standards*** *which are used in several parts of the world. These standards range from process standards like the NCTM standards to product oriented standards as we define and use them in Germany or in Austria.*

The second part of my lecture deals with ***the influence of technology in standards****. In our projects we have observed influences in the mathematical contents as well as in the common mathematical performance expectations and also in the complexity of those competences. I will give some examples of fundamental mathematical competence which is necessary in the age of information technology.*

*The third part is closely connected with the second one –****technology and assessment****. We can say it was the starting point for our involvement in standards. Observing students in the Computer Algebra-classes we recognized that the traditional exams are not suitable to new ways of learning. One of several experiments is a so called two-tier exam. That means, on the one hand, short exams for testing the fundamental competence or looking for building stones of the central goal of problem solving and, on the other hand, longer written exams with an emphasis on problem solving. This discussion about fundamental mathematical competence leads us to the definition of standards*

Content

1. Standards in general – our way of developing standards

If you ask 10 people "what are standards" you will get more than 20 different answers. My first attempt is always to look at an encyclopaedia. There you may read:
"Guiding principle"; "a uniformed fixed measure for certain products of equal quality"; "standardising creates norms".

When we started developing standards in Austria, especially the teachers asked: "Why standards? – we have a curriculum, we are responsible for examining!

We tried to give some answers to such questions;

1.1 Why standards? How to use standards?

Why should we implement standards?

- **Standards as a necessity of globalizing:**
  Internationalisation and globalisation call for a better comparability of educational results. Especially within the European Union a better compatibility of the various educational systems is necessary.

- Standards as a change **from input- to output-steering** of the educational system:
  Due to the growing autonomy of the schools, the input-steering by the curriculum cannot guarantee a comparable output of the educational system.

- **Standards as a reaction to problems of the educational system:**
  International studies like PISA or TIMSS showed us irrefutably that the input steering with curricula is not enough for getting expected results. It is necessary to measure the output, the product of the learning process, the knowledge and competence which the students gained at specific years of their education.

- **Standards as a vision:**
  In the introduction of "NCTM Principles and Standards" this interpretation plays a central role. Standards describe an ambitious and comprehensive set of goals for mathematics instruction, they describe the mathematical understanding, knowledge and skills that students should acquire from pre - kindergarden to grade 12

- **Standards as a reaction to the changes caused by the use of technology.**
  The use of technology caused an insecurity about the goals and results of mathematics education, about the indispensable competence and skills which the students still need when using the power of technology.

These justifications lead us to different ways of using standards.

How to use standards?

- Standards as a contribution to international **comparability and compatibility** of educational systems.
- Standards as a **steering instrument**, as an educational mission of the community. This sort of standards should not only describe goals of a certain subject but also global competences like methodological competence, social or personal competence.
- Standards as a **basis for evaluation**: Those standards cannot only consist of verbal goals they also need instruments for measuring the quality of the product of the educational process.

- **Standards as an instrument for the award of authorizations**: In Austria and in many parts of Germany we have no central final exams. The exams are constructed and corrected by the single teacher. This system causes a significant diversity of the educational results. Standards used as an instrument for awarding authorizations should describe the competence and knowledge of the students at their final exams also by offering suitable examples. In Austria we do not plan central exams but I am planning a project of **a partly central exam which consists of two parts**: The first part is a centrally designed paper examining the fundamental mathematical competence. These are building stones which are necessary for problem solving. The second part is given by the teacher consisting of problem solving examples which reflect the emphasis of the actual educational process.

1.2 Interpretations of the concept of standards

In my lecture I am concentrating on educational standards for a certain subject – mathematics. There are several possible differentiations.

(1) One possible differentiation is:

- **Content oriented standards**: Describe expectations according to goals and contents of the mathematical education. These standards are formulated in our national curriculum.

- **Product oriented standards or achievement oriented standards**: Describe essential competences which student should be able to document at certain steps of their educational process.

- **Process oriented standards**: Describe expectations in the teaching and learning process which are the prerequisite or the condition for reaching the product oriented standards

(2) Another point of view is to differentiate according to **the complexity** of the expected standards:

- **Minimum standards** describe minimum conditions which are expected from all students

- **Regular standards** describe the expected competence which most of the students will gain at a certain point e.g. at the final exam at the end of 12^{th} grade or they describe the normally expected situation of the teaching and learning process.

- **Maximum (ideal) standards** describe the ideal state which will be reached by a certain number of students or the ideal state of the educational process.

(3) We have also to distinguish between **short-term available knowledge and competence** which is necessary for the students during the current learning process and for the current exams and **the long term competence and knowledge** which is the actual goal of the learning process, the competence which permits the students to apply what they have learned at later times or in new contexts.

A weakness of many educational systems is that because of methods of assessment, the short term competence dominates. Students study shortly before the next written exam and after it they quickly forget what they have learned. A very important expectation in the implementation of standards is that we will reach more emphasis on the long term competence in the learning process. But if we want to measure the long term competence we have to be more modest as far as our expectations are concerned.

The first two interpretations of the concept of standards can be visualized in a two dimensional table including some examples:

Interpretation of the concept of standards

	Minimum standards	Regular standards	Maximum standards
Content oriented standards	core curriculum		
Product oriented standards		PISA examples	NCTM principles a. standards
Process oriented standards			

Figure 1.1

Some examples describing the individual fields of the table:

- Our new curriculum of the secondary level I is a core curriculum: It describes the minimum volume of mathematical contents which has to be taught.
- The examples of the PISA study are product or achievement oriented standards which describe the regular expectations.
- The NCTM standards cannot be clearly assigned to one particular field. We can find statements reflecting basic precepts that are fundamental to a high-quality mathematics education – a vision for improving mathematics education => NCTM standards are process standards and ideal standards. On the other hand standards are descriptions of what mathematics instruction should enable students to know and to do => NCTM Standards are just as product oriented - a fact which is also confirmed by the examples.

The German and Austrian interpretation

We are developing product oriented or achievement oriented standards. They determine what competence a student should have acquired at a certain step of his education. Competences must be described so precisely that they can be turned into concrete examples which can be used as measuring instruments for several types of tests.

Characteristics of standards:

- Standards should be derived from consensual educational goals and from a certain picture of the science of mathematics and from the following role of school mathematics.
- Standards should be developed on the basis of a theoretical competence model which offers a classification of the several sorts of competence and which also describes differing complexity levels.
- Standards describe long term competences
- Standards describe only a part of all competences which students should gain during their mathematics education – the central part of fundamental competences
- Standards as an evaluation tool also need concrete examples or better concrete tests which can be used as measuring instruments.

1.3 The competence model for standards in mathematics education

Characteristics of a competence model

The foundation for defining standards is a competence model. A competence model distinguishes among several partial dimensions within a subject and describes varying competence levels. Each competence level is specified by certain cognitive processes or activities of a certain quality which students of a certain grade should gain but not students of a lower grade.

The educational mission of the subject of mathematics

The prerequisite and starting point for formulating standards is the description of the educational mission of the subject mathematics. We named three roles of mathematics which express this educational mission in the best way:

- **Mathematics - the science of problem solving by reasoning**
  The problem solving process mostly consists of 3 phases: modelling, operating and interpreting
- **Mathematics a language**
  Students should learn 3 sorts of languages: their mother tongue, foreign languages and the language of mathematics
- **Mathematics a thinking technology**
  Logical thinking and the heuristic strategies which are necessary for doing mathematics are also necessary, helpful and applicable in many areas of life – often more necessary than certain contents.

The 3 dimensions of our competence model

Dimension 1: The performance dimension

It highlights ways of acquiring and using content knowledge. It describes subject oriented activities which are typical for doing or using mathematics. These activities can be derived from the role and the goals of mathematics and mathematics education.

Dimension 2: The content dimension

The Content Standards explicitly describe the mathematical contents that students should learn in certain phases of their education (primary school, secondary level I, secondary level II).

Dimension 3: The complexity dimension

It specifies cognitive achievements with different complex levels. Reaching such a level expresses what cognitive activities or mental operations can probably be carried out correctly.

Competence classes within the dimensions:

Within the several dimensions we distinguish among several classes These sets of competences do not neatly separate the school mathematics curriculum into non-intersecting subsets. Because mathematics as a discipline is highly interconnected, the areas described by the standards overlap and are integrated.

Dimension 1: The performance dimension

The performance dimension consists of 4 classes:

Modeling, Representing	Includes all activities concerning the translating process from general language into the language of mathematics, the competence of planning, deciding on a certain method of solving, deciding on a certain algorithm
Operating, Calculating	Includes the performance of an algorithm, the usage of a calculating method which leads to a mathematical solution of a problem. It is more than gaining calculating skills. Operating can also mean the competence of visualizing – finding a solution graphically or using a table a.s.o.
Interpreting and Documenting	Includes several aspects of interpreting: Analyzing the usability of the mathematical model and the mathematical solution or mathematical interpretation of the correctness of the solution. It also includes the competence of documenting the problem solving process and representing the results
Arguing and Reasoning	Includes the heuristic strategies of arguing and supposing as well as the competence of reasoning mathematically by using deductive strategies.

Dimension 2: The content dimension

Most of the competence classes — Number and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability — apply across all grades, some like Analysis or Analytic Geometry can be only found in secondary level II. Each class comprises a small number of goals that describes the students´ knowledge and sophistication in absolutely necessary mathematical contents.

Dimension 3: The complexity dimension

The complexity dimension reflects the complexity of the thinking pocess, the number of the thinking steps which are necessary for solving a problem.

Level I: Fundamental competence, building stones for problem solving

It includes elementary skills and abilities, reproductive knowledge. The suitable examples consist of processing a given algorithm, one step modelling, knowing definitions or formulas, verbal explaining of such formulas or definitions, giving reasons for the chosen model.

Level II: Simple combinations of fundamental competence

Based on earlier acquired knowledge, students should make the correct choice. The operations are more complex, the combination of several operations is necessary, solving reproductively acquired simple proofs, arguing on a higher mathematical level.

Level III: Complex combinations of fundamental competence

It includes a synthesis of different mathematical areas, creative independent problem solving, open problems, the use of several proving strategies, the competence of different heuristic strategies like generalizing or specializing.

Complexity must not be confused with difficulty which depends on the single individual. A problem with a certain level of complexity can be very difficult for one student while another student finds it rather easy.

Visualization of the competence model (secondary level II):

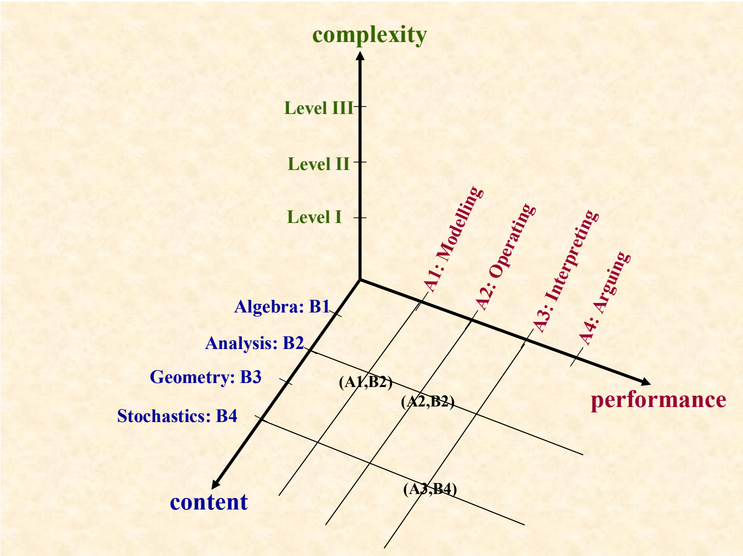


Figure 1.2

The content and the process dimensions are inextricably linked. The process of problem solving cannot happen without using mathematical content. Therefore standards appear in the graphic as pairs consisting of a performance and a content element.

Based on the theoretical foundation of the competence model the actual **standards describe a set of fundamental competences** which students should have at a certain point of their school career (end of primary school, secondary level I and II) and so **standards describe long term competences**.

To realize the standards in the classroom situation, the development of **a pool of examples** is necessary. The purpose of this pool is to offer orientation and self evaluation. It must not be confused with **the secret pool of test items** for external evaluation.

2. Technology and standards

Some mathematics becomes more important – because technology requires it

Some mathematics becomes less important – because technology replaces it

Some mathematics becomes possible – because technology allows it

Bert Waits

This statement of Professor Bert Waits – in Europe we call him "father of the use of Computer Algebra" – shows in the best way the changes which are caused by the use of technology.

In Austria we overslept the phase of graphic calculators but on the other hand we started very early using computer algebra systems (CAS). Since 1992 we have carried out five research projects. Most of our theses were investigated by observing students in experimental classes from 7th to 12th grade. On average 2000 students in about 80 classes took part.

One emphasis of my work during the last five years has been the exam situation. One of our main results was, that the traditional written exams which predominantly assess calculating skills are not suitable to the experimental and pupil oriented way of learning in the "technology classes". These observations and experiments, which I will speak about in the third part of my lecture, lead us to the next open questions:

- Which fundamental, long term competences are necessary as building stones of the main goal of mathematics education namely problem solving?
- How will the fundamental competences change when using technology?

This discussion about fundamental mathematical competence in the age of technology led us to the definition of standards. At the same time and independent of the use of technology, we noticed the world wide discussion about standards for mathematics education starting with the initial activity of the NCTM Standards up to the recent discussion in Germany and Austria. The connection of these two topics forms the emphasis of my recent research work: "standards and technology."

Briefly, some results of our investigations which are important for my thesis in the fields of standards and assessment:

- Most important are not the changes in the content area but rather the changes in the learning process: We are observing **a more pupil centred, experimental way of learning**
- We are observing a shift from doing to planning
- **A shift** of the emphasis from operating **to modelling, interpreting and arguing**
- **A more application oriented mathematics education** is possible and makes mathematics more meaningful and interesting for the students
- The **technology** does not only support cognition, it **becomes part of cognition** [Dörfler, 1991].

2.1 Changes of the competence model influenced by technology

In such a short lecture it is impossible to treat all aspects of this topic. I will not speak about the content dimension, my emphasis in this article is the performance dimension. A first attempt at taking technology into consideration was to add a new performance class - the "**tool competence".**

After a lot of discussions and observations we decided not to separate the tool competence from the other four mathematical performance classes but to integrate it. Tool competence is also a mathematical performance competence and is necessary for all the other performance activities when using technology. Tool competence is just as necessary when modelling or operating, interpreting or arguing.

So we describe the changes of the competence model by "zooming" into the four performance classes and distinguishing at each of them four "subclasses" which express the influence of technology in the mathematical performance activities.

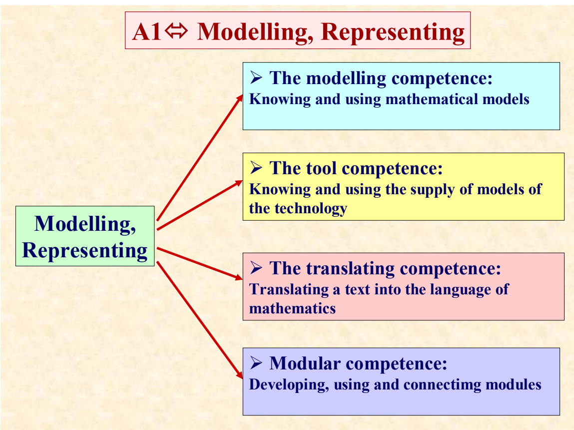
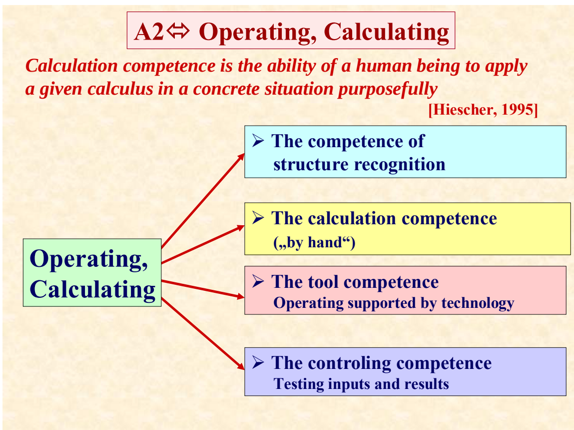


Figure 2.1



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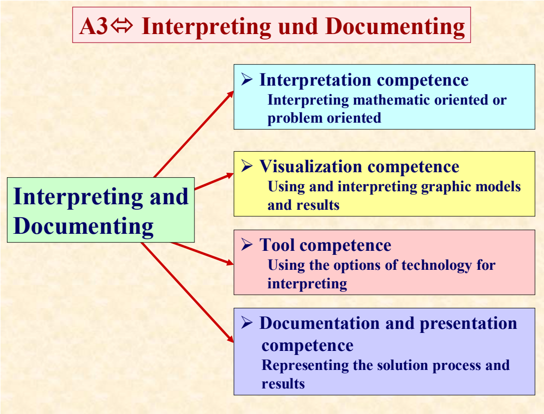


Figure 2.3

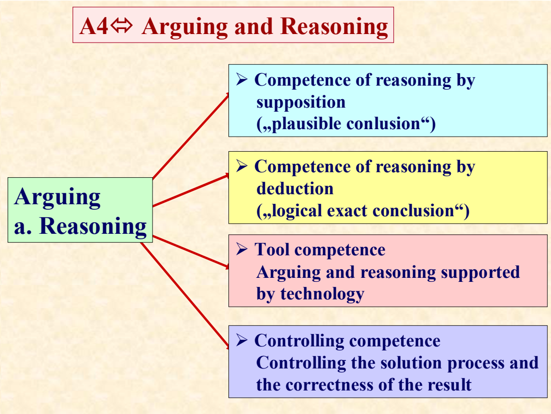


Figure 2.4

Comments on the influence of technology in some of the “subclasses”

A1 ⇔ Modelling, Representing

➢ Modelling competence – Tool competence:

Technology allows a greater variety of prototypes of a formula or especially of functions and also offers some which were not available before. While in traditional mathematics education often only one prototype is available and used, now technology offers several prototypes parallely in several windows. This fact causes a new quality of mathematical thinking. We name this mathematical activity the “Window Shuttle Method”.

Without the necessary tool competence the use of these new prototypes (like recursive models, spreadsheets, regression functions a.s.o) is not thinkable.

➢ Translation competence

The translation process normally proceeds in two steps: At first the information about a certain problem (text, data, graph, a.s.o.) is transformed into a compressed version, which we call “word formula”. And afterwards this word formula is translated into the language of mathematics.

Technology allows the students to transform the word formula directly into a symbolic object of the mathematical language by defining variables, terms or functions, using commands of the technology or writing programs.

Technology also offers and allows a greater variety of testing strategies, in this case testing if the formula is suitable for the problem and mathematically correct.

➢ Modular competence

Using modules is not new for the learners. Every formula used by the pupils can be seen as a module e.g. Hero's formula for the area of a triangle or the use of the cosine rule in trigonometry.

While the modules of traditional math education mostly are the starting point for calculations, the CAS-modules often also do the calculations. W. Dörfler [Dörfler, 1991, pp71] calls any module

"knowledge-unit"
in which knowledge is compressed and
in which operations can be recalled as a whole package.

Creating modules means building a cognitive scheme, condensing cognitive experience. Using modules causes cognitive relief and a reduction of complexity, operations and complex knowledge can be activated as a unit at the same time.

The computer, and especially CAS, opens a new dimension of modular thinking and working. The modular way of thinking is typical of informatics science and in informatics applications. Watching teachers and students in our research project we also found a change of mathematical thinking and working when creating and using modules.

The programming features of the CAS allow the students to create modules which can be used, on the one hand, in the White Box phase as didactic tools and, on the other hand, as Black Boxes for problem solving.

Depending on their source we distinguish **three sorts of modules**:

- Modules produced by the students
- Modules created by the teacher
- Modules which are made available by the CAS

Working with modules causes a modular thinking

A special expert in using modules is Eberhard Lehmann from Berlin [Lehmann, 2002]. He uses the module concept in the classroom starting in the 7th grade. He calls this didactical concept "**concept of building stones**". Walking on the spiral into mathematics means developing a pool of building stones which can be used for problem solving.

Goals of a module oriented mathematics education:

- Defining modules
- Analyzing modules, using modules for experimental learning
- Developing a pool of modules as a source for modelling, for problem solving
- Using modules as black boxes
- Connecting modules, building new more complex modules by using existing modules as building stones

The results of Lehmann´s investigations show that students familiar with modules, use them as new language elements and demonstrate a new quality of mathematical thinking. It is vital to not simply construct a module only to forget it afterwords, but rather to see the opportunity to use the constructed module in several ways.

A2 ⇔ Operating, Calculating

Before discussing what sort of competences are necessary for the students to have we need to first define what calculation competence is:

Definition: **Calculation competence is the ability of a human being to apply a given calculus in a concrete situation purposefully [Hischer, 1995].**

This definition shows that calculation competence does not only mean to execute a certain operation "by hand". Most important for us is the distinction between the goals "performing an operation" (to some extent this can be delegated to a calculator) and "choosing a strategy" (this cannot be done by the calculator.)

The influence of CAS in the calculation competence:

- A shift in emphasis from calculating skills to more conceptual understanding, to modeling and interpreting.
- A shift from doing to planning.

- A reduction of the complexity of manually calculated expressions.
- A shift from calculation competence to other algebraic competences, like structure recognition competence or testing competence.
- A better connection between the formal aspect of mathematics and the aspect of contents.

The definition of the calculation competence shows that manual calculating skills are a branch of the calculation competence, because having calculation competence could also mean being able to decide on the suitable algorithm and to delegate the execution to the computer.

➢ The competence of structure recognition

This competence is necessary when developing a term, when deciding upon or entering a certain operation and also when interpreting or testing. This competence has always been of great importance as research, such as that of Günter Malle, Professor for Didactics of Mathematics in Vienna, has shown us that the most commonly made mistakes during algebraic operating are those of recognizing structures.

Recognizing equivalence of terms which the learners have developed or recognizing results of calculations done by the CAS is a part of the competence of recognizing structures.

A prerequisite for this competence is the knowledge of basic algebraic laws. Such decisions cannot be done successfully by using the CAS as a black box without this mathematical knowledge.

The influence of CAS:

- When using a CAS, the first step, the input of an expression, needs a structure recognition activity.
- Using the CAS as a black box for calculating a recognition of the structure of the expression is necessary before entering the suitable command. Blind usage of commands like *factor* or *expand* is mostly not successful.
- The learner must interpret results and recognize their structure which he himself did not produce.
- The individual results of various students doing experimental learning must often be checked for their equivalence.
- CAS sometimes produces unexpected results and students do not know whether they are equivalent to their expected results or whether they differ.

➢ The calculation competence ("by hand")

We cannot completely leave the calculations to the computer as a black box. We are still dedicated to the following thesis:

> ***For mathematics to develop within a learner certain calculation skills ("by hand") are still needed.***

When I say "we" I also mean the group Herget, Lehmann, Kutzler Heugl [Herget,W. a.o., 2000]. We expressed our position into the paper entitled

Indispensable Manual Calculation Skills in a CAS Environment

We assume a fictitious, written, technology-free exam. We look for questions and classes of questions which we would include in such an exam.

Drawing the border line between questions to be asked in a technology-free exam and questions which would not be asked in such an exam is equivalent to listing the indispensable manual calculation skills.

After reconsidering the meaning and importance of calculation skills and restraining their role in teaching and learning, it is crucial to discuss the consequences for mathematics teaching.

Three Pots

The border line we are looking for clearly depends on many parameters. We try to give a universally applicable answer by creating three pots, which we name –T, ?T, and +T.

The first **pot, –T** (= no technology), contains those questions which we would ask in a technology-free exam. Hence these are the questions which we expect students can answer without the help of *any* calculator or computer.

The third **pot, +T** (= with technology), contains questions which we would not ask in such an exam. Hence in situations in which such problems would occur, we would allow students to use powerful calculators or computers with CAS for their solution.

The second pot, **?T**, reflects our doubts, our different views, and partly also the inherent difficulties of this topic. We either were divided over the questions which ended up in this pot, or we agreed that we would not or could not put them into one of the other two pots. This pot shows how fuzzy the border line (still) is – at least for us.

Whenever feasible we outlined the spectrum and the border line of a class of questions by providing comparable examples for both –T and +T.

The calculation skills needed to answer the questions from pot –T should be mandatory from school year 8, or starting from the school year in which they are taught. The students are supposed to maintain these calculation skills throughout the remaining school years (and, hopefully, beyond school) hence teachers may assess them at any time.

Expressions – With and Without Parentheses

We mentioned above that the formulation of a question is decisive for its value. In the following table we deliberately did without the usual request "expand" and instead requested "eliminate the parentheses." While the first formulation seems to suggest the application of the distributive rule, the second is non-suggestive, which hence increases the value of the question.

	–T (no technology)	*?T*	*+T (with technology)*
01	eliminate parentheses: $a-(b+3)$	eliminate parentheses: $(5+p)^2$	eliminate parentheses: $3a^2(5a-2b)$
02	eliminate parentheses: $2(a+b)$		eliminate parentheses: $(a^2-3b)(-3a+5b^2)$
03	eliminate parentheses: $2(ab)$		eliminate parentheses: $(2a+t)^2$
04	eliminate parentheses: $3(5a-2b)$		eliminate parentheses: $(5+p)^3$
05	eliminate parentheses: $(3+a)(b-7)$		
06	find equivalent forms of: $2a+2b$		
07	simplify $x^2y^2+(xy)^2$		
08	factor $3ab+6ac$		
09	factor x^2-4	Factor x^2+4x+4	factor x^2-x-6

Figure 2.5

–T09: This question is important because it helps to develop the abilities 'deciding' and 'justifying.' Both abilities are needed for sensibly using a calculator's "factor" key or command.

The distributive rule $a\cdot(b+c)=a\cdot b+a\cdot c$ is a background goal here.

We had a long discussion about questions ?T01 and ?T09. Part of our group thought that the ability of recognizing structures needs this calculation skill. On the other hand, the Austrian CAS projects produced some evidence that using technology supports the ability to choose a strategy without requiring the development of corresponding calculation skills.

Linear Equations

	–T (no technology)	?T	+T (with technology)
01	solve w.r.t. x : $x-6=0$		
02	solve w.r.t. x : $5-x=2$		
03	solve w.r.t. x : $3x=12$		
04	solve w.r.t. x : $5x-6=15$		solve w.r.t. x : $5x-6=2x+15$
05	solve w.r.t. y : $\frac{y}{3}=5$		solve w.r.t. x : $2x+3=\frac{4}{3}$
06	solve w.r.t. x : $a \cdot x=5$	Solve w.r.t. x : $a \cdot x-6=15$	
07	solve w.r.t. x : $x+1=x$	Solve w.r.t. x : $2(x+1)=2x$	
08	solve w.r.t. x : $x+1=x+1$	Solve w.r.t. x : $2(x+1)=2x+2$	
09	solve w.r.t. t : $s=v \cdot t$	Solve w.r.t. x : $K=k \cdot x+F$	
10	solve w.r.t. r : $U=2r\pi$		
11	solve w.r.t. x : $\lvert x\rvert=1$		

Figure 2.6

–T06: This example is important, because currently available CAS does not make the necessary case distinction for values of a.

–T11: CAS often produces answers involving the absolute value function. Therefore students should know this function and handle simple applications technology-free.

We do not advocate a simplification or trivialization of teaching mathematics. The suggested low level of manual skills reflects our belief that CAS will become standard tools for mathematics teaching and learning. It also reflects what we observe in our CAS-classrooms: We observe the necessity of a much higher level in other competences like modeling, arguing or interpreting.

➢ The Tool Competence

… to do mathematics means to transform thinking into operating (and then transfering to the computer).

But the essential fact is the entire process and not simply the counter position of contemplating on the one hand and operating on the other.

B. Buchberger

Not only the "thinking act" about the mathematical problem leads to an expected result, also a technology competence is necessary. Without knowing the possibilities of the technology, without knowing the suitable commands and their effects the operation cannot be planned and carried out.

The influence of technology:

- The use of CAS causes additional demands and problems for the students. The operation of the electronic tool needs additional skills which also have to be practiced as calculation skills.
- The evaluation of our last project shows that the measured growing joy and interest in mathematics is significantly higher by those pupils who have no problems with the tool competence, operations of the computer.
- Another significant result is the gap between boys and girls. Both groups show a growing joy and interest in mathematics but boys significantly more so than girls. Girls more often have problems handling the calculator.
- The necessary commands, operations and modes have to be offered to the students in small portions. Practicing and repeating in regular intervals is necessary.

- The use of technology as a Black Box for problem solving demands an agreed documentation of the way of solution, especially in the exam situation.

Some handling skills which are necessary for the fundamental algebraic competence when using a TI-92:

- Input ⇔ recognizing the structure of the expression.
- Storing and recalling variable values.
- Most important commands in the algebra menu are *factor*, *expand* and *solve*.
- Substituting numbers, variables and expressions.
- Setting modes which are necessary for algebra, like the decision exact/approx.
- Defining functions for graphing, displaying Window Variables in the Window Editor. Using Zoom and Trace to explore the graph.
- Generating and exploring a Table, Setting Up the Table Parameters.

➢ The controlling competence

Ever since math has been used as a problem solving technique, it has been necessary to corroborate the correctness of solutions and to interpret them. The teacher has to offer the learners testing strategies or make them able to find some themselves.

In traditional mathematics education testing means activities like substituting numbers, trying another way of solution, checking the usefulness of the mathematical solution for the applied problem, remembering the definition of a concept a.s.o.

A central result of our CAS projects is a more experimental and independent learning process, whereby the expert is not so much the teacher as the CAS. This means that testing becomes even more in important. The stronger emphasis on modelling and interpreting also demands a higher competence in testing.

The influence of CAS:

- The CAS enables the learner to carry out tests both more effectively and quickly.
- Completely new possibilities are available as far as algebraic and graphic testing are concerned
- Using CAS causes a new problem: The learner has to examine and to interpret results which he himself did not produce. The expectation of the sort of the solution or the form of the algebraic term sometimes differs between the learner and the machine.
- The variety of paths leading to solutions and therefore the number of different results increase dramatically. One will not often find the "algorithmic obedience" of the classical math classroom, in which the majority of the students simply imitate the strategies presented by the teacher. Therefore the equivalence of the numerous results has to be tested.
- The more applied mathematics which we see in the CAS-classrooms demands more testing of the correctness of the model, testing of the usefulness of the mathematical solution according to the given problem and testing of the influence of parameters.

Due to the growing importance of testing, new strategies are necessary which show the learner how to make use of the potential possibilities of the CAS. All those who fear that the use of the computer will lead the learner to experimenting with black boxes without the faintest comprehension of what he is doing should realize that in order to carry out an activity on the computer, the learner must, very definitely, have a grasp for algebra and the underlying algorithm. In fact he needs a wider comprehension than if he were to do the problem by hand. A coincidental trial and error method would not be successful.

A2 ⇔ Interpreting and Documenting

➢ Interpretation competence

Interpretation is necessary in two ways: Interpretation concerning the mathematical correctness of the process and interpretation concerning the suitability of the process for the given problem.
The three phases of the problem solving process do not appear in a serial way, they are connected during the whole problem solving process:

- the text in the colloquial language has to be interpreted in order to come to a compressed version – the “word formular”,
- the pool of models which are available for the learner has to be interpreted, to decide a suitable model for the given problem,
- the solution process has to be interpreted concerning the mathematical correctness, several ways have to be analyzed comparing their complexity,
- the solution has to be interpreted concerning the mathematical correctness and the suitability for the given problem,
- the influence of several parameters has to be investigated and interpreted,
- the circle is closing when several improvements of the model have to be interpreted.

➢ Visualisation competence

A special quality of mathematics is the possibility of graphic representation of abstract facts. Apart from free hand drawings, it is difficult to develop graphs without using a computer. Finding the most important points and characteristics of functions in order to be able to draw the graph is the main goal of the discussion of curves in analysis. Technology allows us to draw the graph faster and more directly than the data that is supplied by the curve discussion. Visualizing is one of the most interesting contributions of technology towards a better comprehension of abstract problems. In addition, when observing pupils working with CAS we have found that especially in group or partner work, the visual communication is an important prerequisite and support of spoken and written communication.

➢ Documentation and presentation competence

Using technology means that a main part of operations is carried out by the tool as a black box. Therefore it is absolutely necessary (especially in the exam situation) to demand a precise documentation of the way of solution and such a documentation should also include justifications or references to the used theory.

Especially when there are applied problems to be solved the mathematician (in schools the learner), once having solved the problem, must be able to present his solution clearly either to the questioner or the future user. This should be done in a language which is familiar to the audience.

Presentation competence also includes the use of the presentation tools which are offered by technology. By that mathematics supports the methodology competence, which today is important far above and beyond mathematics.

A2 ⇔ Arguing and Reasoning

> ***"Mathematical thinking technology is the essence of science and the essence of a technology based society" (Buchberger)***

Teachers who read our curriculum start (and end) reading at the contents (Algebra, Analysis, Stochastics …), they forget to read the main part: The educational mission of the subject. There they would find out that

> **the thinking technology which is necessary when doing mathematics, independent from special contents, is the main contribution to the general education in our society.**

The importance of the role of mathematics as a thinking technology correlates to the general goals of teaching laid down in the curriculum:

Pupils should

- become familiar with mathematical methods and manner of thinking
- gain insight into the field of mathematics, whereby the aspects of problem solving, algorithms and the theoretical aspect should be well balanced.
- recognize problems of defining, proving and exactness.

In a memorable lecture at a conference at the University of Klagenfurt in 1991 Prof. Willi Dörfler formulated this thesis concerning the influence of technology in mathematics [Dörfler, 1991]:

- *If we understand cognition as a functional system which encompasses man and tools and the further material and social context, then* ***new tools can change cognition qualitively and generate new competences****. Learning is then not simply the development of existing competences but rather a systematical construction of functional cognitive systems*
- *The computer and computer software must therefore be seen as an* expansion and a strengthening of cognition.

➢ Competence of reasoning by supposition („plausible conlusion“)

Hans Freudenthal said:

"Before students can learn to prove exactly they should learn to suppose"

This heuristic phase of learning – a phase of experimenting, of drawing plausible conclusions - is especially supported by technology and has growing importance when using technology. We can say we often observe the heuristic phase only when students use technology.

Some examples which show the influence of technology on the competence of reasoning by supposition. Technology supports :

- Visualisation,
- Building of Tables,
- Testing of the impact of parameters,
- Zooming,
- Simulating,
- a.s.o.

➢ Competence of reasoning by deduction („logic exact conclusion“)

By proving and substantiating, in other words, by exactifying, assumptions found in the heuristic phase can now be corroborated.

In the abstract phase of the learning process, calculating very often dominates in traditional mathematics education and so the learner often cannot recognize the central goals of this phase like proving or arguing.

Technology allows a shift from doing to planning. The idea, the proof is developed by the students but the necessary calculations can be transferred to the tool.

- **Tool Competence**

Just as in all performance classes tool competence is also needed (and therefore part of standards) when arguing and reasoning.

Some examples:
- Tool competence in the field of algebra: Transforming terms, solving equations.
- Tool competence in the field of vector calculation.
- Tool competence in the field of calculus: Calculating sums and limits, solving differential equations.
- Tool competence in the field of functions: Deciding and using the suitable prototype
- Tool competence in the field of programming.

2.2 Examples for "technology-influenced" Standards

A1 ⇔ Modelling, Representing

- **Modelling competence:**
  Knowing and using mathematical models

Example 1: Prototypes of functions

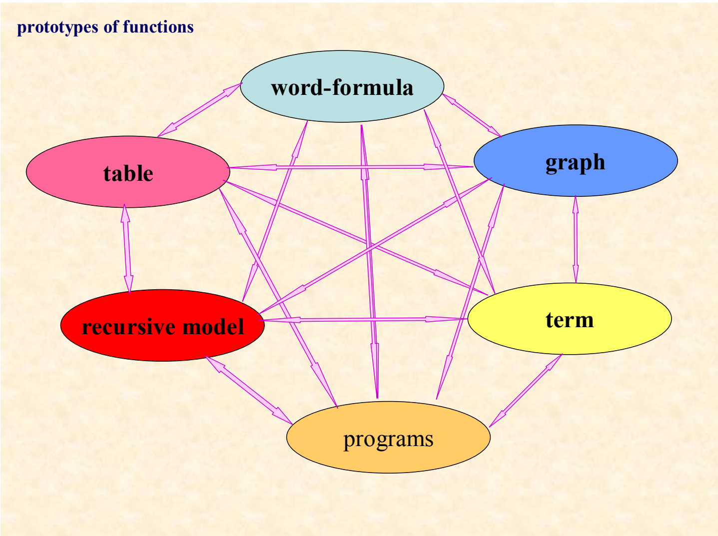


Figure 2.7

The computer as a medium for prototypes makes several prototypes parallely available. This option strengthens and widens the modelling competence.

Standards:
- ***Knowing prototypes of functions offered by the tool.***
- ***Deciding on the suitable prototype for the given problem***

- **Tool competence:**
  Knowing and using the supply of models of the technology

Example 2: Cost and Revenue determine the Profit

[Böhm, J.; 1999]

The Analysis of production cost c for a certain product shows the following total production cost for various production quantities x:

quantity x	10	20	30	40	50	60	70	80	90
Cost c	160	188	210	220	235	255	284	330	390

- Set up a model cost(x) of a total production cost function.
- Produce a table for the total production cost for 0 ≤ x ≤50 with an increment of 5.

Standards:

- ***Generating a table by using technology*** [Figure 2.8]
- ***Setting suitable window-variables*** [Figure 2.9]
- ***Drawing graphs by using technology*** [Figure 2.10]

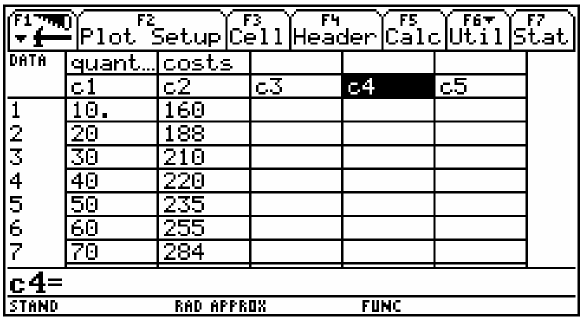


Figure 2.8

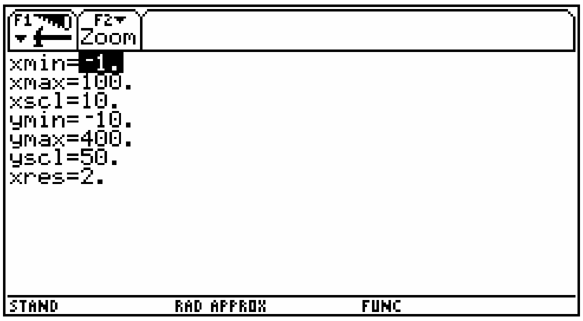


Figure 2.9

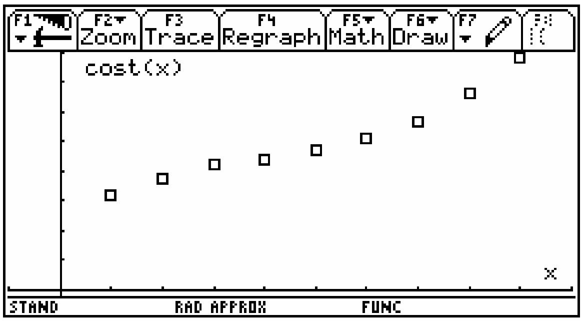


Figure 2.10

Standards:

- ***Choosing the model "Cubic polynomial regression"*** [Figure 2.11]
- ***Analyzing the statistic variables*** [Figure 2.12]
- ***Storing and drawing the graph*** [Figure 2.13]

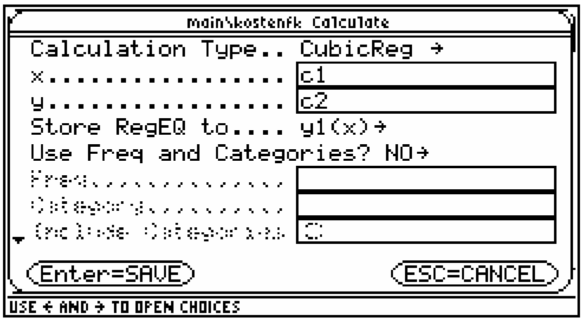


Figure 2.11

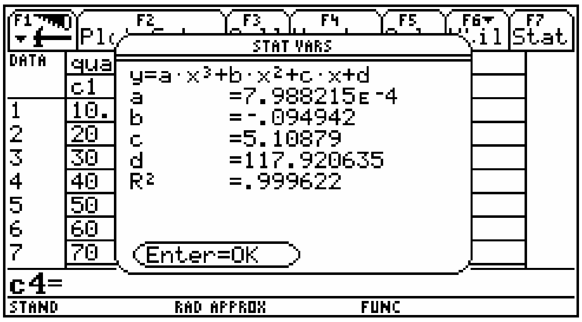


Figure 2.12

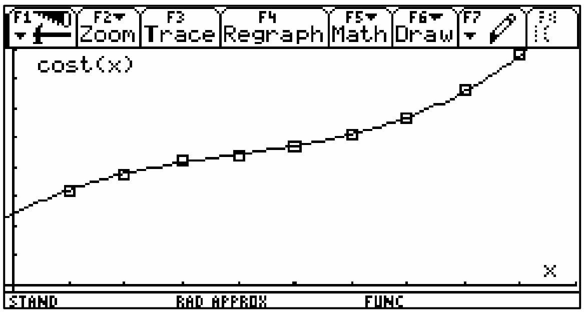
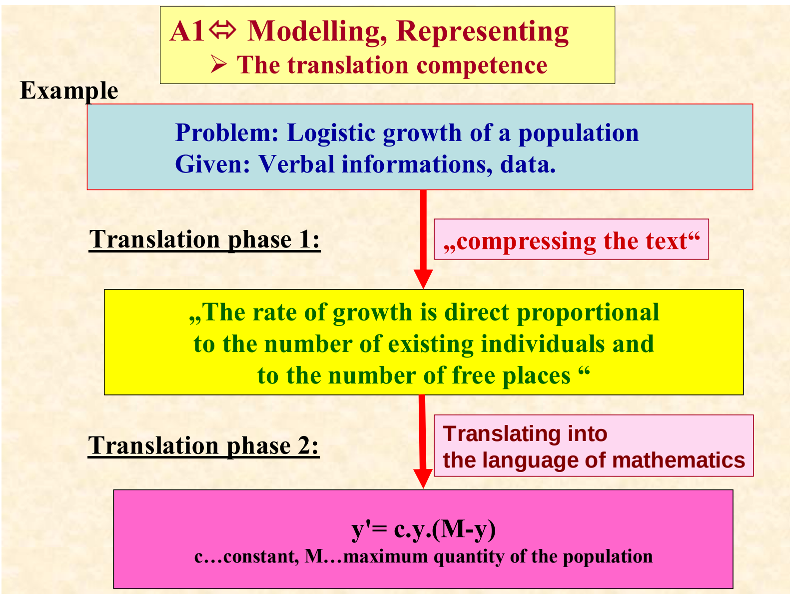


Figure 2.13

- **Translating competence:**
  Translating a text into the language of mathematics

Example 3: Logistic growth of a population

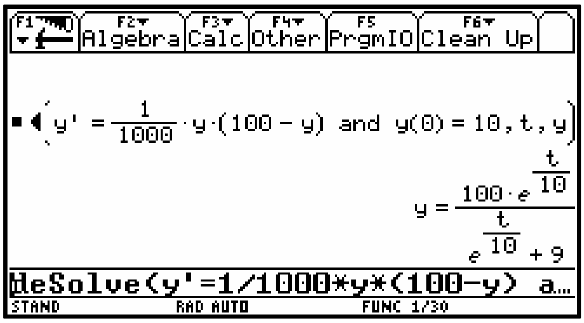
The translation process proceeds in two steps:



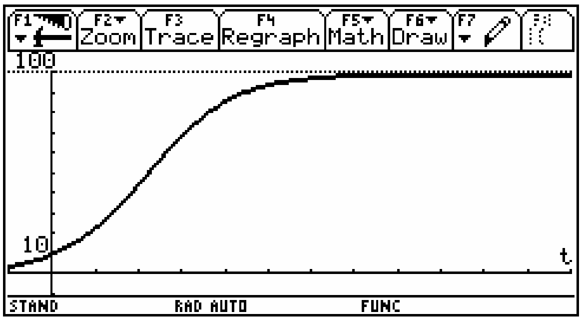
[Figure 2.14]

Standards:

- ***"Compressing" the text*** [Figure 2.14]
- ***Translating into the language of mathematics*** [Figure 2.14]
- ***Calculating by using technology*** [Figure 2.15; Figure 2.16]



[Figure 2.15]



[Figure 2.16]

- **Modular competence:**
  Developing, using and connecting modules

Standards:

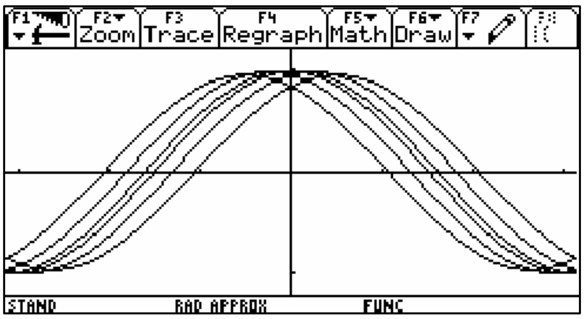
- ***Defining modules by storing*** [Figure 2.17]
- ***Experimenting with modules, formulating conjectures*** [Figure 2.18]
- ***Building new modules by connecting given modules*** [Figure 2.19]

Example 4: Defining a module "Difference Quotient"
[Lehmann, E. 2000]

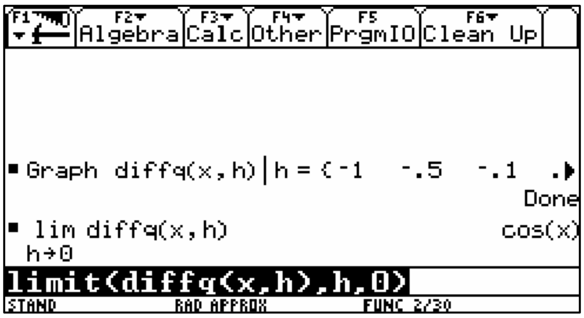
E.g.: Given f(x) = sin(x)

$$\frac{f(x+h)-f(x)}{h} \rightarrow diffq(x,h)$$

[Figure 2.17]



[Figure 2.18]



[Figure 2.19]

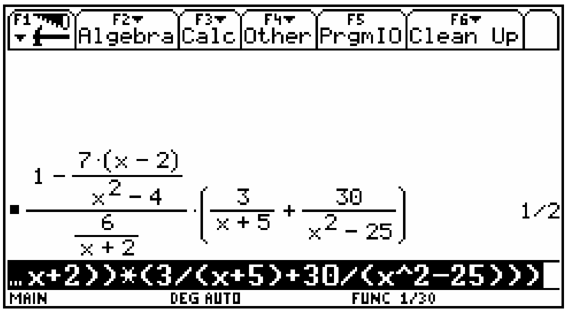
A2⇔ Operating, Calculating

➢ Competence of structure recognition

Standards:

- ***Using suitable parenthesis when entering an expression in the linear entry line*** [Figure 2.20]

Example 5: Structure recognition when entering an expression in the linear entry line



[Figure 2.20]

➢ Calculation competence („by hand“)

Standards:

- ***Calculating without using technology*** [Figure 2.05 and 2.06]

➢ Tool competence

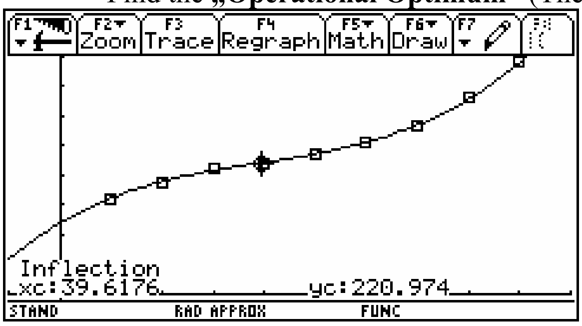
Standards:

- ***Operating in the Graphic Window*** [Figure 2.21; 2.24 and 2.25]
- ***Operating in the Algebra Window*** [Figure 2.22]
- ***Operating by using a table*** [Figure 2.23]

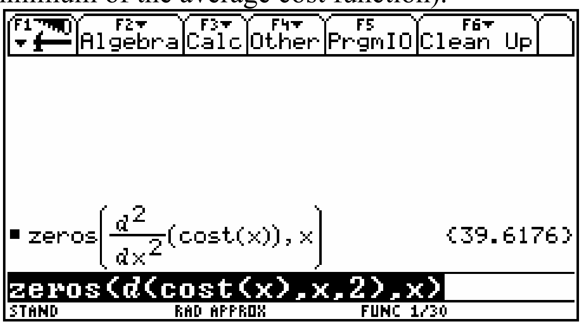
Example 6: Cost and Revenue determine the Profit
[Böhm, J.; 1999]

Prerequisite: The total production cost function cost(x) and the average cost function cost(x)/x have been found.

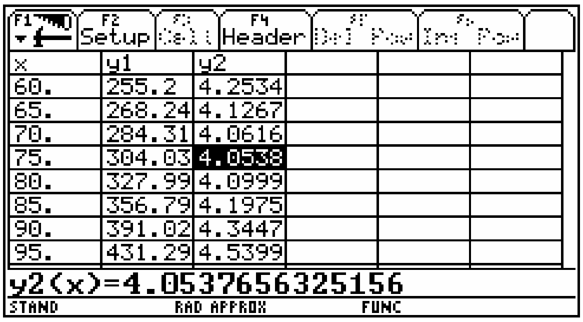
- Find the **„Cost Turn“** (The inflection point of the cost function cost(x)).
- Find the **„Operational Optimum“** (The minimum of the average cost function).



[Figure 2.21]

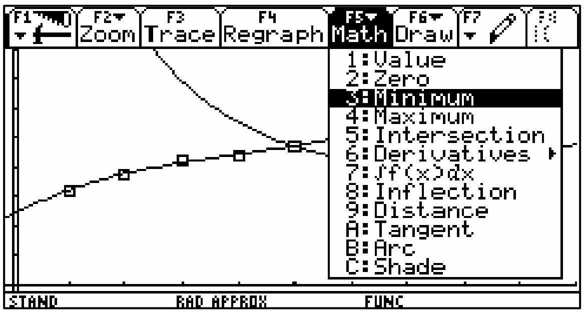


[Figure 2.22]

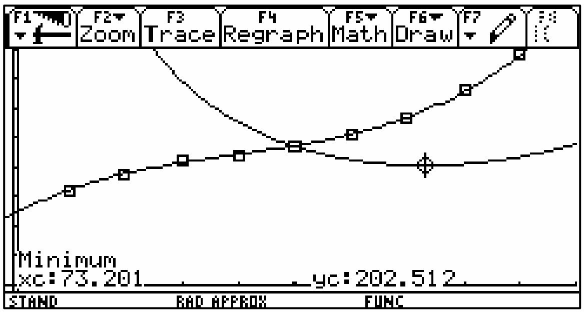


x	y1	y2
60.	255.2	4.2534
65.	268.24	4.1267
70.	284.31	4.0616
75.	304.03	4.0538
80.	327.99	4.0999
85.	356.79	4.1975
90.	391.02	4.3447
95.	431.29	4.5399

[Figure 2.23]



[Figure 2.24]



[Figure 2.25]

A3⇔ Interpreting und Documenting

- **Visualisation competence**
  Using and interpreting graphic models and results

Standards:

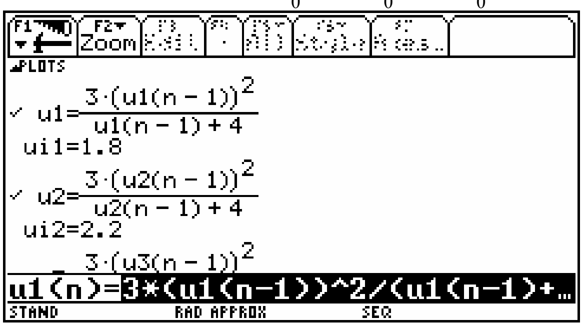
- ***Visualizing in several graphic modes of the technology*** [Figure 2.27 and 2.28]
- ***Investigating the influence of parameters in the graphic window*** [Figure 2.26 and 2.27]
- ***Interpreting results by "shuttling" between several representations (Window – Shuttle-Method)*** [Figure 2.26; 2.27 and 2.28]

Example 7: Sterile Insect Technique (SIT)
[Timischl, W. 1988]

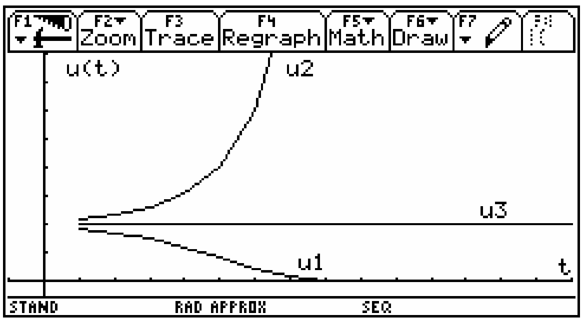
An insect population with u_0 female and u_0 male insects at the beginning may have a natural growth rate r. To fight these insects per generation a certain number s of sterile insects is set free.

Investigate the effect of the method SIT by interpreting the growth function for several parameters u_0, r, s.

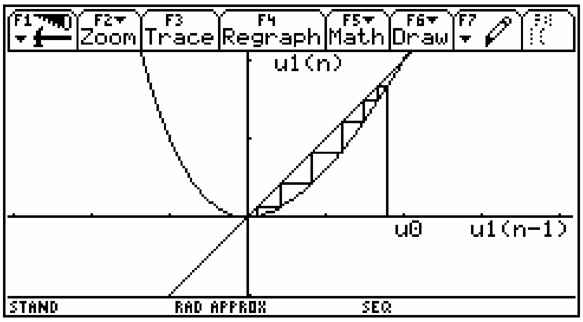
- Model assumption: r=3; s=4
- Initial values: u_0=1,9; u_0=2,2; u_0=2,0 (e.g. million insects)



[Figure 2.28]



[Figure 2.27]



[Figure 2.28]

- ➢ **Tool competence**
  Using the options of technology for interpreting

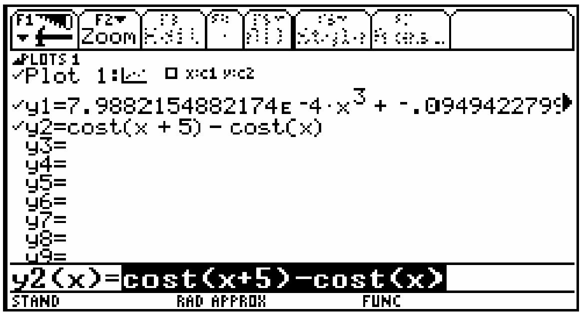
Standards:

- ➧ ***Interpreting results by "shuttling" between several representations (Window – Shuttle-Method)*** [Figure 2.29 and 2.30]

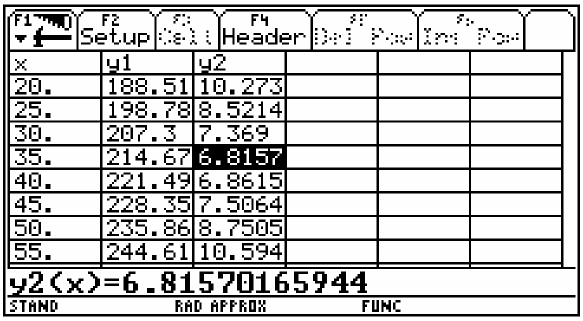
Example 8: Cost and Revenue determine the Profit
[Böhm, J.; 1999]

Prerequisite: The cost function cost(x) was found by using the cubic polynomial regression.

- Determine as accurately as possible the range where production cost exhibits the slowest increase.



[Figure 2.29]



[Figure 2.30]

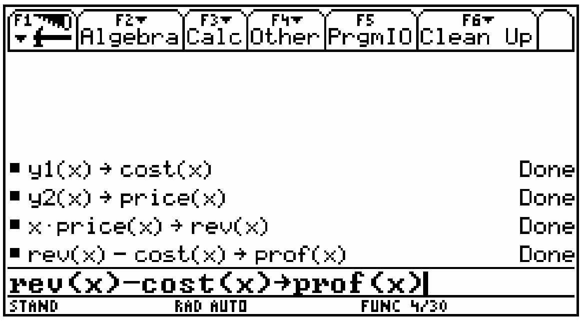
- ➢ **Documentation and presentation competence**
  Representing ways of solution and results

Standards:

- ➧ ***Using technology for presentation of results*** [Figure 2.32]

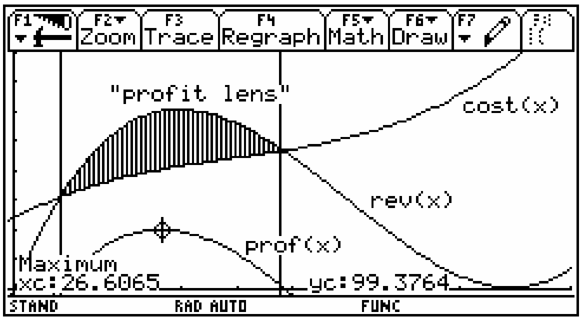
Example 9: Cost and Revenue determine the Profit
[Böhm, J.; 1999]

Prerequisite: The cost function cost(x), the revenue function rev(x) and the profit function prof(x) have been found and stored [Figure 2.31].



[Figure 2.31]

- Represent the result in the Graphic Window (including text labels, profit zone and profit´s maximum point).



[Figure 2.32]

A4⬄ Arguing and Reasoning

➢ **Competence of reasoning by supposition**
(„plausible conlusion")

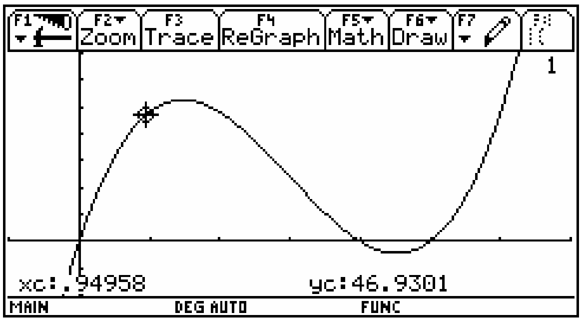
Standards:

➧ ***Getting assumptions by experimenting supported by technology (e.g. "Zooming")***
[Figure 2.33 to 2.37]

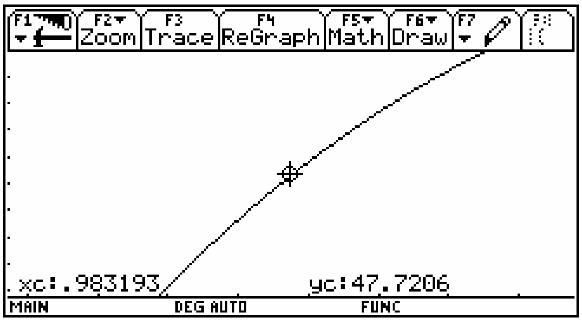
Example 10: Calculus – the idea of linearization

Given: A polynomial function y=f(x).

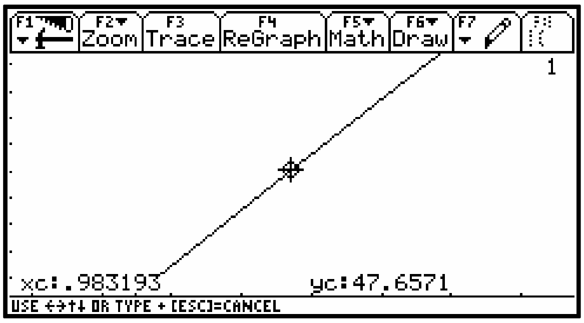
- Try to visualize the assumption of the idea linearization by zooming [Figure 2.33 to 2.35].
- Take the absolute value of f(x) and try again to come to an assumption about linearization by zooming [Figure 2.36 to 2.37].



[Figure 2.33]

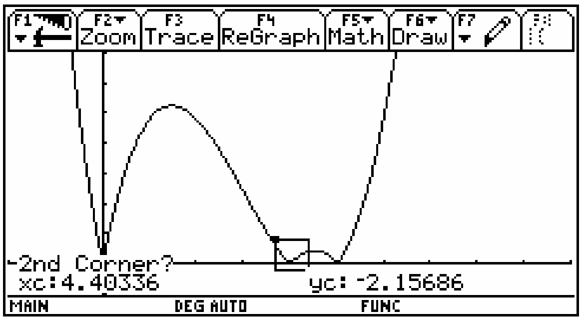


[Figure 2.34]

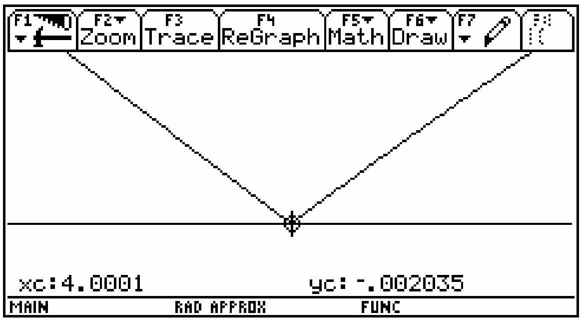


[Figure 2.35]

Assumption: $|f(x)|$ is continuous but not differentiable



[Figure 2.36]



[Figure 2.37]

➢ **Competence of reasoning by deduction**
(„logic exact conclusion“)

Standards:

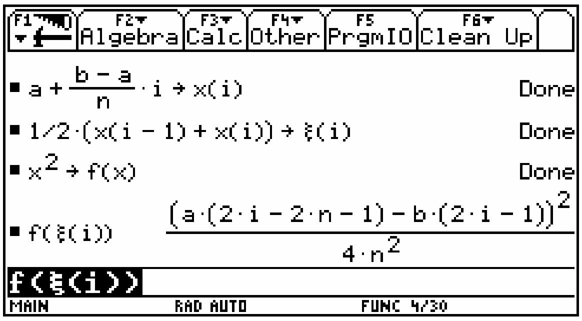
➧ ***Planning the way of solution – operating supported by technology*** [Figure 2.38 to 2.40]

Example 11: Riemann sums - exactly calculated

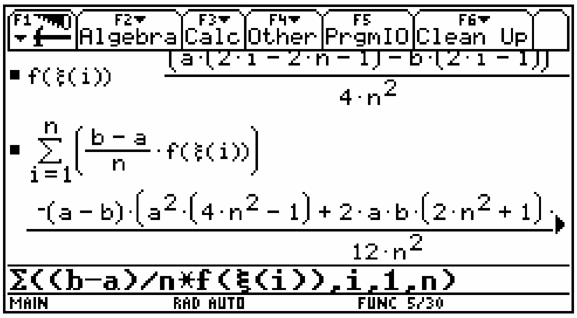
Given: $\int_a^b x^2\,dx$

- Calculate the definite integral by using the definition of the definite integral e.g. use the idea of „midsums“.

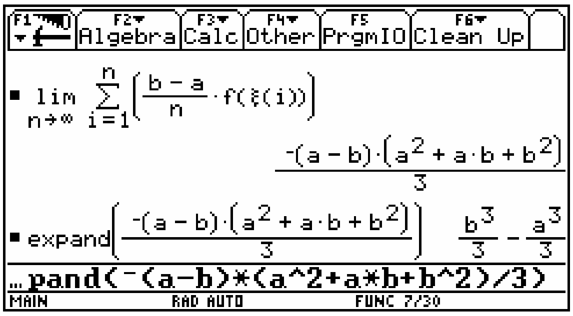
After dividing the interval [a,b] into n equal parts and calculating the mid points ξ(i) the function values f(ξ (i)) have to be computed [figure 2.38)]. Using the technology the sum is available very quickly and also the limit of the sum [figure 2.39 and 2.40].



[figure 2.38]



[figure 2.39]



[figure 2.40]

- **Tool competence**
  Arguing supported by technology

Standards:

➧ ***Explaining results produced by technology*** [Figure 2.41]

Example 12: Explaining ways of solution carried out by ***Derive***

Calculate the first derivative of a complex function. Use the "step mode" of ***Derive*** and explain the comments (blue at the right side) to the single steps offered by the technology.

$$\frac{d}{dx}\sqrt{(x\cdot SIN(x))}$$

$$\frac{d}{dx}F(x)^{n} \Rightarrow n\cdot F(x)^{n-1}\cdot\frac{d}{dx}F(x)$$

$$\frac{\frac{d}{dx}(x\cdot SIN(x))}{2\cdot\sqrt{(x\cdot SIN(x))}}$$

$$\frac{d}{dx}(F(x)\cdot G(x)) \Rightarrow G(x)\cdot\frac{d}{dx}F(x) + F(x)\cdot\frac{d}{dx}G(x)$$

$$\frac{x\cdot\frac{d}{dx}SIN(x) + SIN(x)\cdot\frac{d}{dx}x}{2\cdot\sqrt{(x\cdot SIN(x))}}$$

$$\frac{d}{dx}SIN(x) \Rightarrow COS(x)$$

$$\frac{x\cdot COS(x) + SIN(x)\cdot\frac{d}{dx}x}{2\cdot\sqrt{(x\cdot SIN(x))}}$$

$$\frac{d}{dx}x \Rightarrow 1$$

$$\frac{x\cdot COS(x) + SIN(x)}{2\cdot\sqrt{(x\cdot SIN(x))}}$$

[Figure 2.41]

3. Technology and Assessment

[ACDCA Report of research projects CAS III and CAS IV]

Which is the more valid question ?

- **Do the new ways of mathematics learning and teaching influence the exam situation?**
  or
- **Does the exam situation influence new ways of learning and teaching?**

In the past, the exam situation has always had a great influence on the content and the didactic concept of mathematics education. So the emphasis sometimes placed on a specific math topic can only be explained because it is easy to construct a suitable test.

Therefore it is comprehensible that one of the principles formulated by the American NCTM is called the "**Assessment Principle**"

Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.

Some tasks of the Assessment Principle:

- Assessment should be more than merely a test at the end of instruction to gauge learning.
- Teachers should be continually gathering information about their students.
- Assessment should focus on understanding as well as procedural skills.

- Assessment should be done in multiple ways, and teachers should look for a convergence of evidence from different sources.
- Teachers must ensure that all students are given the opportunity to demonstrate their mathematical learning.

Already in our former CAS projects we recognized that the common methods of testing were not suitable to the new ways of learning which we observed in our CAS classes.

In traditional mathematics education written exams (5 or 6 one-hour-tests per year) dominate. As far as content is concerned the emphasis is on calculation skills. This way of testing is suitable to the dominating style of teacher centered teaching, which causes a more reproductive way of learning. Often in two or three of the four examples of a test the same skills are tested – students have to be busy for one hour.

A special weakness of our system of assessment which was painfully pointed out to us in the PISA-study is that **our students are above all trained to gain "short term competences" for the next exam.**

Some **significant changes of the learning process in our CAS-classes** which strengthen the necessity of changes in the exam situation are:

- A more pupil-oriented learning process. More frequently mathematical discussions among the students. The teacher is not the only source of knowledge, he supports the independent acquisition of knowledge by the students.
- Experimenting, the trial- and error method: We seldom find the "algorithmic obedience" where the teacher shows one way which all the students then accept and follow
- Working in pairs or groups can be seen much more frequently.
- Beside the teacher there exits now a new, very competent expert – the tool CAS. That means pupils do not always need the teacher for examining the correctness of their ideas and results.
- Phases of "open learning" where the students are individually organizing the speed and the contents of their learning process.
- The CAS is not only a calculation tool, students can also store knowledge by defining modules or using the text editor. Therefore it is senseless to forbid the use of learning media, like books or exercise books during the tests.
- New emphasis of fundamental competence. A shift from calculation skills to other competence and skills.
- A clearer emphasis on problem solving
- A more application oriented mathematics.
- More frequent cross curriculum teaching

3.1 New ways in assessment when using technology

Based on this recognition we started a project dealing with new ways in assessment. Task was to investigate the consequences of the following models of examining in the students learning:

Model 1: A "two phase model" for written exams

In contrast to the Austrian National Curriculum which prescribes a certain number of hourly written exams per year, we offer to the teachers a "years´time" for written exams. Our model allows the teachers to test a total of 250 minutes in the academic year, thus permitting them to assess their students in two different ways:

- **Shorter tests** (15 to 30 minutes) to examine certain fundamental competence like calculation competence, visualization competence or also abilities of using the available

CAS. In some tests the use of any electronic tool, especially CAS, was forbidden, in others it was allowed.

- **Problem-solving-examinations** (50 to 120 minutes) measure the competence of problem solving with more application-oriented examples, with more open questions, with more emphasis on argumentation, reasoning or interpreting. During these examinations students mostly are allowed to use their learning media like their math school books or their exercise books.

Both the model and the content of the tests were influenced by our discussion about fundamental mathematical competence:

Thesis:

The fundamental competence examined by short tests is the basis and the prerequisite for the most important task of mathematics, the solving of problems.

Significant for this model is the idea of the two phases:

- At first building the foundation by focussing on a certain mathematical fundamental competence like an algebraic competence and then
- in a second phase using several fundamental competence for problem solving.

The separation of the two sorts of examinations strengthens the “two phase learning model”. By making conscious of the necessity of “fundamental competences” (or “building stones” for problem solving) and by demanding it continually in these short tests we expect a strengthening of “long term competences” – the central goal of product oriented standards.

Model 2: “Project work”

A certain number of the classic written exams are substituted by projects which are partly done during the lessons but the larger part of the work the students have to do at home. In some classes every single student has to deal with his special theme, in others the mission of the project work is given to a group of students. This way can be especially observed in larger classes because the presentation of the project work of every individual student of the class costs a lot of time.

The content of such project work are themes which allow students to apply formerly learned contents but there are also spheres where the student has to deal independently with new problems or contents. Students have to produce term papers focusing on their theme which are offered to the other students of the class and they have also to carry out a presentation of their results.

The assessment takes place in two ways:

Process oriented as well as product oriented

- Observation of the learning process: The independent activity, the ideas of the student, the necessary inputs of the teacher.
- Assessment of the results: The quality of the term papers, the quality of the presentation, the competence during the discussion about the results.

The advantage of this model of examination is that the learning process and the phase of assessment are not separated. Examination is not a singular event which often causes a lot of stress for the students and the result of which often depends on the momentary state of mind of the students. Anyway this model encourages not only the mathematical competence, it also strengthens the other key qualifications, like methodological competence, social competence and personal competence.

Another advantage is that this model allows an inner differentiation which is not possible when using common written tests: More gifted students can work more on demanding problems than not so gifted students

If we could increase the number of such models of learning and examining, the subject mathematics would make a much better contribution to the necessary general education in the age of information technology and the age of life-long learning.

Model 3: Cross curriculum tests

One of the main tasks of the school of the future is a greater emphasis on training of networked-thinking. One sort of exams within the final exam, the Matura, is an oral exam connecting two subjects. Only a few students choose this way, because before the final exam they cannot experience very often cross curriculum phases of learning and a cross curriculum test until now is not planned.

In comparison with traditional classes, we observe a growing importance of cross curriculum phases in our CAS-classes and therefore it is reasonable to consider this fact in the exam situation.

Using the possibilities of the TI-92, especially CBR and CBL, a connection of mathematics and science is obvious. The questions of such cross curriculum tests are dealing with both subjects the results are assessed for the grades of both subjects.

3.2 Examining "building stones" for problem solving and problem solving itself

3.2.1 Examples of short tests examining certain fundamental competence:

[ACDCA Report of research projects CAS III and CAS IV]

As I mentioned before fundamental competence is more than calculating skills. Those tests shall assess fundamental mathematical competence like algebraic competence but also the competence of applying certain heuristic strategies like arguing, reasoning a.s.o.

Example 3.1: Mag. Ingrid Schirmer-Saneff, 9th grade Gymnasium Berndorf

Short test: 30 minutes, without using TI 92

Goals: Algebraic competence:

- Visualization competence
- Calculation competence
- Numerical competence

1.) Visualise the following sets in a coordinate system:

a.) $\{(x,y) \in R^2 | 0 \le x \le 3 \wedge y = 1\}$

b.) $\{(x,y) \in R^2 | 3 \le x \le 7 \wedge -4 \le y \le -2\}$

2.) Solve the unequation in ℝ. Name the used equivalence transformations

$$\frac{6y+5}{7} - 4 < 4 - 4y$$

3.) Solve the following equation:

$$\frac{1}{R_2} = \frac{1}{R} - \frac{1}{R_1} \quad ; \quad R_1 = ?$$

4.) Transform into decimals

a.) $4{,}7 * 10^{9}$

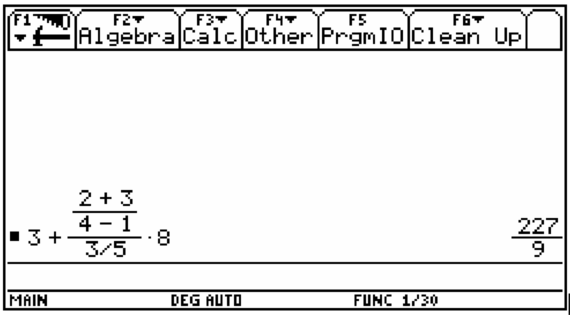
b.) $8{,}59 * 10^{-10}$

Example3.2: Mag. Christian Hochfelsner, 7th grade Gymnasium Stockerau

Goal: Structure recognition:

Usage of the TI 92 is allowed

Which keys were activated for the input of this term?

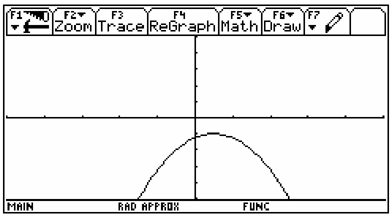


[Figure 3.1]

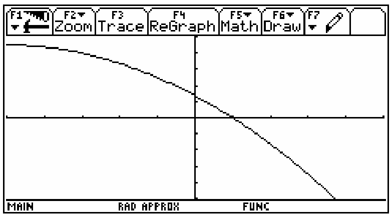
Example 3.3: Dr. Hildegard Urban Woldron, Gymnasium Preßbaum

Goal: Visualisation competence

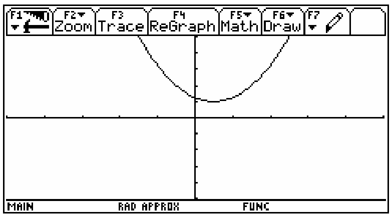
No use of the TI 92 is allowed



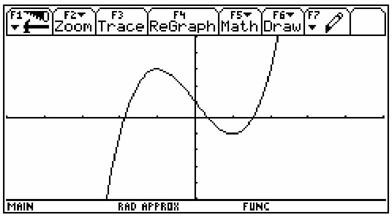
[Figure 3.2]



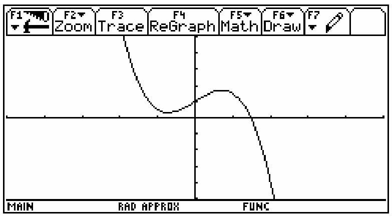
[Figure 3.3]



[Figure 3.4]



[Figure 3.5]



[Figure 3.6]

Which of the graphs have the following qualities:?

f’(0)>0 or f’(1)<0 or f’’(x) is always negativ.

3.2.2 Problem-solving-exams

The single fundamental competence can be seen as modules which are the necessary building stones for solving problems. After having acquired such a competence, the suitable skills e.g. calculation skills can be processed by the tool CAS. The durable, long-term competence is the relational understanding, the competence of planning and deciding on a suitable method or algorithm.
Actually the tool CAS often makes the solving of interesting, realistic problems at first possible. The separate examination of the fundamental competence and the possibility of using all learning media including instructions about this competence makes the significance of these building stones conscious for the students. Therefore they are able to concentrate their activities on the problem solving strategies.

The following example is typical of a problem solving exam:

Example 3.4: Mag. Martin Dangl, Gymnasium Waidhofen/Thaya

Goal: Problem solving

Necessary fundamental competence:

- Modeling competence
- Tool competence
- Translating competence
- Modular competence
- Competence of recognizing structures
- Controlling competence
- Calculating competence
- Visualization competence
- Working with modules (the used formula can be seen as modules)
- Documentation and presentation competence

Problem:

1500 fish are released into a river. The annual growth rate of the fish population is 35%. This growth of the fish population naturally depends on the amount of fish caught annually and should be discussed using the following three models.
Describe each model by using the appropriate difference equation. Graph the three functions with respect to the time for the first twelve years and describe shortly in words the differences of the three systems with respect to the time.

a) **Model A:** The number of fish caught per year remains constant at 400
- Solve the difference equation for this model. (Derive an explicit representation of the population number x_n from the difference equation)
- Calculate the number of years which it takes for the population number to exceed 100 000.
- Calculate how many fish need to be caught annually if no fish are to be left at the end of twelve years.

b) **Model B:** Beginning with a fish population of 400, the amount of fish caught annually should increase by 10% per year.
- Calculate how many fish in total are caught in the first 12 years.

c) **Model C:** Determine how many fish should be caught annually so that the following conditions are fulfilled:
(1) The amount of fish caught annually at the beginning should be greater than that in model B.
(2) The amount of fish caught annually should grow at a constant percentage rate of p.
(3) The population should grow continuously during the first 12 years.
(4) A long term unlimited increase in the population must be prevented.

4. Final conclusion

Conclusion 1: Implementation of standards

We expect the implementation of standards for orientation and evaluation to cause an increasing consciousness for necessary students´ long-term competences and a more meaningful mathematics.

We expect that discussing and evaluating the product oriented performance standards will also cause consequences for an improvement of the educational process.

Conclusion 2: Use of technology

We are sure that the use of technology will increase the joy and interest of the students and they will experience the learning of mathematics in a more meaningful way because we can offer them a more meaningful mathematics.

We expect the influence of technology on standards to be considered. Tool competence as an integral part of mathematic competences also has to be measured and evaluated. A recent problem is that we cannot expect that every student is familiar with technologies like Computeralgebra systems. We hope that this problem will be solved in the near future. The prerequisites are suitable technological tools within parents´ or the schools´ means.

Conclusion 3: Assessment

We expect a shift from "short term" competences to "long term" competences and more balance between process and product oriented assessment.

We expect the familiar technology to be used in every exam situation. This can only be realised when the sort of exam questions will change (e.g. traditional "curve discussions" can no longer be exam questions when CAS is available).

Conclusion from (1), (2) and (2):

Technology has to be considered when implementing standards and when implementing new ways of assessment.

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