What is Happening with CAS in Classrooms?
Example Austria

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Abstract: In this paper a short overview is given how CAS is used in Austrian classrooms. The main part deals – according to the wishes of the organizers of this CAME strand – with real classroom examples and experiences. They are embedded in a kind of framework to explain the status of CAS-supported learning, teaching and assessing in Austria. I left most of the examples in their original form (in German) to stay as authentic as possible. This overview does not pretend to be complete – it is my personal sight of the matter.

1 CAS in Austria

Introduction and propagating the use of CAS in math education is in a very tight connection with the Austrian Centre for Didactics in Austria which was founded by school authorities and enthusiastic teachers in the early nineties. The ACDCA is hosted by the Pedagogical Institute of Lower Austria in Hollabrunn. 1991 the Austrian Government purchased a general licence of DERIVE for all general secondary school (grammar schools). The ACDCA was founded with the intention to attend and support teachers in using this revolutionary tool in a meaningful and responsible way unlike to the introduction of the pocket calculator several years before.

There was a fruitful cooperation with the T^3-organization for several years. Useful synergy-effects made pre- and in-service trainings very successful. Unfortunately this cooperation was ended by Texas Instruments in 2005. It is a fact that Austria – seen as a whole - was never a graphing calculator country, which was caused by the early use of DERIVE.

The ACDCA initiated and supervised a couple of Austrian wide projects:

Austrian ACDCA Projects

Austrian Centre for the Didactics of Computer Algebra, PI Lower Austria, Hollabrunn

CAS I **1993 – 1994** **DERIVE Project**

CAS II **1997 – 1998** **TI-92 Project**
Creating teaching materials – Influence of CAS on teaching
44 schools, 70 classes (65 teachers & 680 female, 1570 male students) involved

CAS III **1999 – 2000** **2nd TI-92 Project**
Electronic Learning Media in Maths Education
Influence on Teaching, Learning, Curriculum and Assessment
94 classes with more than 2000 students involved

CAS IV **2001 – 2002** **CAS Project**
New Media and Methods (New culture of problems, Supervision,
Bilingual Teaching, Establishing a Service Centre for Teachers, …
140 classes with more than 2200 students involved

CAS V **2003 – 2005** **Variety of Media Project**
together with GeoGebra and Mathe-Online
e- & Online-Learning, Self responsible Learning,
Standards for Mathematics, Teaching in "Laptop classes", …

CAS VI **in preparation**

The most used CAS in Austria is DERIVE (General Secondary schools and Secondary Colleges for Business Administration, followed by the handheld CAS (TI92/TI89/V200). The Secondary Colleges for Engineering use MathCAD (general licence) and the Voyage 200. There is a minority using MATHEMATICA and MuPAD. Other Secondary Vocational Schools use – if they use CAS – handheld CAS or DERIVE.

2 CAS and Curriculum

Curricula demand the use of technology for all secondary schools. The respective part in the curriculum for general secondary schools reads:

Technologies like computer algebra systems, spreadsheet programs, dynamic geometry software shall play an important role in mathematics education. Students shall be able to use these programs up to an amount which is relevant for the mathematical content. Minimum realisation is introducing these technologies not only for some examples but using them for acquiring some fundamental concepts. Maximum realisation is the meaningful use of them as an integrative and consequent part of the teaching and learning process.

This is the respective paragraph of the curriculum of Secondary Colleges of Business Administration:

Die Schülerinnen und Schüler sollen

…..

- Computer Algebra Systeme und/oder Tabellenkalkulation bzw. grafikfähige Taschenrechner in allen Jahrgängen einsetzen und mathematische Problemstellungen damit lösen können.

The students shall be able to

…..

- use Computer Algebra Systems and/or Spreadsheet programs or graphing calculators respectively in all forms and to solve mathematical problems by using these tools.

The Secondary Colleges for Engineering have been using CAS for a long time (MathCAD and TI-CAS).

The Austrian school system provides freedom in teaching methods. As there are no central final exams it is in the responsibility of the teachers if and in which amount CAS is permitted or even demanded – or if it is completely banned.

3 CAS and Assessment

If CAS is used in learning and teaching then it is also used for assessment. I show some examples from my teacher career. Problems and questions changed compared with traditional assessment problems. All the following problems had not been trained. The students saw problems like these during the test the first time. (Form 10 = age 16, form 11 = age 17)

Example 1 (form 10)

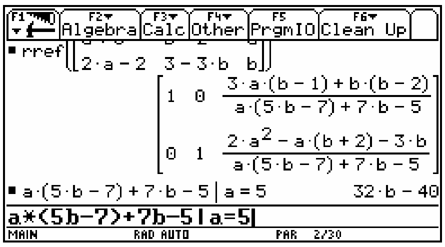
Solve the simultaneous equations for *x* and *y* and find at least two pairs (*a*,*b*), which make the system non soluble!

$$a x + b y = a + 2y - 3x$$
$$2a x - 3b y = b - 3y + 2x$$

This problem was given in times when the TI-92 was not able so solve systems of linear equations with one solve-command. I expected the students to bring the unknown and the parameters in a right order and apply any solution algorithm (Cramer´s Rule, Row Echelon Form, …) and then find out two pairs (*a*,*b*) which equate the denominator (see the TI-92 screen shot):

$$(a+3)\,x + (b-2)\,y = a$$
$$(2a - 2)\,x + (3-3b)\,y = b$$

For example $a = 5$ and $b = 5/4$ form one such pair.



I was very much surprised that one girl was ready within a minute – without using the calculator. I thought that she was cheating and asked her what she did: ‘Oh’, she said, ‘that´s easy. Take the first line: *a* = -3 and *b* = 2 make a contradiction 0 = -3. This is the first pair. The second line gives the other one.’ This was one of the – many moments – when I was happy to be a math teacher.

Example 2 (form 10)

a) Enter the equation and explain the TI´s "simplification".

b) What is the equation´s general solution and which conditions are necessary for its validity?

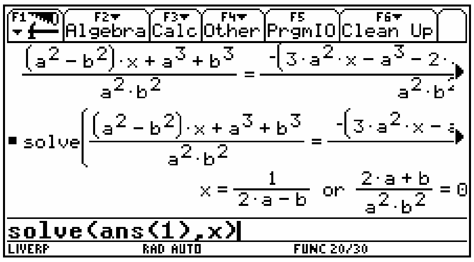
c) Find two special solutions!

d) What is the result of the TI?

e) Explain the second part of the TI-result using a self chosen example.

f) Can you explain where this strange second part is coming from?

$$\frac{b-x}{a^2} + \frac{a+x}{b^2} = \frac{b}{a^2} - \frac{3x-a}{b^2} + \frac{b+2a}{a^2b^2}$$



Example 3 (form 11)

Five spheres (R = 3.17 inch) made of brass are melted down and a cylinder with r = 6.34 inch is cast. What is the length of the cylinder?

I had the students encouraged to compile their individual electronic collection of formulae stored on their handheld CAS-devices. It was very interesting to observe that among the 18 students more than 10 various ways appeared to solve this easy problem. The range reached from calculating like in secondary 1 to the following elegant “One-liner”. Working with CAS encourages working with variables and with functions.

```
solve(5*sh_v(3.17)=cyl_v(6.34,l),l)
```

Example 4

Given is a set of solutions L = {3, -1, ½}
Find two equations of degree 5 with L = set of solutions.
I had not done this before, but the students knew about multiple solutions appearing in equations of higher degrees.

Example 5

Find the solutions of the following equation without applying the solve()-command:

$$36x^6 + 132\,x^5 - 263x^4 - 638x^3 + 926x^2 - 770x + 1225 = 0$$

There was one partial project group with teachers testing alternative assessment forms (problem solving tests, group tests, group presentations, “Facharbeiten”, …The very interesting report can be found in [11].

4 CAS and Standards

The question of establishing education standards in general and of standards for math education in particular is in the focus of discussion in many countries. So it is in Austria, too. As a consequence of TIMSS and PISA and of opening the borders for students in Europe we need comparability of the education systems.

Problems and questions for testing standards have been developed for Secondary 1 and a group of experts with Helmut Heugl in a leading position are working on the standards for Secondary level 2. They are considering including CAS into the standards according to the demands of our curricula.

The work of the team of experts is based on Heugl´s Competence Model. He says:

> CAS-supported mathematical education supports and encourages the 4 key qualifications
>
> - Subject competence
> - Methodological competence
> - Social competence
> - Personal competence
>
> much better than traditional mathematics education.

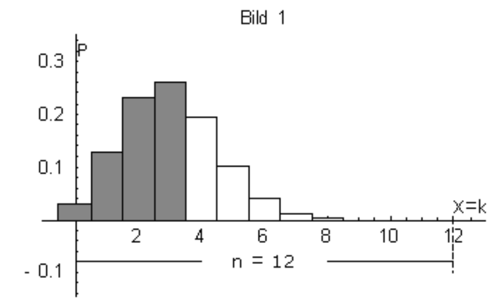
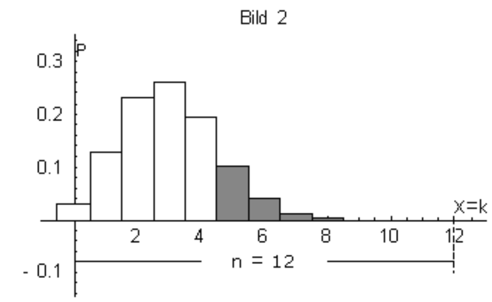
and then he continues about "the necessary fundamental algebraic competence in the age of CAS" addressing competences for

- Recognizing structures
- Finding terms and formulae
- Testing
- Calculating
- Visualizing
- Modular working
- Finding the appropriate tool

Here is one test item for secondary 2:

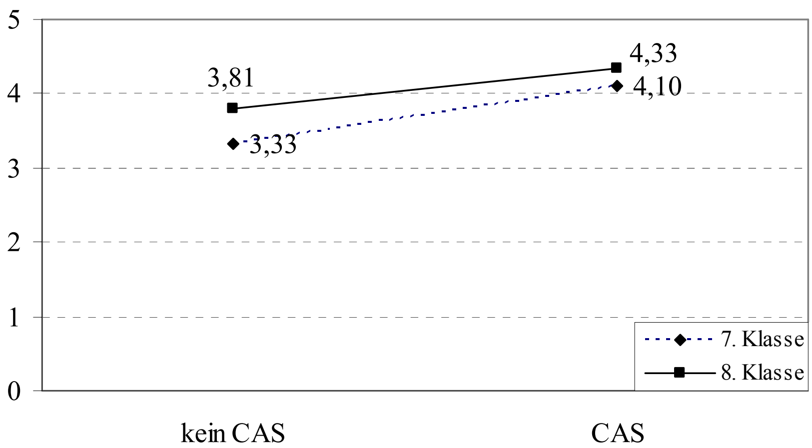
1. A test consists of 12 multiple choice questions with 4 answers for each of them. One answer is correct. The answers are marked randomly. X is the number of correct marked answers. The following diagrams show the distribution of X.

   a) What is represented by the shaded area? Give your answer in words and in an appropriate mathematical form.

	
umgangssprachlich (in words):	umgangssprachlich (in words):
mathematisch:	mathematisch:

The first test of the standard test items brought an unexpected side result. 21 classes of grammar schools were tested (randomly chosen). It turned out that 10 of them were CAS classes. The evaluation showed that the CAS-classes performed much better than the traditional educated classes. (All problems were solved without any tool and they had not been designed considering CAS).

gewichteter Gesamtindex



This is encouraging and leads immediately to the next crucial point of discussion:
(It must be said, that 21 classes is not so much, to deduce a significant statement. Another result was that the standard deviation of correct answered questions was greater in the CAS –Group.)

5 CAS and Basic Skills

One of the main obstacles which must be overcome by propagating CAS in math education is the fear of teachers (and parents, too), that the so called "basic skills" might get loss. Question: "Which are Basic Skills?" Read more about this in the "provocative" paper ([2])

Indispensable Manual Calculation Skills in a CAS Environment

Wilfried Herget (Halle, Germany), Helmut Heugl (Wien, Austria). Bernhard Kutzler (Leonding, Austria) and Eberhard Lehmann (Berlin, Germany)

Many of us use CAS to train these „Basic (Manipulating) Skills" up to a certain amount, because it is our opinion that even in times of a CAS some manipulating skills should remain, because otherwise the pupils cannot develop a certain "mathematic feeling" for solving problems supported by a powerful machine and they will too much work by "try and error".

We can provide the students with programs to train manipulation skills and we made the experience that these training programs were highly appreciated by the students and by the colleagues.

This is a screenshot of a DERIVE program to train factoring expressions and polynomials:

$$qb(3) = \begin{bmatrix} 200 \cdot f^4 \cdot o^3 \cdot w^2 + 400 \cdot f^3 \cdot o^3 \cdot w^3 + 200 \cdot f^2 \cdot o^3 \cdot w^4 \\ -128 \cdot o^6 \cdot r^3 \cdot v - 128 \cdot o^4 \cdot r^3 \cdot v^4 - 32 \cdot o^2 \cdot r^3 \cdot v^7 \\ -540 \cdot c^7 \cdot p \cdot t^3 + 1080 \cdot c^5 \cdot p \cdot t^6 - 540 \cdot c^3 \cdot p \cdot t^9 \end{bmatrix} \qquad ch = \begin{bmatrix} 200 \cdot f^2 \cdot o^3 \cdot w^2 \cdot (f + w)^2 \\ -32 \cdot o^2 \cdot r^3 \cdot v \cdot (2 \cdot o^2 + v^3)^2 \\ -540 \cdot c^3 \cdot p \cdot t^3 \cdot (c^2 - t^3)^2 \end{bmatrix}$$

The students are given randomly created expressions. They have to factor them using paper and pencil. ch = presents the correct answers. (Here for three problems, but the number of problems can be varied).

Factoring random polynomials offers three options: According to the math level you can ask for factoring for rational, irrational or complex roots:

$$\mathrm{polb}(5) = \begin{bmatrix} -3\cdot x^5 - 9\cdot x^4 + 54\cdot x^3 \\ 4\cdot x^4 - 24\cdot x^3 - 36\cdot x^2 + 496\cdot x \\ 49\cdot x^2 + \frac{42\cdot x}{5} + \frac{9}{25} \\ -9\cdot x^2 - 18\cdot x - 18 \\ 4\cdot x^3 - 14\cdot x^2 - 20\cdot x + 48 \end{bmatrix} \qquad \mathrm{ch} = \begin{bmatrix} 3\cdot x^3\cdot(3 - x)\cdot(x + 6) \\ 4\cdot x\cdot(x + 4)\cdot(x^2 - 10\cdot x + 31) \\ \frac{(35\cdot x + 3)^2}{25} \\ -9\cdot(x^2 + 2\cdot x + 2) \\ 2\cdot(x + 2)\cdot(x - 4)\cdot(2\cdot x - 3) \end{bmatrix}$$

$$\mathrm{chw} = \begin{bmatrix} 3\cdot x^3\cdot(3 - x)\cdot(x + 6) \\ 4\cdot x\cdot(x + 4)\cdot(x^2 - 10\cdot x + 31) \\ \frac{(35\cdot x + 3)^2}{25} \\ -9\cdot(x^2 + 2\cdot x + 2) \\ 4\cdot(x + 2)\cdot(x - 4)\cdot\left(x - \frac{3}{2}\right) \end{bmatrix} \qquad \mathrm{chk} = \begin{bmatrix} 3\cdot x^3\cdot(3 - x)\cdot(x + 6) \\ 4\cdot x\cdot(x + 4)\cdot(x - 5 + \sqrt{6}\cdot i)\cdot(x - 5 - \sqrt{6}\cdot i) \\ \frac{(35\cdot x + 3)^2}{25} \\ -9\cdot(x + 1 + i)\cdot(x + 1 - i) \\ 4\cdot(x + 2)\cdot(x - 4)\cdot\left(x - \frac{3}{2}\right) \end{bmatrix}$$

The next screen shows how to train expanding expressions. Squaring and cubing binomials and calculating other products can be done – and immediately by checked, accompanied by a kind of error analysis.

#1: $\mathrm{qu} = (4\cdot x - 3\cdot u)^2$

#2: $\mathrm{ch}(16\cdot x^2 + 9\cdot u^2)$ = Doppeltes Produkt prüfen!

#3: $\mathrm{ch}(16\cdot x^2 - 24\cdot x\cdot u + 9\cdot u^2)$ = R I C H T I G!

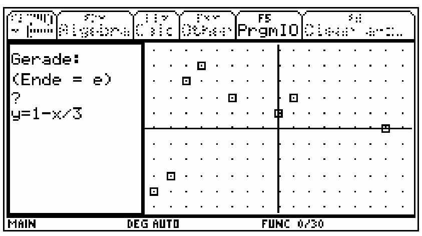
#4: $\mathrm{ku} = (x + 10\cdot y)^3$

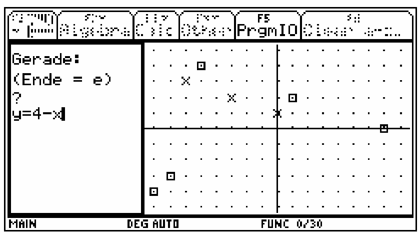
#5: $\mathrm{ch}(x^3 + 30\cdot x^2\cdot y + 300\cdot x\cdot y^2 + 100\cdot y^3)$ = Kuben prüfen!

#6: $\mathrm{ch}(x^3 + 30\cdot x^2\cdot y + 300\cdot x\cdot y^2 + 1000\cdot y^3)$ = R I C H T I G!

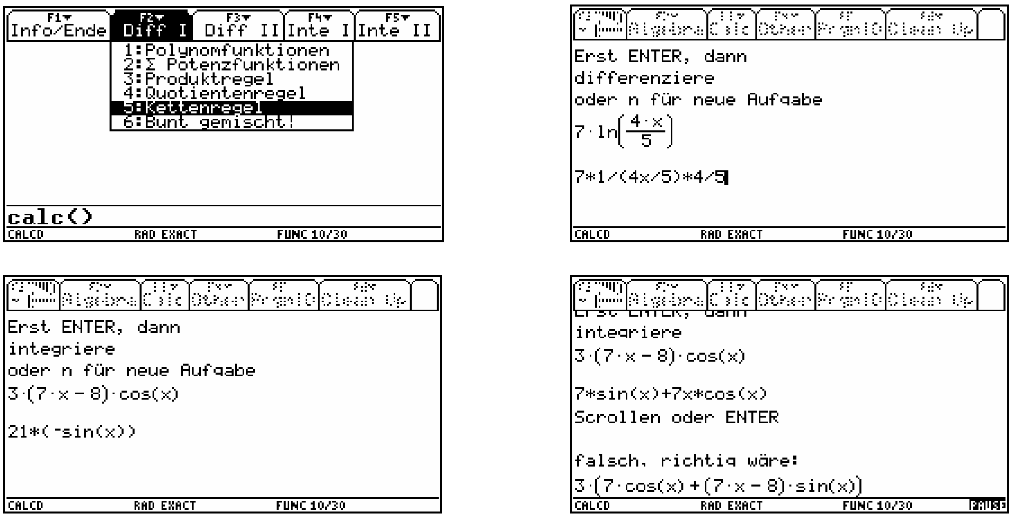
Similar program packages exist for the CAS-TI-devices

With "**Catch the Points!**" I am offering a „shooting game“. Randomly generated grid points are presented. The task is to hit all points with as less straight lines as possible. Later we can repeat the game using parabolas or other function types.





I´d like to underline that this is not the main purpose to use CAS, but sometimes a useful side effect.



Training of skills in differentiating and integrating (randomly generated problems of different levels of complexity). The first row shows an example for applying the chain rule (level I) which was answered correctly, the second row shows a problem for applying integration by parts (level I). After two false answers the system gives the correct solution.

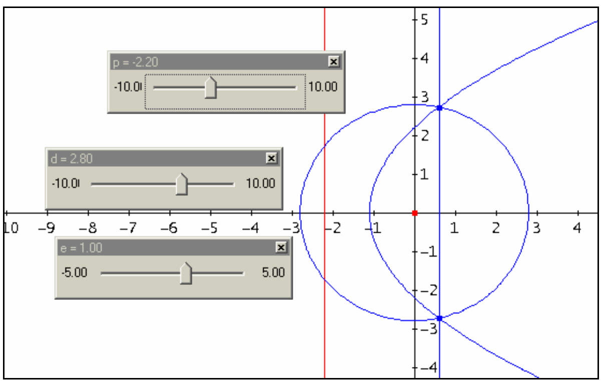
We know from our experience that students really appreciate tools like these and they practise much more compared doing exercises from textbooks. They feel responsible for their success.

6 CAS and Dynamic Geometry

Fortunately geometry is becoming more and more important in our schools again. I am sure that the availability of excellent dynamic geometry programs is responsible for this revival. (Cabri, Euklid, Geolog and especially GeoGebra in recent times). What we are still missing is a dynamic geometry program which is in a tight connection with CAS. Working with slider bars in DERIVE helps in a surprising way to narrow this gap between algebraic and geometric representation form and can easily be done in classroom. Here is one example:

It is well known that the parabola is defined as the locus of all points which have the same distances from a fixed point and a fixed straight line. *We generalize and ask for the locus of all points with a constant ratio of these distances.*

The figure shows the usual definition of a parabola with the ratio mentioned above $e = 1$. p is the variable distance between the fixed point M and the fixed line l and d is the variable radius of the circle with its center in the fixed point M. Which is the locus of points X with $e = \text{MX} : l\text{X} = \text{const}$ (= 1 in the figure).



We work in the Algebra Window with variables and each of the variables is represented by a slider bar in the 2D-Plot Window. Each step in the "Calculation Area" is followed by the parallel step in the "Plot Area". So we have a form of "Integrated Circuit" between abstract algebraic calculation and realistic geometric visualisation.

This is a part of the calculation performed in the Algebra Window:

We try to find the explicit form the locus.
Obviously the curve consists of two branches -
and we start proceeding with the first branch (the first row of #7).

#8: $\left[d + p,\ \sqrt{d^2\cdot(e^2 - 1) - 2\cdot d\cdot p - p^2}\right]$

Parameter d must be eliminated from x = d+p and y = √(.....
This can be done step by step as doing it with PaP, substituting in the 2nd coordinate for d the expression x – p. But with a trick we leave the whole work - even in much more complicated cases the work for DERIVE:

#9: SOLUTIONS$\left(x = d + p \wedge y = \sqrt{d^2\cdot(e^2 - 1) - 2\cdot d\cdot p - p^2},\ [y,\ d]\right)$

#10: $\left[\left[\sqrt{x^2\cdot(e^2 - 1) - 2\cdot e^2\cdot p\cdot x + e^2\cdot p^2},\ x - p\right]\right]$

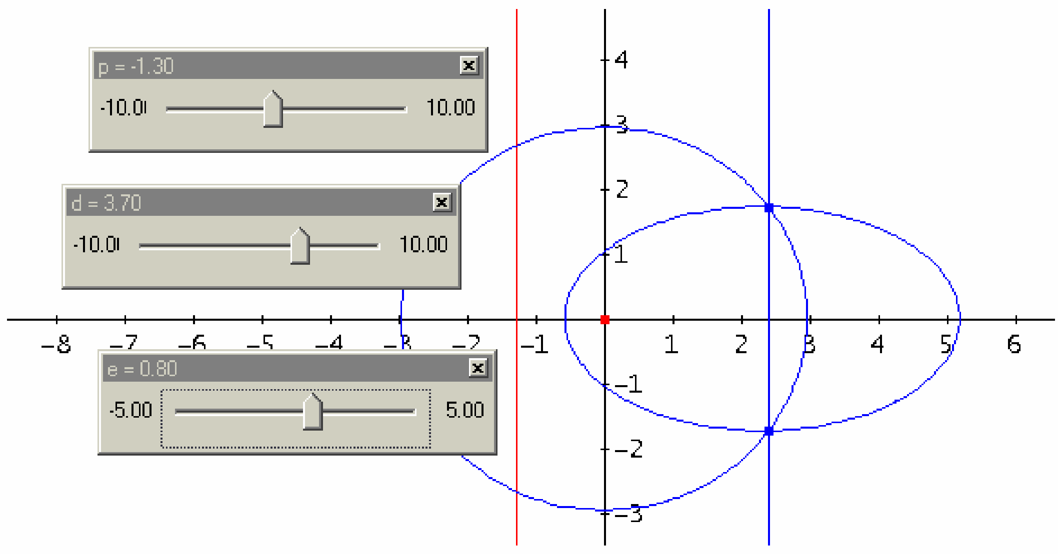
The first component is the requested solution for y (without parameter d!!).
The plot shows the upper branch of the locus.

#11: $y = \sqrt{x^2\cdot(e^2 - 1) - 2\cdot e^2\cdot p\cdot x + e^2\cdot p^2}$

Squaring #11 results in the locus in implicit form:

#12: $y^2 = x^2\cdot(e^2 - 1) - 2\cdot e^2\cdot p\cdot x + e^2\cdot p^2$

Which locus do you find now for e < 1 and e > 1??



And the result is surprising for all students (and for many teachers, too). We obtain one single formula for all conics. (the Apollonian property of the conics). Many other examples (easier and much more complicated) can be given. Changing the value for *e* we can obtain all forms of conics.

7 CAS and In-/Pre-Service Trainings

As I mentioned earlier the ACDCA was and still is one of the driving forces propagating CAS in classroom. ACDCA is represented by members in all federal states. They have good connections to the Pedagogical Institutes in their countries. Because of financial restrictions things have changed a bit. In some countries the colleagues organize individual courses in the frame of their Working Groups. Things will change much more in the future because of establishing the Pedagogical Universities and Mathematical Competence Centres. We don´t know at the moment where the way will lead. It is an exciting time for all of us.

Same happened with the Pre-Service Courses. We had "Technology Courses" for our young teachers in all federal states (Half day introduction for each: didactical use of Spreadsheet, Dynamic Geometry, Graphing Calculator, CAS) supported by T^3. At the moment I do know about one course in Lower Austria (organized by the Pedagogical Institute) and one course in Tyrol (organized by Heiner Juen, P.I. Tyrol).

8 CAS and (Teacher-) Students

There are special lectures for teacher students on some universities (University of Vienna, Technical University of Vienna, University Graz, University Klagenfurt as far as I do know) which shall introduce working with technology in mathematics education – with a special impact on the change in the way of teaching. The students are presented meaningful use of Internet, spreadsheet programs, graphing calculators and CAS. In our lecture (TU Vienna) the students have to deliver a final paper in order to obtain a certificate. (Lecture is held together with Gaby Bleier.)

Just recently I received a final paper on Sequences and Series including investigating monotony and convergence. See what the student presented on the work sheet for pupils (among other examples):

> Let $b_n = 3 + \frac{1}{2n}$.
> Show that for each arbitrary small number ε there exists an index n_0 that $|b_n - 3| < \varepsilon$ for all $n \geq n_0$.

This was the expected solution given by the student: (first for ε = 0.001, then for ε = 0.0001 and finally for a generalized ε):

$$\#1: \quad b(n) := 3 + \frac{1}{2 \cdot n}$$

$$\#2: \quad n :\in \text{Integer } (0, \infty)$$

$$\#3: \quad \text{SOLVE}\left(|b(n) - 3| < \frac{1}{1000}, n\right) = (n < -500 \vee n > 500)$$

The student used CAS only as a calculation tool – but she added a verbal answer. This might be appropriate in many cases. But CAS offers much more possibilities to do mathematics. Without a CAS the pupils had to solve the inequality and I know from my experience that this is hard enough for many of them (considering the absolute value, considering equivalence transformations in inequalities, …)

DERIVE offers the unique and unfortunately widely unknown and underestimated "Stepwise Simplification Tool". Why not ask the students (the classroom students) to simplify the solution process stepwise and to explain each performed step. (Which rule applies? What is its purpose? What is the effect? …). Colleagues using this tool in the right situation are very excited about the enrichment of teaching. (We include it in our technology-oriented textbook, too.)

Using the "Steps" we can combine "Basic Skills", Verbalising, Interpretation and other objectives of math education in a wonderful way. Let´s follow the steps together with our students in the classroom and let´s give comments on the rules which are lying behind the simplification process:

Using the "Stepwise Simplification Tools":

#4: $\text{SOLVE}\left(|b(n) - 3| < \frac{1}{1000}, n\right)$

$|z \cdot w| \rightarrow |z| \cdot |w|$

#5: $\text{SOLVE}\left(\frac{\left|\frac{1}{n}\right|}{2} < \frac{1}{1000}, n, \text{Real}\right)$

$|x| \rightarrow x \cdot \text{SIGN}(x)$

#6: $\text{SOLVE}\left(\frac{1}{2 \cdot n \cdot \text{SIGN}(n)} < \frac{1}{1000}, n, \text{Real}\right)$

$\frac{1}{\text{SIGN}(x)} \rightarrow \text{SIGN}(x)$

#7: $\text{SOLVE}\left(\frac{\text{SIGN}(n)}{2 \cdot n} < \frac{1}{1000}, n, \text{Real}\right)$

If n>0,

$n \cdot x < y \rightarrow x < \frac{y}{n}$

$F(|x|) < y \rightarrow (x \ge 0 \wedge F(x) < y) \vee (x \le 0 \wedge F(-x) < y)$

#8: $\text{SOLVE}\left(\left(n \ge 0 \wedge \frac{1}{n} < \frac{1}{500}\right) \vee \left(n \le 0 \wedge -\frac{1}{2 \cdot n} < \frac{1}{1000}\right), n, \text{Real}\right)$

If y>0 and n>0 is odd,

$\frac{1}{x^n} < y \rightarrow x < 0 \vee x > y^{1/n}$

#9: $\text{SOLVE}\left((n \ge 0 \wedge (n < 0 \vee n > 500)) \vee \left(n \le 0 \wedge -\frac{1}{2 \cdot n} < \frac{1}{1000}\right), n, \text{Real}\right)$

If n>0,

$n \cdot x < y \rightarrow x < \frac{y}{n}$

If n>0,

#10: $\text{SOLVE}\left(n > 500 \vee \left(n \le 0 \wedge -\frac{1}{n} < \frac{1}{500}\right), n, \text{Real}\right)$

$-x < y \rightarrow x > -y$

#11: $\text{SOLVE}\left(n > 500 \vee \left(n \le 0 \wedge \frac{1}{n} > -\frac{1}{500}\right), n, \text{Real}\right)$

If y<0 and n>0 is odd,

$\frac{1}{x^n} > y \rightarrow x > 0 \vee x < -(-y)^{1/n}$

#12: $\text{SOLVE}(n > 500 \vee (n \le 0 \wedge (n > 0 \vee n < -500)), n, \text{Real})$

#13: $n < -500 \vee n > 500$

#14: $n > 500$

Then we can generalize:

$$\text{SOLVE}(|b(n) - 3| < \varepsilon,\ n)$$

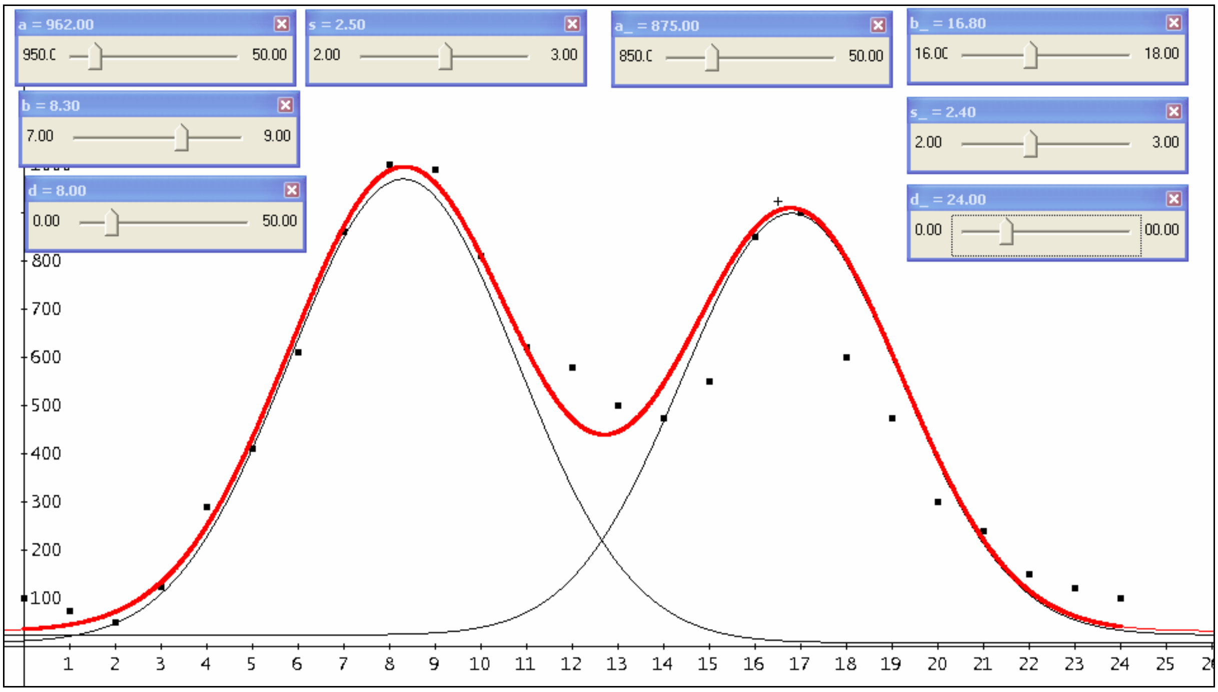
I am quite sure that even many (university) students and maybe also experienced teachers might face problems in verbalizing the single steps.

Extending stepwise simplification was a main task for developing future DERIVE releases but unfortunately by ending the support of DERIVE we cannot expect a wider range of stepwise simplifications. The (weak) hope remains, that other CAS will follow this example!

9 CAS and Teachers

CAS in the classroom is impossible without teachers who like to work with CAS and who hopefully are using CAS not only as calculation tool. Unfortunately there are also teachers – and as I heard just recently – not only in Austria who use CAS to work through the traditional textbooks example for example supported by powerful CAS-tools.

I admit that it is not easy to convince hesitating teachers to work with a CAS and I also admit that there are a lot of good reasons to hesitate changing well established traditional teaching methods. I also admit that teaching without CAS can be excellent math teaching and teaching with CAS can be very bad. Let me tell one nice experience made in an in-service course. We built the model of a traffic density measurement for one day. After some experiments we came up to superimpose two bell shaped functions and using the slider bars we found a satisfying model function (for further investigations – differentiate, integrate, …)

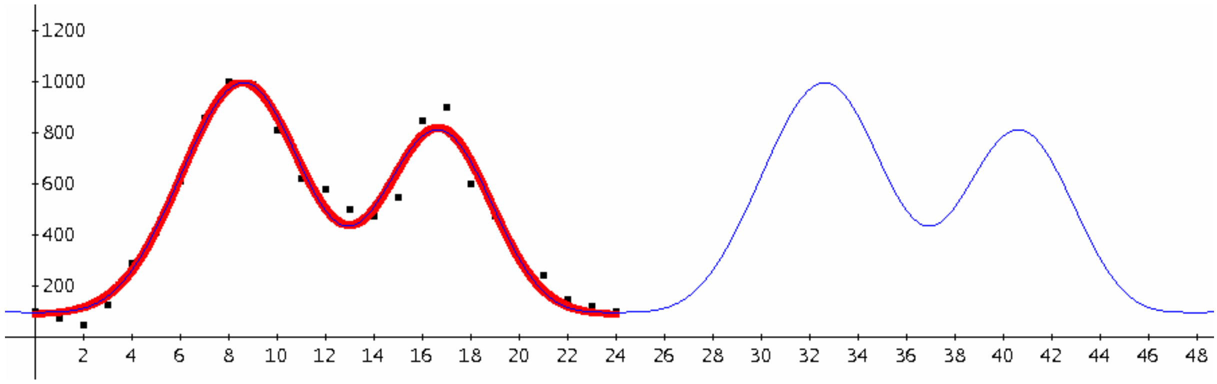


#13: $$\text{veh\_dens}(t) := 962\cdot e^{-1/2\cdot((x - 8.3)/2.5)^2} + 875\cdot e^{-1/2\cdot((x - 16.8)/2.4)^2} + 32$$

Finally we wanted to extend this daily model for more days. What to do? After a while somebody brought into discussion: ´”This is periodical! Trig functions?”. And after another while another colleague whispered (very uncertain) “Fourier?”.

But nobody could remember how to perform a Fourier analysis because this is not in the curriculum of the schools where the participants came from. So we opened the Online Help and searched for FOURIER.

This was the result after calling `FOURIER(veh_dens(t),t,0,24,4)`, which was obtained in a part of a second:



The next surprise was the bulky expression for the Fourier series (which I don´t want to copy!)

I want to tell the end of the story: One – very experienced and very traditional – teacher who had started the course with many reservations became more and more enthusiastic when he saw how we used CAS. When he saw the curve for the week he exclaimed: “That’s really great Josef, I am feeling like an apprentice of a sorcerer (Zauberlehrling)”. The same happens sometimes in classroom – maybe not with `FOURIER` but with other examples. Black Boxes can and should be used to make pupils (and teachers) curious about the contents of the Black Boxes.

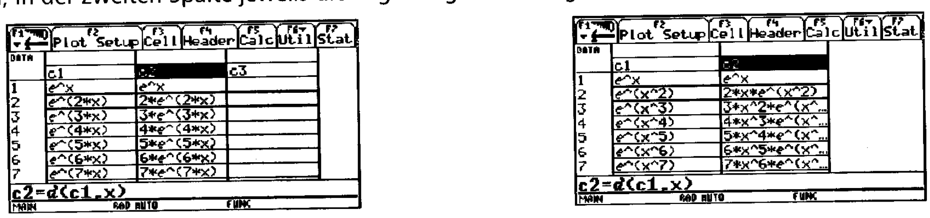
10 CAS and Textbooks

CAS in the classroom needs CAS in the textbooks. Many teachers complain that the textbooks don´t take the use of CAS into account in the right way. If the textbooks make use of CAS then in many cases developing the concepts and the problems don´t change, but there is an appendix – either at the end of the chapters or at the end of the textbook – how to use CAS. (Mostly DERIVE and/or the TI). There is need for textbooks which integrate CAS (and other technologies as well) to teach mathematics (the concepts), to encourage experimenting, arguing, working on open ended questions in addition to a sound base of traditional mathematics.

Three colleagues and I have been asked by a publisher to write a series of textbooks according to the curriculum (which demands the use of technology). Three volumes are ready, volume 4 will appear in fall 2007. The textbooks are for Secondary Colleges for Business Administration.

x_0 $x_0 \ln a$

Betrachte nun folgende zwei Beispiele, realisiert mit dem CAS-Rechner. In der ersten Spalte siehst du die Funktion, in der zweiten Spalte jeweils die zugehörige Ableitung:



Was fällt dir bei den Ergebnissen auf?

Die Argumente der natürlichen Exponentialfunktion sind Funktionen in *x*. An den Ergebnissen ist abzulesen, dass auch hier die Kettenregel angewendet werden muss. Dies gilt natürlich ebenso für die allgemeine Exponential- und für die Logarithmusfunktionen.

The figure shows how students can make up their conjectures how to find the derivative of $e^{f(x)}$.

The next figure demonstrates the solution of a finance problem using the various tools. Working with CAS we underline the use of self defined functions (here for present value, future value, etc.).

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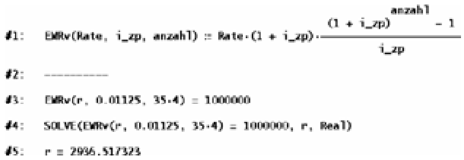
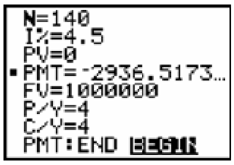
Beispiel: Wir haben geschätzt, dass die Ausbildung eines Akademikers den Staat ca. 1 Million Euro kostet und mit ca. 30 Jahren abgeschlossen ist. Wenn man den Pensionsantritt mit 65 Jahren ansetzt, wie hoch müssten die vorschüssigen vierteljährlichen Ratenzahlungen auf ein Bildungskonto sein, damit die Million bei Pensionsantritt zurückgezahlt werden kann ($i_4 = 4{,}5\,\%$)? Die € 1 000 000,00 bleiben dabei unverzinst; man hat gleichsam ein zinsenloses Darlehen vom Staat erhalten!

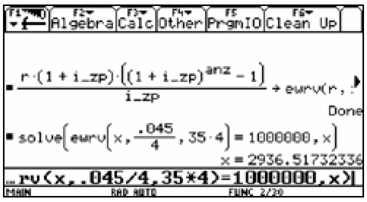
Lösung: Man hat nach seiner Ausbildung 35 Jahre Zeit, um das Geld anzusparen, also muss die Million (Endwert) mit $35 \cdot 4$ Einzahlungen bei einem relativen Zinssatz $i_4 = 1{,}125\,\%$ erreicht sein.



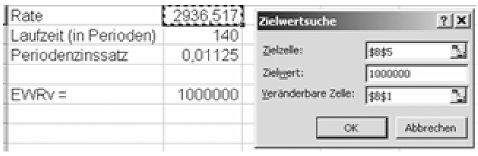
Berechnung der vorschüssigen Quartalsrente mit Derive



Rentenberechnung mit dem TVM-Solver am GTR

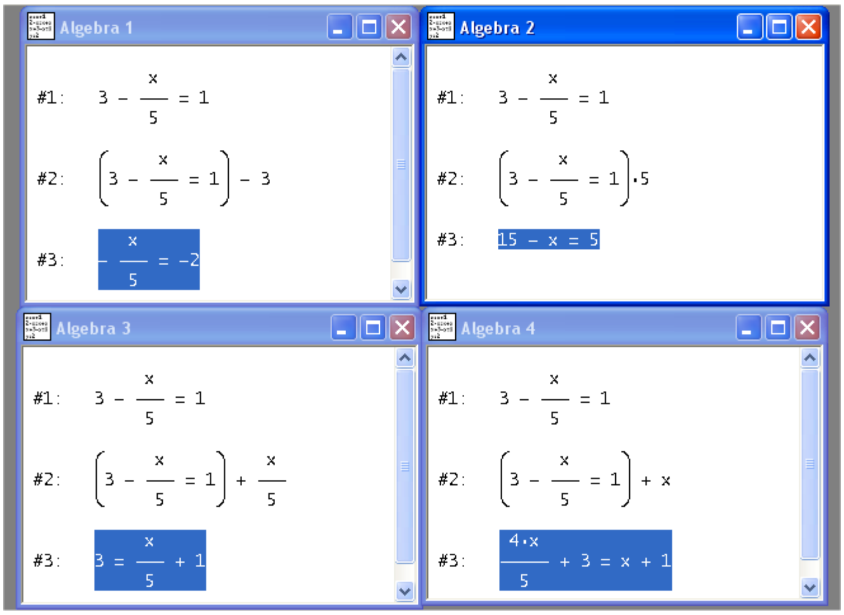


Dieselbe Berechnung am CAS-Rechner, diesmal ohne TVM-Solver

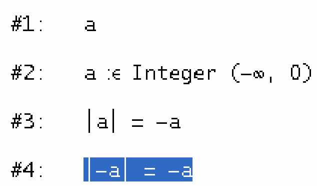


Ratenberechnung mithilfe der Zielwertsuche in Excel

11 CAS and a new "Teaching and Problem Culture"



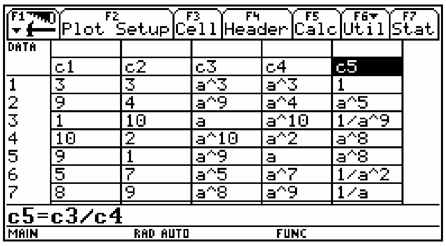
The above screenshot is a favourite procedure for many teachers using CAS in a very early stage: compare useful and not useful equivalence transformations of equations.



The right DERIVE-terms are one of (many) small examples to start mathematics discussions. An absolute value being negative?? Impossible!! Or not!! (Thanks to Walter Klinger for this nice problem.)

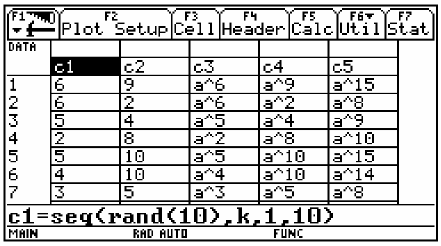
Many classroom examples from secondary I can be found in [1] where special emphasis is given to experimenting, conjecturing and visualizing.

The following screens are very similar to the figures from paragraph 10 but not from the field of Calculus. Using – and generating - random exponents the pupils should find out the rules for manipulating powers. We find it important that in many cases the pupils are able to create the tools by themselves and they do not depend on tools prepared by the teacher. Show them how to investigate multiplication and then they shall find out how to investigate the remaining rules …



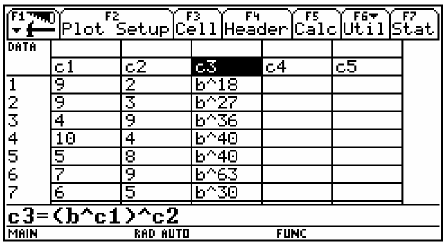
DATA	c1	c2	c3	c4	c5
1	3	3	a^3	a^3	1
2	9	4	a^9	a^4	a^5
3	1	10	a	a^10	1/a^9
4	10	2	a^10	a^2	a^8
5	9	1	a^9	a	a^8
6	5	7	a^5	a^7	1/a^2
7	8	9	a^8	a^9	1/a

c5=c3/c4



DATA	c1	c2	c3	c4	c5
1	6	9	a^6	a^9	a^15
2	6	2	a^6	a^2	a^8
3	5	4	a^5	a^4	a^9
4	2	8	a^2	a^8	a^10
5	5	10	a^5	a^10	a^15
6	4	10	a^4	a^10	a^14
7	3	5	a^3	a^5	a^8

c1=seq(rand(10),k,1,10)



DATA	c1	c2	c3	c4	c5
1	9	2	b^18		
2	9	3	b^27		
3	4	9	b^36		
4	10	4	b^40		
5	5	8	b^40		
6	7	9	b^63		
7	6	5	b^30		

c3=(b^c1)^c2

This is not the end of the story. The experimenting phase is followed by the exactifying phase which is the proof for the conjecture(s).

Some examples for possible "new" questions are:

- Rewrite $\frac{a^2b}{15c^3}$ in at least 4 different ways.
- Fill in the form:

$x^4 - 16x^3 + 96x^2 - ......$

$a^6b^3 + 6a^4b^5$ .... = (...........................)^3

For the next give more than one solution!!

.............$-180x^3z$.................... = (..........................)^2

$729x^{12} - 5832x^{10}y + 19440x^8y^2$ .... =

Find equations of lowest degrees to the given solution sets:

L = {-2, -3, -4, 0, 1}

L = $\{x_{1,2} = \frac{2}{3}, x_{3,4} = \frac{1}{a}\}$

L = {a + 2b, a – 2c, b – c}

L = $\{2 + 3\sqrt{5}, 2 - \sqrt{5}, -1, 2, 0\}$

L = $\{\frac{2}{3}, \frac{2}{5}, -\frac{1}{4}, -\frac{3}{5}\}$

Find equations of given degree to the presented solutions sets:

L = {1, -2}; degree 4

L = {a}; degree 3

L = {0, 2, 3} degree 2

L = {-5, 2} degree 3

I am very often asked how to find problems for CAS-supported teaching. My answer is:

Make New from Old

The original version reads:

All graphs of a family of functions $f_k(x)$ of degree 4 which are symmetric with respect to the y-axis intersect the graphs of another family of curves $g_k(x) = \frac{1}{16k}x - \frac{1}{32k}x^2$ orthogonal in P(2/0) and also have the origin in common.

a) Show that $f_k(x) = k\,x^4 - 4k\,x^2$.
b) Sketch the graphs of both functions for $k = 1/4$ including the zeros, extremal values and inflection points.
c) The graphs of f_k and g_k form an area for $k \geq 0$. Show that this area is given by $A(k) = \frac{1}{16k}x + \frac{64k}{15}$.
d) For which positive k will we receive the minimum area A?

This is my proposal for a new version:

Given is a family of funktions $g_k(x) = \frac{1}{16k}x - \frac{1}{32k}x^2$.

a) Which is the form of all curves of the family?
b) What are the common properties of all graphs? Give reasons!
c) What is the influence of parameter k on form and position of the graphs. Present your findings using an appropriate survey.
d) Find a "partner family" $f_k(x)$ so that each g_k is intersecting the corresponding f_k orthogonally.
e) Show for any k that condition d) is fulfilled.
f) What is the common area enclosed by two "partners"? Shade this area on your device for any appropriate k. Explain how you can achieve this.
g) Which value of k makes this area extremal? Is it a Maximum or a Minimum?

Compare the worked through traditional “edition” (next page) with the problems set in the new version. The traditional version is a problem from a final exam from the 1980’s.

If you compare the two versions carefully you will find out that the new one does without asking for the quartic. But there are some new questions and problems from a) to c) which emphasize visualization, verbalisation, organisation of working.

Question f) addresses the Tool Competence.

Problem g) is not so easy because the 2^{nd} derivative test gives a ”wrong” answer. One has to take the “sign” of the area into account to give the right interpretation.

A4) $y = \frac{-x^4+2ax^2+3a^2}{4a}$

Symmetrisch zur y-Achse 1

Nullstellen: $0 = -x^4+2ax^2+3a^2$

$x^2 = \frac{-2a \pm \sqrt{16a^2}}{-2} = 3a, -a$

$x_{1,2} = \pm\sqrt{3a}$ $\quad N_{1,2}(\pm\sqrt{3a}|0)$ 2

Extremwerte:

$y' = \frac{-4x^3+4ax}{4a} = \frac{-x^3+ax}{a}$ $\quad 0 = -x^3+ax = x(-x^2+a)$ $\quad x_1 = 0$ $\quad x_{2,3} = \pm\sqrt{a}$

$E_1(0|\frac{3a}{4})$; $E_{2,3}(\pm\sqrt{a}|a)$ 2

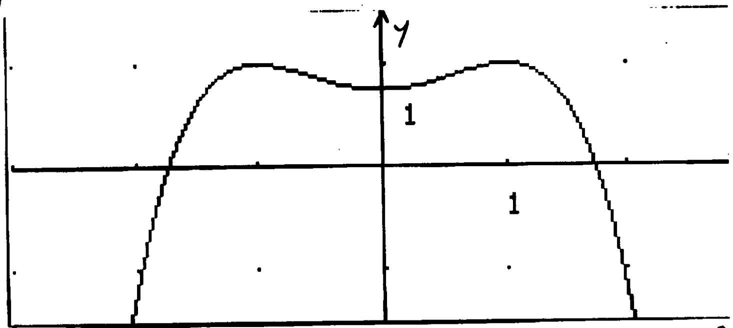
Wendepunkte:

$y'' = \frac{-12x^2+4a}{4a} = \frac{-3x^2+a}{a}$ $\quad 0 = -3x^2+a$ $\quad x_{1,2} = \pm\sqrt{\frac{a}{3}}$

$W_{1,2}(\pm\sqrt{\frac{a}{3}}|\frac{8a}{9})$ 2

$a = 1$: $N_{1,2}(\pm\sqrt{3}|0)$
$E_1(0|\frac{3}{4})$; $E_{2,3}(1|1)$ (7)
$W_{1,2}(\pm\sqrt{\frac{1}{3}}|\frac{8}{9})$

$y = \frac{-x^4+2x^2+3}{4}$



$A = 2\int_0^{\sqrt{3a}} \frac{-x^4+2ax^2+3a^2}{4a}\,dx = \frac{1}{2a}\left| -\frac{x^5}{5}+\frac{2ax^3}{3}+3a^2x \right| = \frac{1}{2a}\left(-\frac{(\sqrt{3a})^5}{5}+\frac{2}{3}a(\sqrt{3a})^3 + 3a^2\sqrt{3a}\right)$

$= \frac{1}{2a}\left(-\frac{9a^2\sqrt{3a}}{5}+\frac{2a\cdot 3a\sqrt{3a}}{3}+3a^2\sqrt{3a}\right) = \frac{1}{2}\cdot\frac{16a}{5}\sqrt{3a}$

$A(a) = \frac{8a\cdot\sqrt{3a}}{5} = ~~100~~ 25$

$a\sqrt{3a} = \frac{125}{2} \quad |^2$

$3a^3 = \left(\frac{125}{2}\right)^2 \rightarrow a = \sqrt[3]{\frac{125^2}{12}}$

$\approx 10{,}92$ (7)

$\frac{-x^4+2ax^2+3a^2}{4a} = \frac{-x^4+2bx^2+3b^2}{4b}$

$-x^4b+2abx^2+3a^2b = -x^4a+2abx^2+3ab^2$

$x^4~~(a-b)~~ = -3ab~~(a-b)~~$

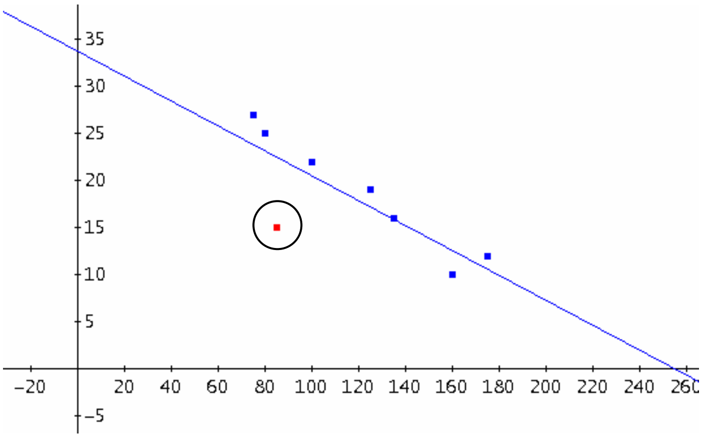
$x^4 = -3ab < 0 \quad L = \{\ \}$ (4)

My personal recipe for designing “new” problems (which is often “reanimating” old problems):

- Include the graphing capabilities for additional tasks - forcing visualization.
- Encourage and ask for numerical and graphical solutions, ask for more than one way of solution, accept heuristic methods, encourage and ask for "Try and Error Methods".
- Let the pupils produce self made tools and give opportunities to apply them.
- Provoke functional thinking.
- Find the problem to presented solutions.
- Let the pupils set up conjectures - but also ask sometimes for the proofs.
- Include the various ways of the CAS´ output into the task, interpret the output and compare with calculation by hands, make the CAS built-in simplification rules to a mathematical subject.
- Use the CAS-capabilities or deficiencies for additional questions (and keep staying within the curriculum!!)

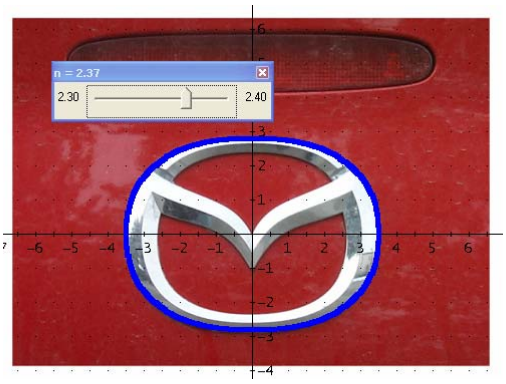
Another example:

Working with Linear Regression is standard – but is this question also standard??



We can find the regression line for the data points (using CAS).
We pick out one point – the red one (≈ (85,15). Moving this point in the plane results in different regression lines. (This can be demonstrated by the use of slider bars). Some of the lines are parallel to the first one. What is the locus of all (red) points which lead to regression lines with the same regression coefficient? Do you know?

Or take this: *Bring your environment into the math classroom.*



This is the MAZDA-Logo. Can you describe the form of this logo.

We use the slider bar for the exponent n:

$$\#5: \quad \left|\frac{x}{3.49}\right|^n + \left|\frac{y}{2.8}\right|^n = 1$$

$n = 2.35$. (It is a Lamé-Curve, Hyper-ellipse).

Following Group Work: Find and model other logos.

As I should focus on the use of CAS in our classrooms all other technologies have been pushed in the background. We know that especially modelling can be done with numerical tools (spreadsheet) in an excellent way, but it seems to be useful to do this also with CAS and to compare the modelling process.

See one example from Business Mathematics:

Supply- and Demand functions ($a(p)$ and $n(p)$) are given (both dependent on price $p(t)$):

$$a(p) = 0,7p^2 + p + 0,3 \text{ and } n(p) = 2,5 - 0,2p - 0,15p^2; \quad p = p(t); \quad p(t=0) = 1,00$$

The rate of change for price p is given by the following differential equation (dependent on time t):

$$p'(t) = k \cdot (n(t) - a(t))$$

Factor of proportionality $k = 1$ and the time interval $\Delta t = 0.1$.

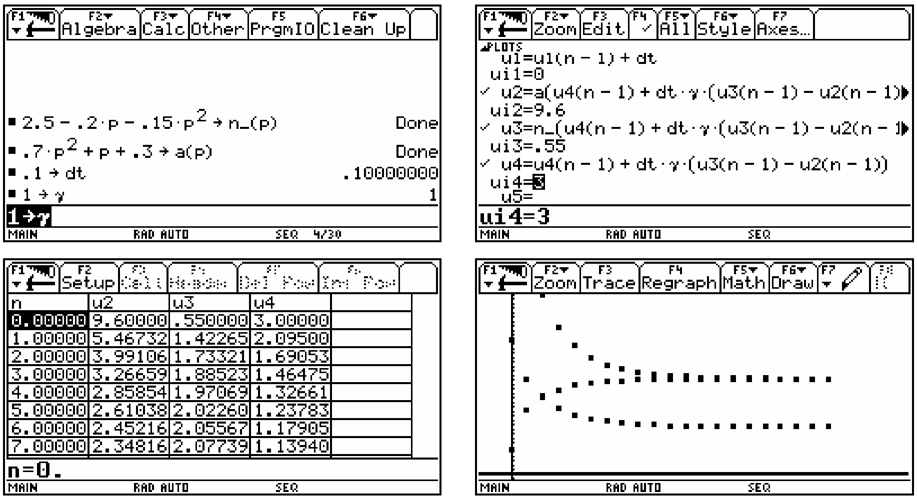
This is the respective difference equation (which can be treated with Excel):

$$\frac{\Delta p}{\Delta t} = k \cdot (n(t) - a(t)) \rightarrow \Delta p = \Delta t \cdot k \cdot (n(t) - a(t))$$

$$p(t + \Delta t) = p(t) + \Delta t \cdot k \cdot (n(t) - a(t))$$

$$p_{new} = p_{old} + \Delta t \cdot k \cdot (n(p_{old}) - a(p_{old}))$$

The difference equation on the CAS-TI (Sequence model)

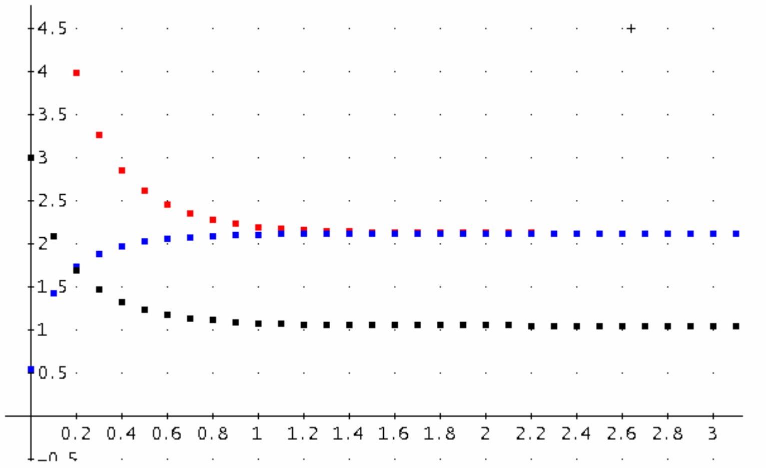


The same with DERIVE (Iterations)

```
#2:  [a(p_) := 0.7·p_^2 + p_ + 0.3, n(p_) := 2.5 - 0.2·p_ - 0.15·p_^2]

#3:  elemmarkt(p0, γ, Δt, n_) := ITERATES([v_1 + Δt, a(v_4 + Δt·γ·(v_3 - v_2)), n(v_4 + Δt·γ·(v_3 - v_2)), v_4 + Δt·γ·(v_3 - v_2)],
       v, [0, a(p0), n(p0), p0], n_)

#4:  elemmarkt(3, 1, 0.1, 20)

     [ 0     9.6           0.55          3
       0.1   5.4673175     1.42264625    2.095
       0.2   3.991063856   1.733208214   1.690532875
       0.3   3.266586590   1.885227835   1.464747310
       0.4   2.858539965   1.970693027   1.326611435
```

- Possible question: which effect has a change of k?

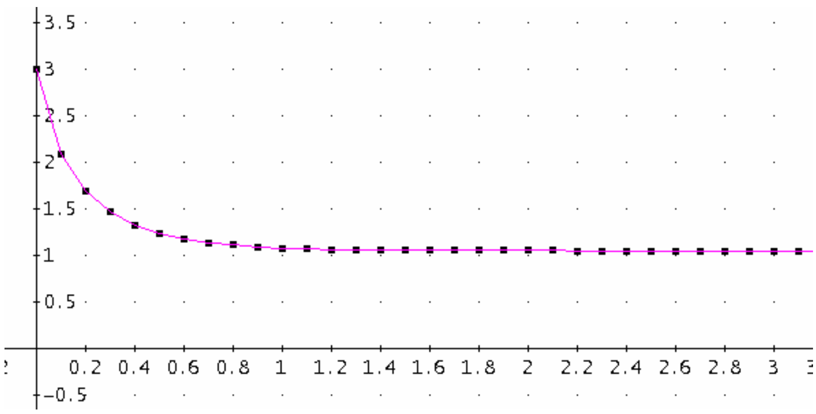
```
#6:   (elemmarkt(3, 1, 0.1, 50))↓↓[1, 2]
#7:   (elemmarkt(3, 1, 0.1, 50))↓↓[1, 3]
#8:   (elemmarkt(3, 1, 0.1, 50))↓↓[1, 4]
```



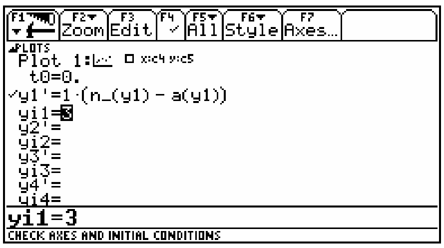
Solving the differential equation with a Black Box:

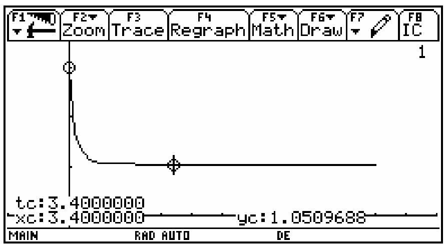
```
#10:   EULER_ODE(1·(n(p_) - a(p_)), t, p_, 0, 3, 0.1, 50)
```

The discrete points from before together with the numerical solution of the DE:

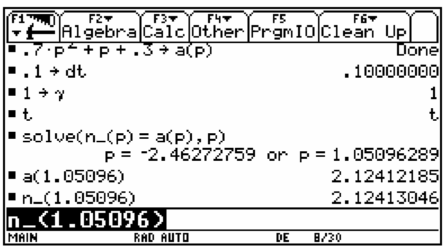


The differential equation on the TI:





Calculation of the equilibrium point:



A final example from just recently (source: my colleague and CAS-fighter Tania Koller, HAK St. Pölten)

Tania and her colleague found a strange (very traditional) example in a textbook:

Find a cubic with a turning point in (-3, 2) and an inflection point for $x = -1$ which passes $P(3,7)$.

The respective system of equations had – surprisingly enough- no solution:

```
SOLVE([f(-3)=2, f'(-3)=0, f(3)=7, f''(-1)=0], [a,b,c,d])=[]
```

What happened? It was a typing error. But fortunately enough, the teachers and the "CAS-infected" students were not satisfied with this answer. They changed one or the other value and then the system had a solution and there was a resulting cubic. Then they generalized the problem:

```
SOLVE([f(-3) = 2, f'(-3) = 0, f(3) = 7, f''(t) = 0], [a, b, c, d])
```

$$\left[a = -\frac{5}{108\cdot(t+1)} \wedge b = \frac{5\cdot t}{36\cdot(t+1)} \wedge c = \frac{5}{12\cdot(t+1)} + \frac{5}{6} \wedge d = \frac{13\cdot t + 18}{4\cdot(t+1)}\right]$$

and they found out, that exactly $t = -1$ causes the problems. Working with the slider bar supported their results visually.

Having a CAS at their disposal they could go further. Why is –1 the bad boy/girl? Go on with generalizing:

```
SOLVE(f(xe) = 1 ∧ f'(xe) = 0 ∧ f(xp) = yp ∧ f''(xw) = 0, [a, b, c, d])
```

$$a = \frac{yp - 1}{2\cdot xe^3 - 3\cdot xe^2\cdot(xp + xw) + 6\cdot xe\cdot xp\cdot xw + xp^2\cdot(xp - 3\cdot xw)} \wedge b =$$

$$\frac{3\cdot xw\cdot(1 - yp)}{2\cdot xe^3 - 3\cdot xe^2\cdot(xp + xw) + 6\cdot xe\cdot xp\cdot xw + xp^2\cdot(xp - 3\cdot xw)} \wedge c =$$

$$\frac{3\cdot xe\cdot(1 - yp)\cdot(xe - 2\cdot xw)}{2\cdot xe^3 - 3\cdot xe^2\cdot(xp + xw) + 6\cdot xe\cdot xp\cdot xw + xp^2\cdot(xp - 3\cdot xw)} \wedge d =$$

$$\frac{2\cdot xe^3\cdot yp - 3\cdot xe^2\cdot(xp + xw\cdot yp) + 6\cdot xe\cdot xp\cdot xw + xp^2\cdot(xp - 3\cdot xw)}{2\cdot xe^3 - 3\cdot xe^2\cdot(xp + xw) + 6\cdot xe\cdot xp\cdot xw + xp^2\cdot(xp - 3\cdot xw)}$$

The same denominator appears!

$$\text{FACTOR}(2\cdot xe^3 - 3\cdot xe^2\cdot(xp + xw) + 6\cdot xe\cdot xp\cdot xw + xp^2\cdot(xp - 3\cdot xw))$$

$$(xe - xp)^2\cdot(2\cdot xe + xp - 3\cdot xw)$$

Factoring the denominator and leaving the trivial case $xe \neq xp$, we find that $xp \neq 3xw - 2xe$.
So we can design many tasks of the same kind which "don´t work". Try to do this without a CAS!!

What did I say, some pages earlier: "Make New from Old!"

12 CAS and Visions

We all have visions for a PeCAS (Pedagogical CAS) or EduCAS (Educational CAS). I´d like to finish this paper offering some ideas which I have collected in several talks with Austrian colleagues:

Main Visions are:

- to make a CAS very intuitive to use. Teachers and students should not waste time in getting lost by considering technical details.
- to develop a CAS-standard language (syntax) for a base set of commands (solve, simplify, plot, differentiate, integrate, …) together with at least partial compatibility between PC and handheld versions.
- to integrate the various technologies which are important for math education: CAS and Dynamic Geometry, Spreadsheet, Statistics Package, Probes, Text processing, Programming, CAD, Portability into web formats.
- to develop and to find new possibilities for teaching and learning mathematics which are based on the power of a CAS. One excellent example for this is in our opinion the *Stepwise Simplification* tool of DERIVE.
- to find some agreement about basic knowledge, basic skills, basic concepts. Which amount is indispensable without CAS? See more in [2].
- CAS should be used in the sense of a tool. Tools can and shall support and increase mathematics activities and insights.

13 References

This is a selection of papers. You can find many additional materials on the ACDCA-website (http://www.acdca.ac.at) and many books on the website of bk teachware. (http://shop.bk-teachware.com)

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Private communication with Gaby Bleier, Peter Hofbauer, Heiner Juen, Walter Klinger, Tania Koller, Josef Lechner, Walter Wegscheider, Otto Wurnig, and others.

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