This nice exploration can be done using either DERIVE or the TI-family. This is the DERIVE-worksheet. At the end of this paper you will find how to adapt the nice part producing fractal patterns on the TI-screen. JB

Exploring the **Binomial Theorem** (Josef Böhm)

We want to calculate - to expand - higher powers of binomials, like

$\left(4x^2y - 3xy^3\right)^8 = ?$ How can we do this with DERIVE?

Edit the expression, then press **E** for **Expand** and the ENTER-key, because we don´t need any special expansion.

The result is impressive! What is the calculation time? . . . .

If you press the Ctrl-key together with the → - key you are able to shift the expression so that you can see the full result. the ← - key (+ Ctrl) will bring you back again.

Write down here the 1st term of the expression: ....

the 4th one: .... and the last one: ....

Let now DERIVE expand the powers of (a + b) and note only the coefficients of the terms. (Write down the numbers neatly one beneath the other):

n = 4: $(a + b)^4$:

n = 5: $(a + b)^5$:

n = 6: $(a + b)^6$:

n = 7: $(a + b)^7$:

If you believe to recognize a system, then add two lines more (for n = 8 and 9). Check the numbers with DERIVE. If you don´t see a system, then don´t worry, let DERIVE do the work.

Complete the scheme of numbers upwards (n = 3, 2, 1)

How should the prime line (n = 0) look like? ....

Which conclusion can be drawn from this line? Check it!

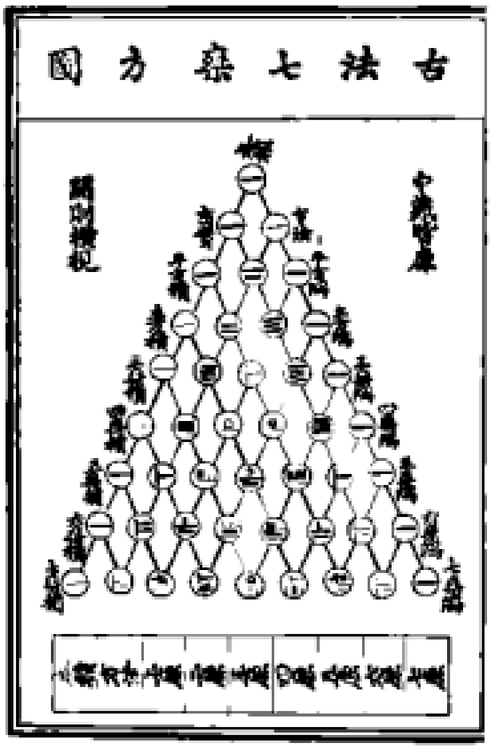
....

Write down the numbers from the coefficients´ scheme in form of a triangle:

(n = 0)	1
(n = 1)	1 1
(n = 2)	1 2 1
(n = 3)	
(n = 4)	
(n = 5)	
(n = 6)	
(n = 7)	
(n = 8)	
(n = 9)	
(n = 10)	
(n = 11)	

Can you find now (or again) an obvious scheme?

We call this triangle of numbers **Pascal´s Triangle** (Blaise Pascal, 1623-1662; but this triangle is found in a Chinese paper from 1303 a.c.)



The other picture is from an old 16th century German book for merchants.

(Both pictures from the - highly recommended - book AGNESI to ZERO, Key Curriculum Press)

Form the sum of the numbers in each line!

What can you notice? ....

Are you able to proof this facts? (A hint: substitute for a and b!)

....

Observe the exponents of a and b in the partial expressions of the expansions of the first five powers of (a + b).

Try to generate $(a + b)^8$ using PASCAL´s Triangle. >Then check your result with DERIVE.

$(a + b)^8$ = ...

Try your experience at $(2x + 3y)^5$:

$(2x + 3y)^5$ = ...

Expand all the partial expressions:

....

Check again with DERIVE:

Expand $(a + b)^5$, then we replace (substitute) in the result for a and b the values 2x and 3y in the following way:

Highlight the result of $(a + b)^5$, then perform a **substitution.** DERIVE will ask you now to substitute for all variables which can be found in the highlighted expression. If you don´t want to substitute for any variable then press ENTER, in the other case you substitute the offered variable by your special quantity. Here for a: 2x and for b: 3y. Compare the result offered by DERIVE with yours one! Expanding this will lead to a final result. You can let expand $(2x + 3y)^5$ directly without intermediate substitution.

Try now $(4z + 2u)^6$ = ... =

= ....

and compare with the CAS-result!

Which is the effect of the changed sign in $(a - b)^n$? Express your findings in your own words:...

Exercises: Calculate without the machine and then check your results:

$(z + 4)^6$ = ... =

= ....

$(4x^2y - 3xy^3)^8$ = ...

= ....

$(5ab + 2\,b^2)^9$ the 3rd element: ....; the 7th element: ....

$(2u + 1{,}5a)^5$ = ...

Find a way to expand $(3\,a^2 - 2\,a\,b + b^2)^5$?

....

The rule or law for calculating the powers of binomials is called

the **Binomial Law**.

The numbers appearing in Pascal´s triangle and there special order have fascinated mathematicians since very long - and they still do. Use the next two sheets to experiment with the numbers in this famous triangle.

Mark all even numbers using a felt pen. Do you observe a special pattern??

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

1 17 136 680 2380 6188 12376 19448 24310 24310 19448 12376 6188 2380 680 136 17 1

1 18 153 816 3060 8568 18564 31824 43758 48620 43758 31824 18564 8568 3060 816 153 18 1

1 19 171 969 3876 11628 27132 50388 75582 92378 92378 75582 50388 27132 11628 3876 969 171 19 1

1 20 190 1140 4845 15504 38760 77520 125970 167960 184756 167960 125970 77520 38760 15504 4845 1140 190 20 1

Now mark all numbers which are divisible by 3! You will find another pattern.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

1 17 136 680 2380 6188 12376 19448 24310 24310 19448 12376 6188 2380 680 136 17 1

1 18 153 816 3060 8568 18564 31824 43758 48620 43758 31824 18564 8568 3060 816 153 18 1

1 19 171 969 3876 11628 27132 50388 75582 92378 92378 75582 50388 27132 11628 3876 969 171 19 1

1 20 190 1140 4845 15504 38760 77520 125970 167960 184756 167960 125970 77520 38760 15504 4845 1140 190 20 1

We can recognize the socalled "self similarity", a property which is investigated in the very modern "Chaostheory".

Supported by *DERIVE* you easily can produce various "pictures" emerging from Pascal´s Triangle.

Start a new *DERIVE* session and then load the utility file PASCAL.MTH in the background.

```
pas_div(n, a) := VECTOR
pas_di(n, a) := VECTOR(
pas_rem(n, a, r) := VEC
  n)
pas_rm(n, a, r) := VECT
pas_rems(n, a) := VECTO
```

`pas_di(n,a)`	generates the first n rows of P.T; all elements, which are divisible by a will be represented by a **`*`**, the others by a space. (eg: **`pas_di(25,2)`**). Which n will "fit into your screen"?
`pas_div(n,a)`	represents the numbers from above by 0 and 1.
`pas_rem(n,a,r)`	marks all numbers of the P.T., which leave the remainder r after a division by a.
`pas_rm(n,a,r)`	mark these positions with a *, eg. `pas_rn(15,5,2)`
`pas_rems(n,a)`	shows all remainders after a division by a, eg `pas_rems(15,5)`.

```
[ ]
[ , ]
[ ,2, ]
[ , , , ]
[ , ,2, , ]
[ , ,2,2, , ]
[ ,2, , , ,2, ]
[ , , , , , , , ]
[ , , , ,2, , , , ]
[ , , , ,2,2, , , , ]
[ ,2, , ,2, ,2, , ,2, ]
[ , , , ,2,2,2,2, , , , ]
[ , ,2, , , , , , , , ,2, , ]
```

```
[1]
[1,1]
[1,2,1]
[1,0,0,1]
[1,1,0,1,1]
[1,2,1,1,2,1]
[1,0,0,2,0,0,1]
[1,1,0,2,2,0,1,1]
[1,2,1,2,1,2,1,2,1]
[1,0,0,0,0,0,0,0,0,1]
[1,1,0,0,0,0,0,0,0,1,1]
[1,2,1,0,0,0,0,0,0,1,2,1]
[1,0,0,1,0,0,0,0,0,1,0,0,1]
```

The functions are given in the appendix.

These nice fractals can easily be produced on the TI-89/92 screens using a short programm (see below)

Using the two programs pas() und invpas() you can generate various "pictures" emerging from Pascal´s Triangle..

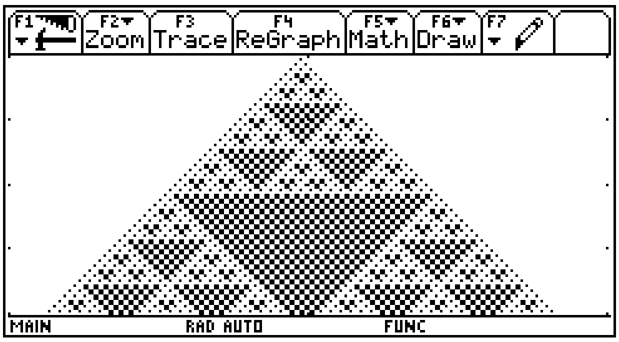
Transmit both programs pas and invpas und das Hilfsprogramm pt auf dein Gerät.

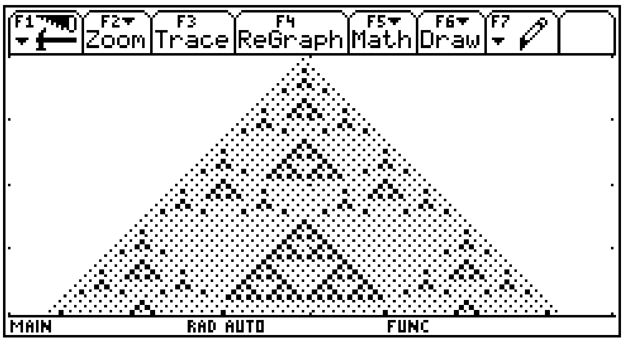
pas(n,divisor,remainder) paints all locations in the first n rows of the P.T. where the division of the number by divisor leaves the remainder.
invpas(n,divisor,remainder) returns black at these positions, where we do **not** receive the remainder.

If one wants to mark the even coefficients in the first 10 rows then call pas(10,2,0).
Why?

Try to reproduce your hand made paintings on the TI screen. How many rows can be represented on this small screen?

Which parameters in pas generate the following pictures?





Try to produce two "extra nice" Pascal pictures.

My Pascal Gallery:

The program code for pas(n,a,r):

```
pas(n,a,r)
Prgm
ClrDraw
FnOff :PlotsOff
setGraph("Axes","Off")
setGraph("Grid","Off")
Local i,j,z,pt
Define pt(x,y)=Prgm
PxlOn x,y:PxlOn x,y:PxlOn x+1,y
PxlOn x,y-1:PxlOn x+1,y-1
EndPrgm

For i,0,n
  For j,0,i
    nCr(i,j)»z
    If mod(z,a)=r Then
    pt(1+2*i,117-2*i+4*j)
    Else
    PxlOn 1+2*i,116-2*i+4*j
    EndIf
EndFor
EndFor
EndPrgm
```

Two explanations and one question:

nCr(i,j) generates the "binomial coefficients" using a rule which you will learn later in the frame of probability theory.

Funktion mod(z,a) - **Modulo Function** - returns the integer remainder after division of the interger z by the integer a.

Find first mentally, then using the *TI*:

mod(87,9) = ..., mod(15,3) = ..., mod(113,7) = ..., mod(200,13) = ...

Which line of the program above you will have to change to change pas() to invpas()?

Appendix:

The *DERIVE* file BINOM.MTH

```
#1:   DisplayFormat:=Compressed
#2:   pas_div(n,a):=VECTOR([VECTOR(IF(MOD(COMB(j,k),a)≠0,1,0),k,0,j)],j,0,n)
#3:   pas_di(n,a):=VECTOR([VECTOR(IF(MOD(COMB(j,k),a)≠0, ,*),k,0,j)],j,0,n)
#4:   pas_rem(n,a,r):=VECTOR([VECTOR(IF(MOD(COMB(j,k),a)=r,r,0),k,0,j)],j,0,n)
#5:   pas_rm(n,a,r):=VECTOR([VECTOR(IF(MOD(COMB(j,k),a)=r,r, ),k,0,j)],j,0,n)
#6:   pas_rems(n,a):=VECTOR([VECTOR(MOD(COMB(j,k),a),k,0,j)],j,0,n)
```

Program invpas(n,a,r)

```
invpas(n,a,r)
Prgm
ClrDraw
FnOff :PlotsOff
setGraph("Axes","Off")
setGraph("Grid","Off")
Local i,j,z,pt
Define pt(x,y)=Prgm
PxlOn x,y:PxlOn x,y:PxlOn x+1,y
PxlOn x,y-1:PxlOn x+1,y-1
EndPrgm
For i,0,n
  For j,0,i
    nCr(i,j)»z
    If mod(z,a)r Then
       pt(1+2*i,116-2*i+4*j)
    Else
       PxlOn 1+2*i,117-2*i+4*j
    EndIf
  EndFor
EndFor
EndPrgm
```