T³ EUROPE

Differentialgleichungen mit CAS
Differential Equations with CAS

Josef Böhm

Ein Unterrichtsbehelf zum Einsatz moderner Technologien im Mathematikunterricht

Austrian Center for Didactics of Computer Algebra

T³ Österreich / ACDCA am PI-Niederösterreich, Hollabrunn

Inhalt / Contents

Alle Dateien können von der DUG-Homepage heruntergeladen werden.

http://www.austromath.at/dug/index.htm

Solving Odes Using DERIVE´s ODE1.MTH

The file ODE1.MTH is helpful to solve first-order ordinary differential equations. I want to explain the capabilities of this file, complete it and make its use a bit easier.

The simplest Odes are those which can be solved by separating the variables. Let's look at an example:

Example 1: $(1 + y^2)dx + xy\,dy = 0$

Give the general solution and give the curve of solution containing then point P(3/2).

Sketch the direction field, given by (1), some curves of the general solution and the curve of the special solution!

It is necessary to bring the given Ode to the form: $\dfrac{dy}{dx} = p(x) \cdot q(y)$

to use the function SEPARABLE:

$$1:\quad \text{SEPARABLE}(p, q, x, y, x0, y0) := \int_{y0}^{y} \frac{1}{q}\,dy = \int_{x0}^{x} p\,dx$$

After separating the variables I get: $p = -\dfrac{1}{x}$ and $q = \dfrac{1 + y2}{y}$

and this leads to: **(in Derive 5!)**

#2: Logarithm := Expand

$$\#3:\quad \text{SEPARABLE}\left(-\frac{1}{x}, \frac{1 + y^2}{y}\right) = \left(\frac{\text{LN}(y^2 + 1)}{2} - \frac{\text{LN}(y0^2 + 1)}{2} = \text{LN}(x0) - \text{LN}(x)\right)$$

#4: [x :∈ Real (0, ∞), y :∈ Real (0, ∞)]

$$\#5:\quad \text{SOLVE}\left(\frac{\text{LN}(y^2 + 1)}{2} - \frac{\text{LN}(y0^2 + 1)}{2} = \text{LN}(x0) - \text{LN}(x), y\right)$$

$$\#6:\quad y = -\frac{\sqrt{x0^2\cdot(y0^2 + 1) - x^2}}{x} \vee y = \frac{\sqrt{x0^2\cdot(y0^2 + 1) - x^2}}{x}$$

$$\#7:\quad \text{SEPARABLE}\left(-\frac{1}{x}, \frac{1 + y^2}{y}, x, y, 3, 2\right) = \left(\frac{\text{LN}(y^2 + 1)}{2} - \frac{\text{LN}(5)}{2} = \text{LN}(3) - \text{LN}(x)\right)$$

$$\#8:\quad \text{SOLVE}\left(\frac{\text{LN}(y^2 + 1)}{2} - \frac{\text{LN}(5)}{2} = \text{LN}(3) - \text{LN}(x), y\right)$$

$$\#9:\quad y = -\frac{\sqrt{45 - x^2}}{x} \vee y = \frac{\sqrt{45 - x^2}}{x}$$

$$\#10:\quad y^2 = -\frac{x^2 - 45}{x^2}$$

Without the declarations in #2 and #4 one would get bulkier expressions.

I prefer the general solution containing a parameter c to get the family of curves presenting the multiplicity of all solution curves.

$$\#11:\quad \text{gen\_sol\_sep}(p, q, x, y) := \int \frac{1}{q}\,dy - \int p\,dx = c$$

$$\#12:\quad \text{spec\_sol\_sep}(p, q, x0, y0, x, y) := \int_{y0}^{y} \frac{1}{q}\,dy = \int_{x0}^{x} p\,dx$$

$$\#13:\quad \text{gen\_sol\_sep}\left(-\frac{1}{x}, \frac{1+y^2}{y}\right) = \left(\text{LN}(x) + \frac{\text{LN}(y^2+1)}{2} = c\right)$$

$$\#14:\quad \text{SOLVE}\left(\text{LN}(x) + \frac{\text{LN}(y^2+1)}{2} = c, y\right) = \left(y = -\frac{\sqrt{(\hat{e}^{2\cdot c} - x^2)}}{x} \vee y = \frac{\sqrt{(\hat{e}^{2\cdot c} - x^2)}}{x}\right)$$

$$\#15:\quad \left[y = -\frac{\sqrt{(\hat{e}^{2\cdot c} - x^2)}}{x}, y = \frac{\sqrt{(\hat{e}^{2\cdot c} - x^2)}}{x}\right]$$

$$\#16:\quad \text{spec\_sol\_sep}\left(-\frac{1}{x}, \frac{1+y^2}{y}, 3, 2\right)$$

after soLving for y we obtain the same results as in #9 above.

Now I want to plot not only the graph of the special solution satisfying the condition y(x = 3) = 2, but also a familiy of curves representing solutions of the given differential equation and bed them into the direction field, given by

$$y' = -\frac{1+y^2}{x\ y}$$

DERIVE provides the function DIRECTION_FIELD (part of DERIVE 2.xx's file ODE_APPR.MTH) for our purpose.

How to use this function is described in the manual:

I've made up my own FIELD-function, because I want all the tangent segments to have the same length.

Try this and you will get fine results, if you consider the following hints.

$$\#20:\quad \text{DIRECTION\_FIELD}\left(-\frac{1+y^2}{x\cdot y}, x, -8, 8, 16, y, -6, 6, 12\right)$$

```
       dir(r, x, y, x0, y0) :=
         If LIM(LIM(1/r, x, x0), y, y0) = 0
#21:        [x0, y0 + t]
            LIM(LIM([x + t/√(1 + r^2), y + r·t/√(1 + r^2)], x, x0), y, y0)
```

```
#22:  feld(r, x, y, xl, xr, xs, yd, yu, ys) := VECTOR(VECTOR(dir(r, x, y, x0, y0), x0,
        xl, xr, xs), y0, yd, yu, ys)
```

r is the slope r(x,y) = y'(x,y); xl and xr give the left and right border, xs the x-increment; y varies from yd to yo, with steps of ys.

$$\#23:\quad \text{feld}\left(-\frac{1+y^2}{x\cdot y}, x, y, -8, 8, 1, -6, 6, 1\right)$$

With feld() you can vary the length of the tangent pieces by setting the values for parameter t, $-0.25 \le t \le +0.25$ is recommended. Plot the direction field in blue.

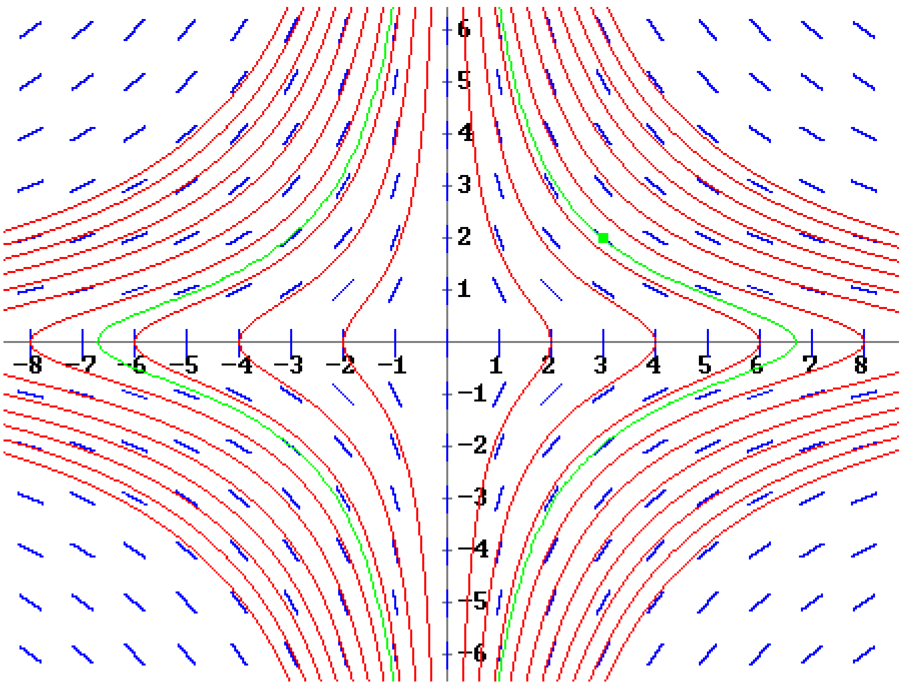
Set the appropriate scale, switch off the automatical change of colors, switch on Approximating Before Plotting and plot .... and then you will watch how the direction field will spread across the defined plane.

I've brought the function term in a more convenient form and build up two vectors of the family of curves, which represent the general solution.

$$\#24:\quad \mathrm{VECTOR}\left(-\frac{\sqrt{(c^2 - x^2)}}{x},\ c,\ -20,\ 20,\ 2\right)$$

$$\#25:\quad \mathrm{VECTOR}\left(\frac{\sqrt{(c^2 - x^2)}}{x},\ c,\ -20,\ 20,\ 2\right)$$

Plot them all in red



and finally add the special solution with the given point:

$$\#36:\quad \left[y^2 = -\frac{x^2 - 45}{x^2},\ [3,\ 2]\right]$$

Many things have changed since 1991: You need not loading the utility files. Derive recognizes all functions which are provided in the utility files (which you can find in the MATH-folder. It was not possible to plot implicitly given functions (#36).

We can do this now, see expression #38 .

$$\#38:\quad \mathrm{VECTOR}\left(y^2 = \frac{c^2 - x^2}{x^2},\ c,\ -20,\ 20,\ 2\right)$$

Example 2: $(1 + e^x)\ y\ y' = e^x$	spec. sol. P(1;1)
Example 3: $y'\sin x = y \ln y$	$y(\pi/2) = 1/2$

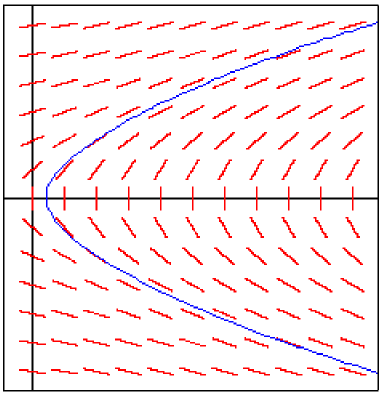
$$\#26:\quad \text{spec\_sol\_sep}\left(\frac{\hat{e}^x}{1+\hat{e}^x}, \frac{1}{y}, 1, 1\right)$$

$$\#27:\quad \frac{y^2}{2} - \frac{1}{2} = \text{LN}(\hat{e}^x + 1) - \text{LN}(\hat{e} + 1)$$

$$\#28:\quad \text{SOLVE}\left(\frac{y^2}{2} - \frac{1}{2} = \text{LN}(\hat{e}^x + 1) - \text{LN}(\hat{e} + 1), y\right)$$

$$\#29:\quad y = -\sqrt{(2\cdot\text{LN}(\hat{e}^x + 1) - 2\cdot\text{LN}(\hat{e} + 1) + 1)} \lor y = \sqrt{(2\cdot\text{LN}(\hat{e}^x + 1) - 2\cdot\text{LN}(\hat{e} + 1) + 1)}$$

$$\#30:\quad (y = -\sqrt{(2\cdot\text{LN}(\hat{e}^x + 1) - 2\cdot\text{LN}(\hat{e} + 1) + 1)})^2$$



$$\#31:\quad y^2 = 2\cdot\text{LN}(\hat{e}^x + 1) - 2\cdot\text{LN}(\hat{e} + 1) + 1$$

$$\#32:\quad \text{feld}\left(\frac{\hat{e}^x}{y\cdot(\hat{e}^x + 1)}, x, y, 0, 5, 0.5, -3, 3, 0.5\right)$$

$$\#34:\quad \text{spec\_sol\_sep}\left(\frac{1}{\text{SIN}(x)}, y\cdot\text{LN}(y), \frac{\pi}{2}, \frac{1}{2}\right)$$

$$\#35:\quad \text{LN}\left(\frac{\text{LN}(y)}{\text{LN}(2)}\right) - \pi\cdot\hat{\imath} = \text{LN}\left(\text{TAN}\left(\frac{x}{2}\right)\right)$$

$$\#36:\quad \text{SOLVE}\left(\text{LN}\left(\frac{\text{LN}(y)}{\text{LN}(2)}\right) - \pi\cdot\hat{\imath} = \text{LN}\left(\text{TAN}\left(\frac{x}{2}\right)\right), y\right)$$

$$\#37:\quad \text{SOLVE}\left(\text{SOLVE}\left(\text{LN}\left(\frac{\text{LN}(y)}{\text{LN}(2)}\right) - \pi\cdot\hat{\imath} = \text{LN}\left(\text{TAN}\left(\frac{x}{2}\right)\right), y\right), y\right)$$

$$\#38:\quad x :\in \text{Complex}$$

$$\#39:\quad y :\in \text{Complex}$$

$$\#40:\quad y = \text{IF}\left(-\pi < \text{IM}\left(\text{LN}\left(\text{LN}(2)\cdot\text{TAN}\left(\frac{x}{2}\right)\right) + \pi\cdot\hat{\imath}\right) \le \pi, \text{IF}\left(-\pi < \text{IM}\left(-\text{LN}(2)\cdot\text{TAN}\left(\frac{x}{2}\right)\right) \le \pi, 2^{-\text{TAN}(x/2)}\right)\right)$$

$$\#41:\quad \text{gen\_sol\_sep}\left(\frac{1}{\text{SIN}(x)}, y\cdot\text{LN}(y)\right) = \left(\text{LN}(\text{LN}(y)) - \text{LN}\left(\text{TAN}\left(\frac{x}{2}\right)\right) = c\right)$$

$$\#42:\quad \text{SOLVE}\left(\text{LN}(\text{LN}(y)) - \text{LN}\left(\text{TAN}\left(\frac{x}{2}\right)\right) = c, y\right)$$

#43: $y = \mathrm{IF}\left(-\pi < \mathrm{IM}\left(\mathrm{LN}\left(\mathrm{TAN}\left(\frac{x}{2}\right)\right)+c\right) \le \pi,\ \mathrm{IF}\left(-\pi < \mathrm{IM}\left(\hat{e}^{c}\cdot \mathrm{TAN}\left(\frac{x}{2}\right)\right) \le \pi,\ \hat{e}^{\hat{e}^{c}\cdot \mathrm{TAN}(x/2)}\right)\right)$

#44: $y = \hat{e}^{\hat{e}^{c}\cdot \mathrm{TAN}(x/2)}$

#45: $\mathrm{SOLVE}\left(\frac{1}{2} = \hat{e}^{\hat{e}^{c}\cdot \mathrm{TAN}(\pi/2/2)},\ c\right)$

#46: $c = \mathrm{LN}(\mathrm{LN}(2)) + 3\cdot\pi\cdot\hat{\imath} \vee c = \mathrm{LN}(\mathrm{LN}(2)) - \pi\cdot\hat{\imath} \vee c = \mathrm{LN}(\mathrm{LN}(2)) + \pi\cdot\hat{\imath}$

#47: $\mathrm{SUBST}\left(y = \hat{e}^{\hat{e}^{c}\cdot \mathrm{TAN}(x/2)},\ c,\ \mathrm{LN}(\mathrm{LN}(2)) + \pi\cdot\hat{\imath}\right) = \left(y = 2^{-\mathrm{TAN}(x/2)}\right)$

#48: $y = 2^{-\mathrm{TAN}(x/2)}$

It´s interesting to compare Derive´s solution from 1991 with its treatment in 2004. Domain declarations are very important.

$\mathrm{SPEC\_SOL\_SEP}\left[\frac{1}{\mathrm{SIN}(x)},\ y\ \mathrm{LN}(y),\ \frac{\pi}{2},\ \frac{1}{2}\right]$

$\mathrm{LN}\left[-\frac{\mathrm{LN}(y)}{\mathrm{LN}(2)(\mathrm{COS}(x)-1)}\right] - \mathrm{LN}\left[\frac{1}{\mathrm{SIN}(x)}\right] - \hat{\imath}\ \pi = 0$

$\hat{e}^{\mathrm{LN}(-\mathrm{LN}(y)/(\mathrm{LN}(2)(\mathrm{COS}(x)-1))) - \mathrm{LN}(1/\mathrm{SIN}(x)) - \hat{\imath}\,\pi} = 0$

$\frac{\mathrm{SIN}(x)\ \mathrm{LN}(y)}{\mathrm{LN}(2)(\mathrm{COS}(x)-1)} = 1$

: $y = 2^{\mathrm{COT}(x) - 1/\mathrm{SIN}(x)}$

Example 4: The logistic growth curve

Logistic growth is described by a difference equation: Increase of a population y during a time interval dx is proportional to its present quantity, to the time interval and a factor (1 – y/k) with k as a value for the environment capacity.

$$dy = dx \cdot r \cdot y \cdot (1 - y/k)$$

r is a prop. factor, the growth rate.

dy/dx = r y (1 - y/k); r = 0.5; k = 3000; y(0) = 100 (Initial population)

#49: $\mathrm{spec\_sol\_sep}\left(1,\ 0.5\cdot y\cdot\left(1 - \frac{y}{3000}\right),\ 0,\ 100\right)$

#50: $\mathrm{SOLVE}\left(2\cdot\mathrm{LN}(y) - 2\cdot\mathrm{LN}\left(\frac{y-3000}{29}\right) + 2\cdot\pi\cdot\hat{\imath} = x,\ y\right)$

#51: $y = \frac{3000\cdot\hat{e}^{x/2}}{\hat{e}^{x/2} + 29}$

I let DERIVE compute the limit of this function to obtain the saturation quantity

dy/dx = r y (1 - y/k); r = 0.5; k = 3000; y(0) = 100 (Initial population)

#49: $\text{spec\_sol\_sep}\left(1,\ 0.5\cdot y\cdot\left(1-\frac{y}{3000}\right),\ 0,\ 100\right)$

#50: $\text{SOLVE}\left(2\cdot LN(y) - 2\cdot LN\left(\frac{y-3000}{29}\right) + 2\cdot\pi\cdot\hat{\imath} = x,\ y\right)$

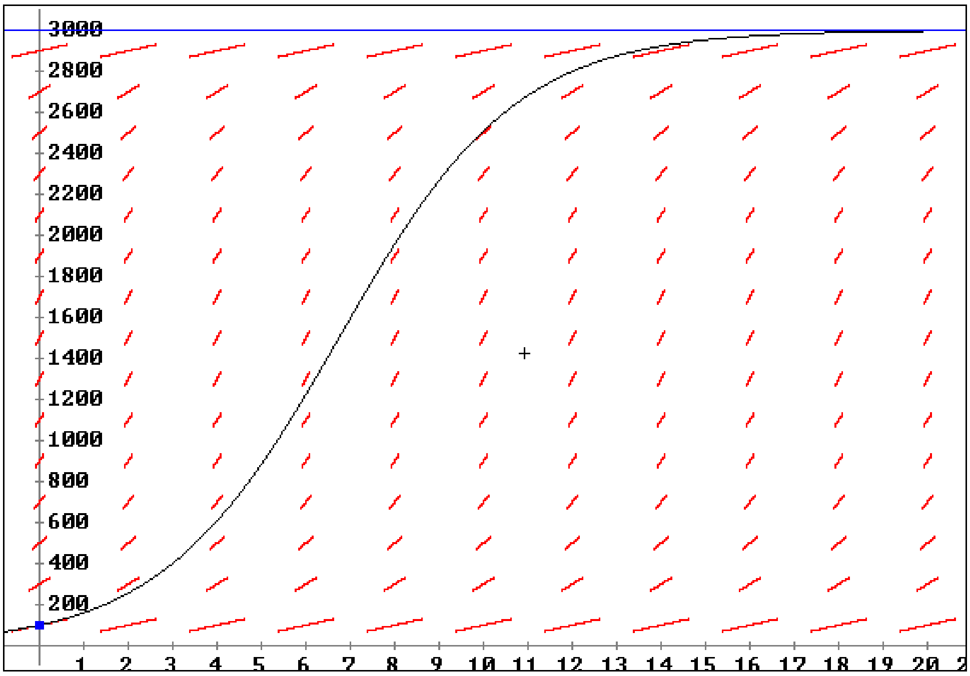
#51: $y = \dfrac{3000\cdot\hat{e}^{x/2}}{\hat{e}^{x/2} + 29}$

#52: $\lim\limits_{x\to\infty} \dfrac{3000\cdot\hat{e}^{x/2}}{\hat{e}^{x/2} + 29} = 3000$

#53: $\text{feld}\left(0.5\cdot y\cdot\left(1-\frac{y}{3000}\right),\ x,\ y,\ 0,\ 20,\ 2,\ 100,\ 2900,\ 200\right)$

-30 <= t <= 30

#54: [[0, 100], 3000]



Example 5: The general equation expressing the relation between electromotive force U and current I in an circuit containing the resistance R and the inductance L is:

$$U = L \cdot I' + R \cdot I$$

Solve this equation first for a constant U = U0.
I(t = 0) = 0
Then for U_0 = 20V, R = 5 Ohm, L = 0.1 Henry!
Solve the equation for U = U_0 sin(w t)! (w = 40)

#55: $u0 :=$

#56: $\text{gen\_sol\_sep}\left(1, \frac{u0 - r\cdot i}{l}, t, i\right)$

#57: $-\frac{l\cdot \text{LN}(i\cdot r - u0)}{r} - t = c$

#58: $\text{SOLVE}\left(-\frac{l\cdot \text{LN}(i\cdot r - u0)}{r} - t = c, i, \text{Real}\right)$

#59: $i = \text{IF}\left(-\pi < \text{IM}\left(-\frac{r\cdot(c + t)}{l}\right) \le \pi, \frac{\hat{e}^{- c\cdot r/l - r\cdot t/l}}{r} + \frac{u0}{r}\right)$

$$\#60: \quad i = \frac{\hat{e}^{- c\cdot r/l - r\cdot t/l}}{r} + \frac{u0}{r}$$

#61: $\text{spec\_sol\_sep}\left(1, \frac{u0 - r\cdot i}{l}, 0, 0, t, i\right)$

#62: $\frac{l\cdot \text{LN}(-u0)}{r} - \frac{l\cdot \text{LN}(i\cdot r - u0)}{r} = t$

#63: $\text{SOLVE}\left(\frac{l\cdot \text{LN}(-u0)}{r} - \frac{l\cdot \text{LN}(i\cdot r - u0)}{r} = t, i\right)$

#64: $i = \frac{u0}{r} - \frac{u0\cdot \hat{e}^{- r\cdot t/l}}{r}$

#65: $i = \frac{20}{5} - \frac{20\cdot \hat{e}^{- 5\cdot t/0.1}}{5}$

$$\#66: \quad i = 4 - 4\cdot \hat{e}^{- 50\cdot t}$$

The next question in this example leads to a linear monic differential equation of the form y' + p(x) · y = q(x). The function LINEAR1(p,q,x,y,x0,y0) helps us to find the solution. My special form LIN1(p,q,x,y) gives the solution containing the parameter c.

#67: $\text{LINEAR1}(p, q, x, y, x0, y0) := y = \frac{y0 + \int_{x0}^{x} q\cdot \hat{e}^{\text{INT}(p, x, x0, x)} dx}{\hat{e}^{\text{INT}(p, x, x0, x)}}$

#68: $\text{LINEAR1}\left(\frac{r}{l}, \frac{u0}{l}\cdot \text{SIN}(w\cdot t), t, i, 0, 0\right)$

#69: $i = \frac{u0\cdot \hat{e}^{- r\cdot t/l}\cdot(l\cdot w - \hat{e}^{r\cdot t/l}\cdot(l\cdot w\cdot \text{COS}(t\cdot w) - r\cdot \text{SIN}(t\cdot w)))}{l^2\cdot w^2 + r^2}$

$$\#72: \quad i = 1.95\cdot \hat{e}^{- 50\cdot t} - 1.95\cdot \text{COS}(40\cdot t) + 2.43\cdot \text{SIN}(40\cdot t)$$

$$\#73:\quad \text{lin1}(p, q, x, y) := y = \frac{\int q\cdot ê^{\int p\, dx}\, dx + c}{ê^{\int p\, dx}}$$

$$\#74:\quad \text{lin1}\left(\frac{r}{l}, \frac{u0\cdot \text{SIN}(w\cdot t)}{l}, t, i\right)$$

$$\#75:\quad i = c\cdot ê^{-r\cdot t/l} - \frac{u0\cdot (l\cdot w\cdot \text{COS}(t\cdot w) - r\cdot \text{SIN}(t\cdot w))}{l^2\cdot w^2 + r^2}$$

$$\#76:\quad 0 = c\cdot ê^{-5\cdot 0/0.1} - \frac{20\cdot (0.1\cdot 40\cdot \text{COS}(0\cdot 40) - 5\cdot \text{SIN}(0\cdot 40))}{0.1^2\cdot 40^2 + 5^2}$$

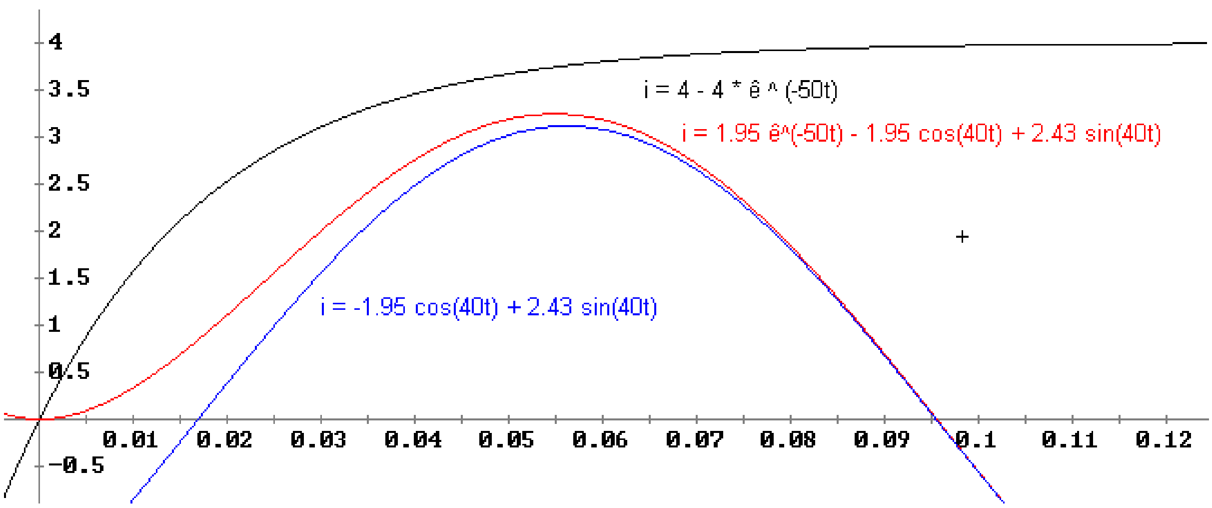
$$\#77:\quad \text{NSOLVE}\left(0 = c\cdot ê^{-5\cdot 0/0.1} - \frac{20\cdot (0.1\cdot 40\cdot \text{COS}(0\cdot 40) - 5\cdot \text{SIN}(0\cdot 40))}{0.1^2\cdot 40^2 + 5^2}, c, \text{Real}\right)$$

$$\#78:\quad c = 1.95$$

$$\#79:\quad i = 1.95\cdot ê^{-50\cdot t} - 1.95\cdot \text{COS}(40\cdot t) + 2.43\cdot \text{SIN}(40\cdot t)$$

$$\boxed{\#80:\quad i = -1.95\cdot \text{COS}(40\cdot t) + 2.43\cdot \text{SIN}(40\cdot t)}$$

This is the plot with both solutions #66(black) and #72 = #79(red). The exponential term in #79 is called transient, because it usually becomes negligibly small after a short lapse of time. Graph of #80 (blue) shows the remaining steady-state term.



(Annotating plots is also a benefit of later Derive releases together with direct simplification using the = sign at the end of an expression.)

Example 6: The ODE y' = x + y defines a nice direction field. Let DERIVE plot this field including a family of curves fitting into it.

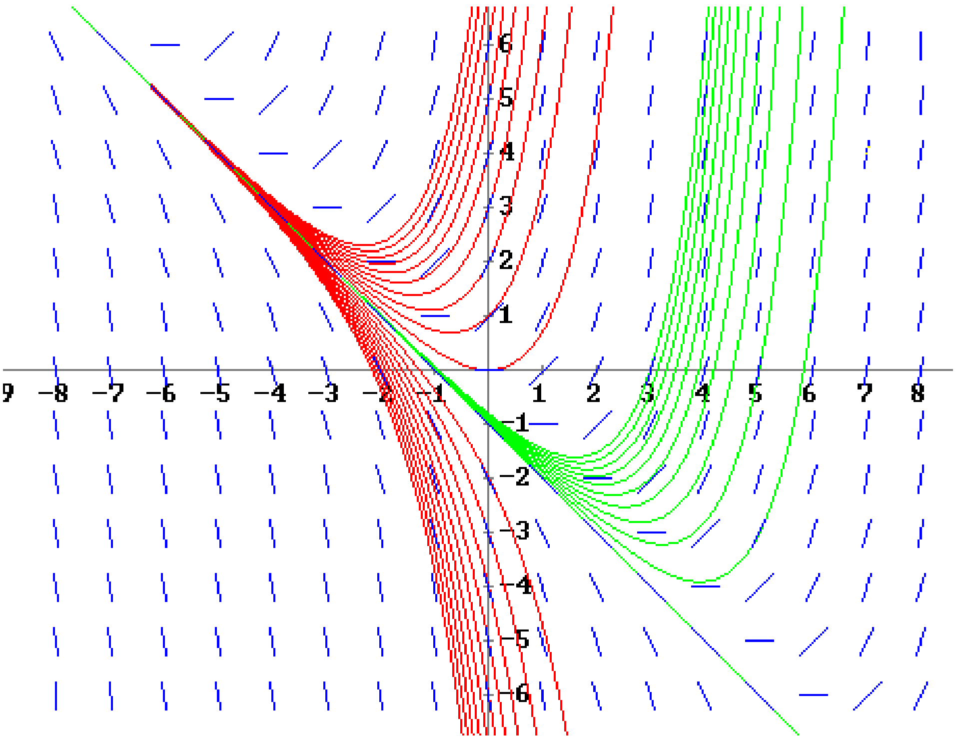
#81: lin1(-1, x)

#82: $y = c\cdot \hat{e}^{x} - x - 1$

#83: DIRECTION_FIELD(x + y, x, -8, 8, 16, y, -6, 6, 12)

#84: VECTOR($y = c\cdot \hat{e}^{x} - x - 1$, c, -10, 10)

#85: VECTOR($y = c\cdot \hat{e}^{x} - x - 1$, c, 0, 0.2, 0.02)



Solving Odes Using *DERIVE*'s `ODE1.MTH`, Part 2

Josef Böhm, Würmla

I tried to extend the file ODE1.MTH in orderto be able to write general solutions with a parameter *c* instead of ($x0$,$y0$) used in ODE1.MTH. You will find a function to express the integrating factor and a function to solve the LAGRANGE-DE. The expressions #1 - #4 have been used in Part 1 and will be used now, too. I collect all the new functions and hints in the file ODE1_EXT.MTH. Having loaded ODE1, I merge (load) ODE1_EXT and start working.

(In later MATH-files ODE1.MTH general solutions using a constant *c* have been added.)

ODE1_EXT.MTH

#1: $$\text{GEN\_SOL\_SEP}(p, q, x, y) := \int \frac{1}{q}\,dy - \int p\,dx = c$$

#2: $$\text{SPEC\_SOL\_SEP}(p, q, x0, y0, x, y) := \int_{y0}^{y} \frac{1}{q}\,dy = \int_{x0}^{x} p\,dx$$

#3: $$\text{DIR}(r, x, y, x0, y0) := \lim_{y\to y0}\lim_{x\to x0} \text{IF}\left(\frac{1}{r} = 0, [x, y + t], \left[x + \frac{t}{\sqrt{(1 + r^2)}}, y + \frac{t\cdot r}{\sqrt{(1 + r^2)}}\right]\right)$$

#4: FELD(r, x, y, xl, xr, xs, yd, yu, ys) := VECTOR(VECTOR(DIR(r, x, y, x0, y0), x0, xl, xr, xs), y0, yd, yu, ys)

#5: $$\text{LINEAR1}(p, q, x, y, x0, y0) := y = \frac{y0 + \int_{x0}^{x} q\cdot ê^{\text{INT}(p, x, x0, x)}\,dx}{ê^{\text{INT}(p, x, x0, x)}}$$

#6: $$\text{GEN\_LIN1}(p, q, x, y) := y = \frac{\int q\cdot ê^{\int p\,dx}\,dx + c}{ê^{\int p\,dx}}$$

#7: $$\text{GEN\_BERN}(p, q, k, x, y) := \text{GEN\_LIN1}((1 - k)\cdot p, (1 - k)\cdot q, x, y^{1 - k})$$

#8: $$\text{GEN\_SOL\_HOM}(r, x, y) := \lim_{y\to y/x} \text{GEN\_SOL\_SEP}\left(\frac{1}{x}, \left(\lim_{y\to x\cdot y} r\right) - y, x, y\right)$$

#9: $$\text{GEN\_SOL\_EX}(p, q, x, y) := \int p\,dx + \int \left(q - \frac{d}{dy}\int p\,dx\right) dy = c$$

#10: $$\text{GEN\_INT\_FCT}(\mu, p, q, x, y) := \text{GEN\_SOL\_EX}(\mu\cdot p, \mu\cdot q, x, y)$$

#11: $$\text{INT\_FCT}(t, v) := ê^{\int t\,dv}$$

#12: $$\text{GEN\_FRAC\_AUX}(r, p, q, x, y) := \lim_{[y, x]\to[y - q, x - p]} \text{GEN\_SOL\_HOM}\left(\lim_{[y, x]\to[y + q, x + p]} r, x, y\right)$$

$$\#13:\quad \text{GEN\_LIN\_FRAC}(r, a, b, c, p, q, k, x, y) := \text{GEN\_FRAC\_AUX}\left(r, \frac{b\cdot k - c\cdot q}{a\cdot q - b\cdot p}, \frac{c\cdot p - a\cdot k}{a\cdot q - b\cdot p}, x, y\right)$$

$$\#14:\quad \text{AUX\_LAGR}(p, q, x, y) := \frac{\int q\cdot \hat{e}^{\int p\,dx}\,dx + c}{\hat{e}^{\int p\,dx}}$$

$$\#15:\quad \text{LAGR}(p, q, x, y, v, c1) := \left[y = \lim_{v\to c1}(x\cdot p + q),\ \lim_{v\to c1} p = c1,\ x = \text{AUX\_LAGR}\left(\frac{\frac{d}{dv}p}{p - v}, -\frac{\frac{d}{dv}q}{p - v}, v, x\right),\ y = \text{AUX\_LAGR}\left(\frac{\frac{d}{dv}p}{p - v}, -\frac{\frac{d}{dv}q}{p - v}, v, x\right)\cdot p + q\right]$$

```
Example 7: Find the general solutions of the given differential equations
           and verify them. Find also the special solutions containing the
           given points.

           a) x·y' - 4y + 2x^2 + 4 = 0;          P(1|-3)

           b) y'·cos x + y·sin x = tan x;        P(π/6|1)

           c) (1 - x^2) y' = 1 + x·y;            P(2|2)
```

These linear monic differential equations must be brought into the form $y' + p(x).y = q(x)$. Then use `LINEAR1` from `ODE1.MTH` or `GEN_LIN1(p,q,x,y)` from ODE1_EXT.MTH to find the general solution written with a single symbolic constant *c*.

Example 7a $\qquad x\,y' - 4y + 2x^2 + 4 = 0 \;\Leftrightarrow\; y' - \frac{4}{x}y = -\frac{2x^2+4}{x}$

$$\#16:\quad \text{GEN\_LIN1}\left(-\frac{4}{x}, -\frac{2\cdot x^2 + 4}{x}\right)$$

$$\#17:\quad y = c\cdot x^4 + x^2 + 1$$

Check for correctness of the solution

$$\#18:\quad x\cdot\frac{d}{dx}(c\cdot x^4 + x^2 + 1) - 4\cdot(c\cdot x^4 + x^2 + 1) + 2\cdot x^2 + 4 = 0$$

$$\#19:\quad -3 = c\cdot 1^4 + 1^2 + 1$$

$$\#20:\quad \text{SOLVE}(-3 = c\cdot 1^4 + 1^2 + 1, c, \text{Real})$$

$$\#21:\quad c = -5$$

Special solution: $\qquad y(x) = -5x^4 + x^2 + 1$

$$\#22:\quad \text{LINEAR1}\left(-\frac{4}{x}, -\frac{2\cdot x^2 + 4}{x}, x, y, 1, -3\right) = (y = -5\cdot x^4 + x^2 + 1)$$

using LINEAR1 gives the special solution promptly.

Example 7b $y'\cos x + y\sin x = \tan x \;\Leftrightarrow\; y' + \frac{\sin x}{\cos x} y = \frac{\tan x}{\cos x} \;\Leftrightarrow\; y' + \tan x \cdot y = \frac{\tan x}{\cos x}$

#23: $\text{LINEAR1}\left(\text{TAN}(x), \frac{\text{SIN}(x)}{\text{COS}(x)^2}\right)$

#24: $y = \text{COS}(x)\cdot\left(\frac{y0}{\text{COS}(x0)} - \frac{1}{2\cdot\text{COS}(x0)^2}\right) + \frac{1}{2\cdot\text{COS}(x)}$

#25: $\text{GEN\_LIN1}\left(\text{TAN}(x), \frac{\text{SIN}(x)}{\text{COS}(x)^2}\right)$

Which solution is more comfortable? #24 or #26?

#26: $y = c\cdot\text{COS}(x) + \frac{1}{2\cdot\text{COS}(x)}$

#27: $1 = \frac{\sqrt{3}\cdot c}{2} + \frac{\sqrt{3}}{3}$

#28: $\text{SOLVE}\left(1 = \frac{\sqrt{3}\cdot c}{2} + \frac{\sqrt{3}}{3}, c, \text{Real}\right)$

#29: $c = \frac{2\cdot\sqrt{3}}{3} - \frac{2}{3}$

#30: $y = \left(\frac{2\cdot\sqrt{3}}{3} - \frac{2}{3}\right)\cdot\text{COS}(x) + \frac{1}{2\cdot\text{COS}(x)}$

or directly with LINEAR1(...)

#31: $\text{LINEAR1}\left(\text{TAN}(x), \frac{\text{SIN}(x)}{\text{COS}(x)^2}, x, y, \frac{\pi}{6}, 1\right) = \left[y = \left(\frac{2\cdot\sqrt{3}}{3} - \frac{2}{3}\right)\cdot\text{COS}(x) + \frac{1}{2\cdot\text{COS}(x)}\right]$

Example 7c $(1-x^2)y' = 1 + x\,y \;\Leftrightarrow\; y' + \left(-\frac{x}{1-x^2}\right)y = \frac{1}{1-x^2}$

#32: $\text{LINEAR1}\left(\frac{x}{x^2 - 1}, \frac{1}{1 - x^2}\right)$

#33: $y = -\frac{\text{LN}(\sqrt{x^2 - 1} + x) - \text{LN}(\sqrt{x0^2 - 1} + x0) - y0\cdot\sqrt{x0^2 - 1}}{\sqrt{x^2 - 1}}$

#34: $\text{GEN\_LIN1}\left(\frac{x}{x^2 - 1}, \frac{1}{1 - x^2}\right)$

#35: $y = \frac{c - \text{LN}(\sqrt{x^2 - 1} + x)}{\sqrt{x^2 - 1}}$

#36: $y(x) := \frac{c - \text{LN}(\sqrt{x^2 - 1} + x)}{\sqrt{x^2 - 1}}$

#37: $(1 - x^2)\cdot y'(x) - (1 + x\cdot y(x))$

#38: 0

#39: $(1 - x^2)\cdot y'(x) = 1 + x\cdot y(x)$

#40: $y(x) :=$

#41: $y :=$

For verfication of the result we define $y(x)$.
For proceeding in this session we "redo" this definition and go on.

In later Derive versions LINEAR1_GEN() was included in ODE1.MTH together with other _GEN()-functions in order to create general solutions using a symbolic constant (c by default).

$$\#42:\quad \text{LINEAR1\_GEN}\left(\frac{x}{x^2-1}, \frac{1}{1-x^2}\right) = \left[y = \frac{c - \text{LN}(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}}\right]$$

and finally the requested special solution

$$\#43:\quad \text{LINEAR1}\left(\frac{x}{x^2-1}, \frac{1}{1-x^2}, x, y, 2, 2\right)$$

$$\#44:\quad y = \frac{2\cdot\sqrt{3} - \text{LN}((2-\sqrt{3})\cdot(\sqrt{x^2-1}+x))}{\sqrt{x^2-1}}$$

Both of the following equations are of the Bernoulli-type:

$$y' + p(x)\cdot y = q(x)\cdot y^k \qquad (k = \text{const.})$$

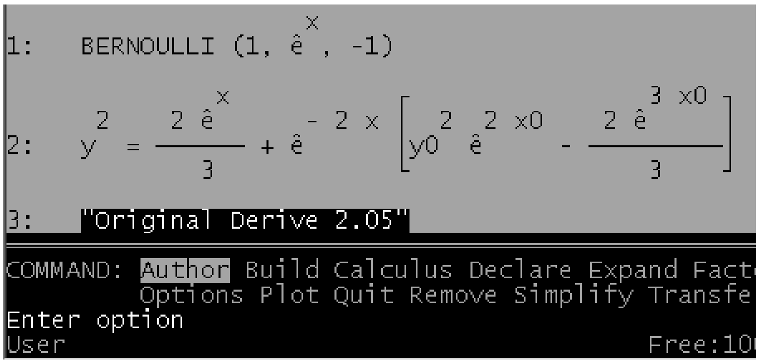
In `ODE1.MTH` we found `BERNOULLI(p,q,k,x,y,x0,y0)`. `GEN_BERN(p,q,k)` gives the general solution with constant *c*. In *DERIVE5*´s `ODE1.MTH` from today we can find `BERNOULLI_ODE()` and `BERNOULLI_ODE_GEN()`.

Example 8:
a) $y'\cdot y = e^x - y^2$; P(0|2)
b) $y'\cdot x^3 + y\cdot x^2 - (x^2+1)\cdot y^3$; verify the solution

BERNOULLI() now gives the BERNOULLI number!

$$\#45:\quad \text{BERNOULLI}(1, \hat{e}^x, -1) = -\frac{1}{2}$$

In 1991 `BERNOULLI()` delivered the solution of the differential equation:



Example 8a

This is *DERIVE*5 and ODE1_EXT.MTH

$$\#46:\quad \text{BERNOULLI\_ODE}(1, \hat{e}^x, -1)$$

$$\#47:\quad y^2 = \frac{2\cdot\hat{e}^x}{3} + \hat{e}^{-2\cdot x}\cdot\left(y0^2\cdot\hat{e}^{2\cdot x0} - \frac{2\cdot\hat{e}^{3\cdot x0}}{3}\right)$$

$$\#48:\quad \text{GEN\_BERN}(1, \hat{e}^x, -1)$$

$$\#49:\quad y^2 = \frac{2\cdot\hat{e}^x}{3} + c\cdot\hat{e}^{-2\cdot x}$$

$$\#50:\quad \text{BERNOULLI\_ODE\_GEN}(1,\ \hat{e}^x,\ -1) = \left[y^2 = \frac{2\cdot \hat{e}^x}{3} + c\cdot \hat{e}^{-2\cdot x}\right]$$

$$\#51:\quad \text{BERNOULLI\_ODE}(1,\ \hat{e}^x,\ -1,\ x,\ y,\ 0,\ 2) = \left[y^2 = \frac{2\cdot \hat{e}^x}{3} + \frac{10\cdot \hat{e}^{-2\cdot x}}{3}\right]$$

$$\#52:\quad 2^2 = \frac{2\cdot \hat{e}^0}{3} + c\cdot \hat{e}^{-2\cdot 0}$$

$$\#53:\quad \text{SOLVE}\left(2^2 = \frac{2\cdot \hat{e}^0}{3} + c\cdot \hat{e}^{-2\cdot 0},\ c,\ \text{Real}\right)$$

$$\#54:\quad c = \frac{10}{3}$$

Compare with the Derive solution from above!

Example 8b

$$\#55:\quad \text{GEN\_BERN}\left(\frac{1}{x},\ \frac{x^2+1}{x^3},\ 3\right)$$

$$\#56:\quad \frac{1}{y^2} = \frac{2\cdot c\cdot x^4 + 2\cdot x^2 + 1}{2\cdot x^2}$$

$$\#57:\quad \frac{1}{\dfrac{1}{y^2} = \dfrac{2\cdot c\cdot x^4 + 2\cdot x^2 + 1}{2\cdot x^2}}$$

$$\#58:\quad y(x) := \sqrt{\left(\frac{2\cdot x^2}{2\cdot c\cdot x^4 + 2\cdot x^2 + 1}\right)}$$

$$\#59:\quad y'(x)\cdot x^3 + y(x)\cdot x^2 - (x^2 + 1)\cdot y(x)^3 = 0$$

$$\#60:\quad 0 = 0$$

$$\#61:\quad y :=$$

$$\#62:\quad y = y$$

Example 9: Test the following differential equations on homogeneity.
Find two ways for obtaining the general solutions.

a) $y' = \dfrac{y+x}{y-x}$; P(1|2)

b) $y' = \dfrac{y\,e^{y/x} - x}{x\,e^{y/x}}$; P(1/2|-3/2) Verify the solution!

Derive 2.05 had a function `HOMOGENEOUS_TEST()`*, which disappeared in later versions.*
The equation is homogeneous if `HOMOGENEOUS_TEST()` *gives back an expression which is free of the independent variable. We can now reanimate this function or make a new one, which uses the definition of homogeneous expressions: substituting k x and k y for x and y should result in an equivalent expression.*

$$\text{HOMOGENEOUS\_TEST}(r, x, y) := \lim_{y \to x \cdot y} r$$

$$\text{HOMOGENEOUS\_TEST}\left(\frac{y+x}{y-x}\right) = \frac{y+1}{y-1}$$

$$\text{HOMOGENEOUS\_TEST}\left(\frac{y \cdot ê^{y/x} - x}{x \cdot ê^{y/x}}\right) = y - ê^{-y}$$

$$\text{HOMOGENEOUS\_TEST}\left(\frac{y \cdot ê^{y/x} - 1}{x \cdot ê^{y/x}}\right) = y - \frac{ê^{-y}}{x}$$

```
my_hom_test(r, x, y) :=
  If r - SUBST(r, [x, y], [k·x, k·y]) = 0
     "homogeneous"
     "non homogeneous"
     "non homogeneous"
```

$$\text{my\_hom\_test}\left(\frac{y+x}{y-x}\right) = \text{homogeneous}$$

$$\text{my\_hom\_test}\left(\frac{y \cdot ê^{y/x} - x}{x \cdot ê^{y/x}}\right) = \text{homogeneous}$$

$$\text{my\_hom\_test}\left(\frac{y \cdot ê^{y/x} - 1}{x \cdot ê^{y/x}}\right) = \text{non homogeneous}$$

Proceed with solving the equation:

$$\#63: \quad \text{HOMOGENEOUS}\left(\frac{y+x}{y-x}\right)$$

$$\#64: \quad \frac{\text{LN}(-x0^2 - y0\cdot(2\cdot x0 - y0))}{2} - \frac{\text{LN}\left(-\frac{x0^2\cdot(x^2 + 2\cdot x\cdot y - y^2)}{x^2}\right)}{2} = -\text{LN}\left(\frac{x0}{x}\right)$$

This was the 1991-recipe for manipulating this expression in order to find the solution:

`Build #64, ENTER, *, 2, ENTER, E(xp), R(ecip), ^ENTER`

`Manage Substitute x0 = 1, y0 = 2, ^ENTER`

$$x^2 + 2\cdot x\cdot y - y^2 = 1$$

Let´s do the same without the "Building procedure" from Derive-for-DOS-times:

#64 * 2

$$\#65: \quad \left(\frac{\text{LN}(-x0^2 - y0\cdot(2\cdot x0 - y0))}{2} - \frac{\text{LN}\left(-\frac{x0^2\cdot(x^2 + 2\cdot x\cdot y - y^2)}{x^2}\right)}{2} = -\text{LN}\left(\frac{x0}{x}\right)\right)\cdot 2$$

$$\#66: \quad \text{LN}(-x0^2 - y0\cdot(2\cdot x0 - y0)) - \text{LN}\left(-\frac{x0^2\cdot(x^2 + 2\cdot x\cdot y - y^2)}{x^2}\right) = -2\cdot\text{LN}\left(\frac{x0}{x}\right)$$

e^(#66)

$$\#67:\quad \hat{e}^{LN(-x0^2-y0\cdot(2\cdot x0-y0))-LN(-x0^2\cdot(x^2+2\cdot x\cdot y-y^2)/x^2)}=-2\cdot LN(x0/x)$$

$$\#68:\quad \frac{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}=\frac{x^2}{x0^2}$$

1/#68

$$\#69:\quad \frac{1}{\frac{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}=\frac{x^2}{x0^2}}$$

$$\#70:\quad \frac{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}=\frac{x0^2}{x^2}$$

#70/(x0^2/x^2)

$$\#71:\quad \frac{\frac{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}=\frac{x0^2}{x^2}}{\frac{x0^2}{x^2}}$$

$$\#72:\quad \frac{x^2+2\cdot x\cdot y-y^2}{x0^2+2\cdot x0\cdot y0-y0^2}=1$$

Substitute for x0 = 1, y0 = 2, leading to

$$\#73:\quad x^2+2\cdot x\cdot y-y^2=1$$

$$\#74:\quad GEN\_SOL\_HOM\left(\frac{y+x}{y-x},\ x,\ y\right)$$

$$\#75:\quad -\frac{LN(-x^2-y\cdot(2\cdot x-y))}{2}=c$$

$$\#76:\quad -\frac{\pi\cdot\hat{\imath}}{2}=c$$

$$\#77:\quad -\frac{LN(-x^2-y\cdot(2\cdot x-y))}{2}=-\frac{\pi\cdot\hat{\imath}}{2}$$

$$\#78:\quad \hat{e}^{(-LN(-x^2-y\cdot(2\cdot x-y))/2=-\pi\cdot\hat{\imath}/2)\cdot(-2)}$$

$$\#79:\quad -x^2-y\cdot(2\cdot x-y)=-1$$

If you exactly know what to do, then it is easy in *DERIVE*5, too:

`SUBST(1/ê^(2*#64)/(x0^2/x^2),[x0,y0],[1,2])` which results in $x^2+2\cdot x\cdot y-y^2=1$

$$\#80:\quad HOMOGENEOUS\_GEN\left(\frac{y+x}{y-x}\right)$$

$$\#81:\quad -LN\left(\frac{\sqrt{(-x^2-y\cdot(2\cdot x-y))}}{x}\right)=LN(x)+c$$

$$\#82:\quad HOMOGENEOUS\left(\frac{y+x}{y-x},\ x,\ y,\ 1,\ 2\right)$$

$$\#83:\quad -\frac{LN\left(-\frac{x^2+2\cdot x\cdot y-y^2}{x^2}\right)}{2}+\frac{\pi\cdot\hat{\imath}}{2}=LN(x)$$

Transform to find a nice reprecentation of the solution!

$$\#84:\quad \text{HOMOGENEOUS}\left(\frac{y\cdot \hat{e}^{y/x}-x}{x\cdot \hat{e}^{y/x}},\ x,\ y,\ \frac{1}{2},\ -\frac{3}{2}\right)$$

$$\#85:\quad \hat{e}^{-3}-\hat{e}^{y/x}=\text{LN}(2\cdot x)$$

$$\#86:\quad \text{SOLVE}(\hat{e}^{-3}-\hat{e}^{y/x}=\text{LN}(2\cdot x),\ y,\ \text{Real})$$

$$\#87:\quad y=x\cdot \text{LN}(1-\hat{e}^{3}\cdot \text{LN}(2\cdot x))-3\cdot x$$

Or do it stepwise:

$$\#88:\quad \text{HOMOGENEOUS\_GEN}\left(\frac{y\cdot \hat{e}^{y/x}-x}{x\cdot \hat{e}^{y/x}}\right)$$

$$\#89:\quad -\hat{e}^{y/x}=\text{LN}(x)+c$$

$$\#90:\quad \text{SOLVE}\left(\text{SUBST}\left(-\hat{e}^{y/x}=\text{LN}(x)+c,\ [x,\ y],\ \left[\frac{1}{2},\ -\frac{3}{2}\right]\right),\ c\right)$$

$$\#91:\quad c=\text{LN}(2)-\hat{e}^{-3}$$

$$\#92:\quad \text{SOLVE}(-\hat{e}^{y/x}=\text{LN}(x)+\text{LN}(2)-\hat{e}^{-3},\ y)$$

$$\#93:\quad y=x\cdot \text{LN}(1-\hat{e}^{3}\cdot \text{LN}(2\cdot x))-3\cdot x-2\cdot \pi\cdot \hat{\imath}\cdot x \vee y=x\cdot \text{LN}(1-\hat{e}^{3}\cdot \text{LN}(2\cdot x))-3\cdot x+2\cdot \pi\cdot \hat{\imath}\cdot x$$
$$\vee\ y=x\cdot \text{LN}(1-\hat{e}^{3}\cdot \text{LN}(2\cdot x))-3\cdot x$$

As we didn´t ask for the real solutions only, we have to select the real one:

$$\#94:\quad y=x\cdot \text{LN}(1-\hat{e}^{3}\cdot \text{LN}(2\cdot x))-3\cdot x$$

Verification of the solution:

$$\#95:\quad y:=x\cdot \text{LN}(1-\hat{e}^{3}\cdot \text{LN}(2\cdot x))-3\cdot x$$

$$\#96:\quad \frac{d}{dx}\,y-\frac{y\cdot \hat{e}^{y/x}-x}{x\cdot \hat{e}^{y/x}}=0$$

The counter example:

$$\text{HOMOGENEOUS\_GEN}\left(\frac{y\cdot \hat{e}^{y/x}-1}{x\cdot \hat{e}^{y/x}}\right)=\text{inapplicable}$$

Example 10: Compare the integral curves of

$$y'=\frac{y^2-x^2}{2xy}\ \text{ and }\ y'=\frac{2xy}{y^2-x^2}\quad \text{intersecting in } (2|-1)$$

Plot both curves together with their tangents in P on the same axes.

$$\#99:\quad \mathrm{GEN\_SOL\_HOM}\left(\frac{y^2 - x^2}{2\cdot x\cdot y}\right)$$

$$\#100:\quad -\mathrm{LN}\left(\frac{x^2 + y^2}{x}\right) = c$$

$$\#101:\quad \hat{e}^{-\left(-\mathrm{LN}((x^2 + y^2)/x) = c\right)}$$

$$\#102:\quad \frac{x^2 + y^2}{x} = \hat{e}^{-c}$$

$$\#103:\quad \mathrm{HOMOGENEOUS}\left(\frac{y^2 - x^2}{2\cdot x\cdot y}, x, y, 2, -1\right)$$

$$\#104:\quad -\mathrm{LN}\left(\frac{4\cdot(x^2 + y^2)}{5\cdot x^2}\right) = \mathrm{LN}\left(\frac{x}{2}\right)$$

$$\#105:\quad \mathrm{SOLVE}\left(-\mathrm{LN}\left(\frac{4\cdot(x^2 + y^2)}{5\cdot x^2}\right) = \mathrm{LN}\left(\frac{x}{2}\right), y\right)$$

$$\#106:\quad y = -\frac{\sqrt{2}\cdot\sqrt{x}\cdot\sqrt{(5 - 2\cdot x)}}{2} \vee y = \frac{\sqrt{2}\cdot\sqrt{x}\cdot\sqrt{(5 - 2\cdot x)}}{2}$$

This turns out to be a family of circles with centers on the *x*-axis. The solution containing point P is one of the semicircles from #106. The direction field for only one single point gives the tangent of the integral in this point. (Plot #106 - #109 and see the circle, the point and the tangent ($-2 \le t \le 2$).

$$\#107:\quad \mathrm{FELD}\left(\frac{y^2 - x^2}{2\cdot x\cdot y}, x, y, 2, 2, 1, -1, -1, 1\right)$$

$$\#108:\quad \left[\left[\left[\frac{4\cdot t}{5} + 2, \frac{3\cdot t}{5} - 1\right]\right]\right]$$

$$\#109:\quad [2, -1]$$

$$\#110:\quad \mathrm{Logarithm} := \mathrm{Expand}$$

$$\#111:\quad \mathrm{HOMOGENEOUS}\left(\frac{2\cdot x\cdot y}{y^2 - x^2}, x, y, 2, -1\right)$$

$$\#112:\quad -\frac{\mathrm{LN}(y\cdot(y^2 - 3\cdot x^2))}{3} + \mathrm{LN}(x) + \frac{\mathrm{LN}(11)}{3} - \mathrm{LN}(2) = \mathrm{LN}(x) - \mathrm{LN}(2)$$

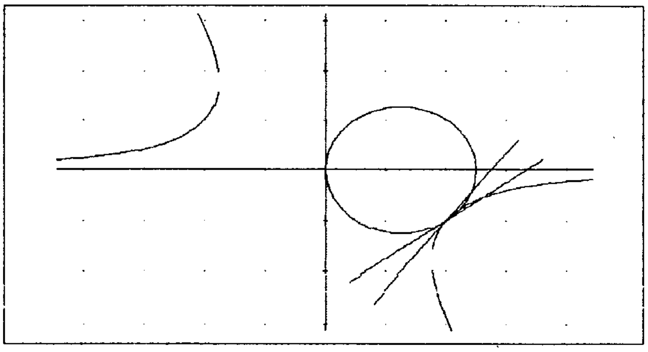
$$\#113:\quad \hat{e}^{(-\mathrm{LN}(y\cdot(y^2 - 3\cdot x^2))/3 + \mathrm{LN}(x) + \mathrm{LN}(11)/3 - \mathrm{LN}(2) = \mathrm{LN}(x) - \mathrm{LN}(2))\cdot 3}$$

$$\#114:\quad \frac{11\cdot x^3}{8\cdot y\cdot(y^2 - 3\cdot x^2)} = \frac{x^3}{8}$$

$$\#115:\quad \left(\frac{11\cdot x^3}{8\cdot y\cdot(y^2 - 3\cdot x^2)} = \frac{x^3}{8}\right)\cdot\frac{8\cdot y\cdot(y^2 - 3\cdot x^2)}{x^3}$$

$$\#116:\quad 11 = y\cdot(y^2 - 3\cdot x^2)$$

In 1991 we couldn´t do implicit plots. Now we can. In DNL#3 we solved equation #116 for y and received three solutions (= three explicit branches of the integral cuve). This is the plot from DNL#1. There is a third part which is outside of this plot window.



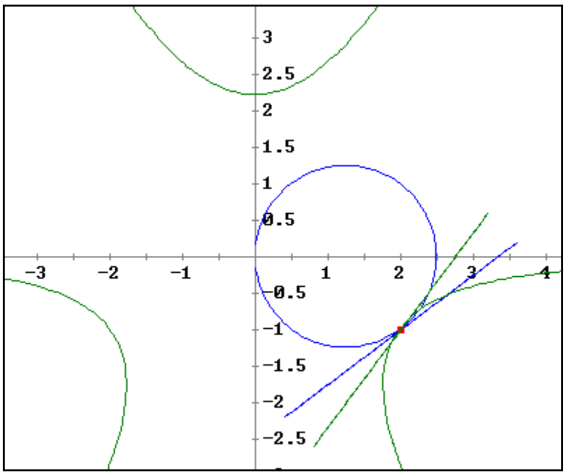
Without any doubt I believed Derive, but today I know a bit more:

Check if both points (2|-1) and (-2|1) really fulfill equation #116?

#117: $\text{FELD}\left(\frac{2\cdot x\cdot y}{y^2-x^2}, x, y, 2, 2, 1, -1, -1, 1\right)$

#118: $\left[\left[\left[\frac{3\cdot t}{5}+2, \frac{4\cdot t}{5}-1\right]\right]\right]$

The final plot shows both curves together with the tangents. And we see that Derive 5 delivers another plot. Solving #116 now for y and plotting the three branches gives exact the same figure – and checking the points works, too.



Example 11: Find the curves which intersect the circles $x^2 + y^2 = r^2$ at angles of $\pi/4$.

Plot some circles and a family of curves representing the general solution of the underlying differential equation.

This problem leads to the homogeneous DE $y' = \frac{y-x}{y+x}$

We start working with rectangular coordinates and switch later to polar coordinates. Compare my "do it yourself" GEN_SOL_HOM with professional HOMEGENEOUS_GEN:

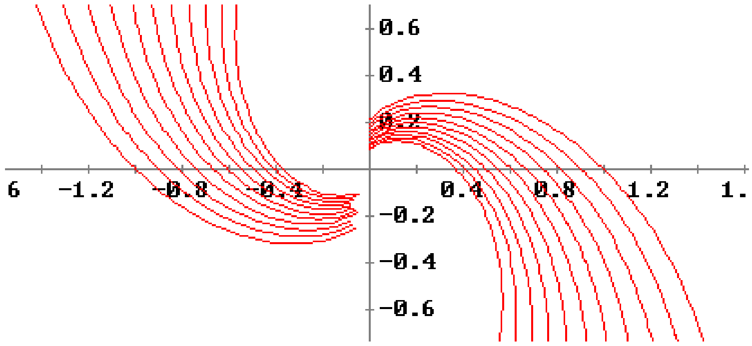
#122: $\text{GEN\_SOL\_HOM}\left(\frac{y-x}{y+x}\right)$

#123: $-\text{ATAN}\left(\frac{y}{x}\right) - \text{LN}(\sqrt{(x^2+y^2)}) = c$

#124: $\text{HOMOGENEOUS\_GEN}\left(\frac{y-x}{y+x}\right)$

#125: $-\text{ATAN}\left(\frac{y}{x}\right) - \text{LN}\left(\frac{\sqrt{(x^2+y^2)}}{x}\right) = \text{LN}(x) + c$

$$\#126:\ \text{VECTOR}\left(-\ \text{ATAN}\left(\frac{y}{x}\right) - \frac{\text{LN}(x^2 + y^2)}{2} = c,\ c,\ 0,\ 1,\ 0.1\right)$$



These partially plotted spirals are not really satisfying me. We switch to polar coordinates substituting for $x = r\cos(\varphi)$ and for $y = r\sin(\varphi)$.

$$\#127:\ r :\in \text{Real}\ (0,\ \infty)$$

$$\#128:\ -\ \text{ATAN}\left(\frac{r\cdot\text{SIN}(\varphi)}{r\cdot\text{COS}(\varphi)}\right) - \frac{\text{LN}((r\cdot\text{COS}(\varphi))^2 + (r\cdot\text{SIN}(\varphi))^2)}{2} = c$$

$$\#129:\ \pi\cdot\text{FLOOR}\left(\frac{\varphi}{\pi} + \frac{1}{2}\right) - \text{LN}(r) - \varphi = c$$

$$\#130:\ \pi\cdot\text{FLOOR}\left(\frac{\varphi}{\pi} + \frac{1}{2}\right) - \text{LN}(r) - \varphi = -\ \text{LN}(c)$$

$$\#131:\ \text{SOLVE}\left(\pi\cdot\text{FLOOR}\left(\frac{\varphi}{\pi} + \frac{1}{2}\right) - \text{LN}(r) - \varphi = -\ \text{LN}(c),\ r\right)$$

$$\#132:\ r = c\cdot\hat{e}^{\pi\cdot\text{FLOOR}(\varphi/\pi + 1/2) - \varphi}$$

$$\#134:\ -\ \text{LN}(r) - \varphi = c$$

$$\#135:\ -\ \text{LN}(r) - \varphi = -\ \text{LN}(c)$$

$$\#136:\ \text{SOLVE}(-\ \text{LN}(r) - \varphi = -\ \text{LN}(c),\ r)$$

$$\#137:\ r = c\cdot\hat{e}^{-\varphi}$$

$$\#138:\ \text{VECTOR}(c\cdot\hat{e}^{-\varphi},\ c,\ 0.5,\ 5,\ 0.5)$$

$$\#139:\ \text{VECTOR}(c,\ c,\ 0.5,\ 5,\ 0.5)$$

$$\#140:\ 2$$

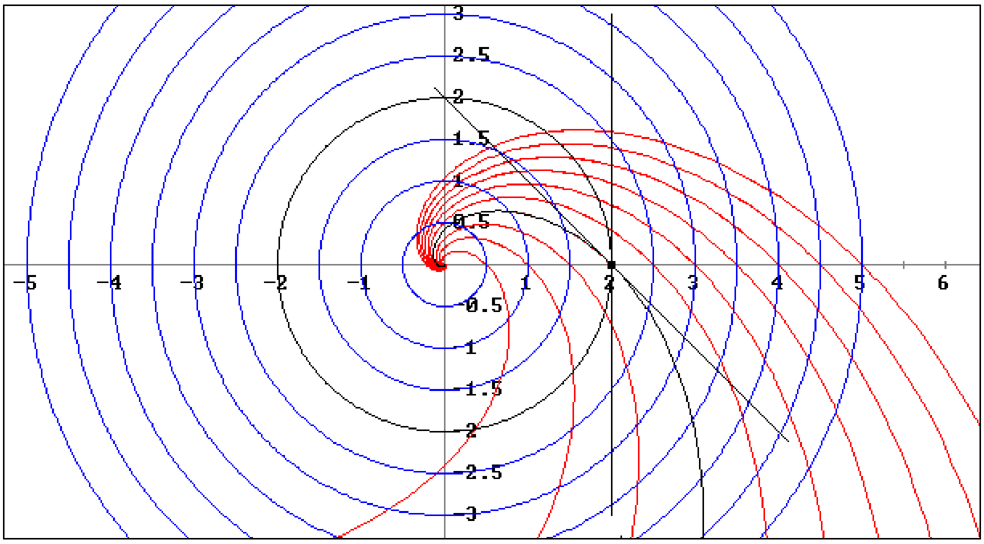
$$\#141:\ 2\cdot\hat{e}^{-\varphi}$$

$$\#142:\ [2,\ 0]$$

$$\#143:\ \text{FELD}\left(\frac{y - x}{y + x},\ x,\ y,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1\right)$$

$$\#144:\ \left[\left[\left[\frac{\sqrt{2}\cdot t}{2} + 2,\ -\ \frac{\sqrt{2}\cdot t}{2}\right]\right]\right]$$

$$\#145:\ [2,\ t]$$



A family of circles together with a family of integral curves. The tangents in point (2|0) are plotted

Example 12: A point moves on a curve in *x*-*y*-plane in such a way that the angle formed by the tangent of the curve and the *x*-axis is three times the angle between the radius vector and the *x*-axis. Find the Cartesian equation of the family of curves satisfying this condition.

Plot a family of curves, and plot the special curve containing point P(-3|-2).

Using $y' = \tan(\alpha) = \tan(3ß)$ and $\tan(ß) = y/x$ you find after some calculations – let Derive do this job – ,applying trigonometric identities the homogeneous differential equation $y' = \dfrac{3x^2y - y^3}{x^3 - 3xy^2}$.

#185: TAN(3·β)

#186: Trigonometry := Expand

#187: $$\frac{4\cdot \text{SIN}(\beta)\cdot \text{COS}(\beta)}{1 - 4\cdot \text{SIN}(\beta)^2} + \frac{\text{SIN}(\beta)}{\text{COS}(\beta)\cdot(4\cdot \text{SIN}(\beta)^2 - 1)}$$

#188: $$\frac{4\cdot \text{SIN}\left(\text{ATAN}\left(\frac{y}{x}\right)\right)\cdot \text{COS}\left(\text{ATAN}\left(\frac{y}{x}\right)\right)}{1 - 4\cdot \text{SIN}\left(\text{ATAN}\left(\frac{y}{x}\right)\right)^2} + \frac{\text{SIN}\left(\text{ATAN}\left(\frac{y}{x}\right)\right)}{\text{COS}\left(\text{ATAN}\left(\frac{y}{x}\right)\right)\cdot\left(4\cdot \text{SIN}\left(\text{ATAN}\left(\frac{y}{x}\right)\right)^2 - 1\right)}$$

#189: $$\frac{y\cdot(3\cdot x^2 - y^2)}{x\cdot(x^2 - 3\cdot y^2)}$$

This is the original file from 1991:

#16: $\mathrm{GEN\_SOL\_HOM}\left(\dfrac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}\right)$

#17: $-\mathrm{LN}\left(\dfrac{x^2 + y^2}{\sqrt{x}\cdot\sqrt{y}}\right) = c$

#18: $-\mathrm{LN}\left(\dfrac{x^2 + y^2}{\sqrt{x}\cdot\sqrt{y}}\right) = -\mathrm{LN}(c)$

#19: $\dfrac{x^2 + y^2}{\sqrt{x}\cdot\sqrt{y}} = c$

#20: We cannot do implicit plots!

#21: So we use Polar coordinates!

#22: $\dfrac{(r\cdot\mathrm{COS}(\varphi))^2 + (r\cdot\mathrm{SIN}(\varphi))^2}{\sqrt{(r\cdot\mathrm{COS}(\varphi))}\cdot\sqrt{(r\cdot\mathrm{SIN}(\varphi))}} = c$

#23: $\dfrac{r}{\sqrt{(\mathrm{COS}(\varphi))}\cdot\sqrt{(\mathrm{SIN}(\varphi))}} = c$

#24: $\dfrac{r^2}{\mathrm{COS}(\varphi)\cdot\mathrm{SIN}(\varphi)} = c^2$

#25: $\dfrac{2\cdot r^2}{\mathrm{SIN}(2\cdot\varphi)} = c^2$

This is Bernoulli´s Leminscate!

#26: $r = -\dfrac{\sqrt{2}\cdot c\cdot\sqrt{(\mathrm{SIN}(2\cdot\varphi))}}{2}$

#27: $r = \dfrac{\sqrt{2}\cdot c\cdot\sqrt{(\mathrm{SIN}(2\cdot\varphi))}}{2}$

#28: $\dfrac{(-3)^2 + (-2)^2}{\sqrt{(-3)}\cdot\sqrt{(-2)}} = c$

#29: $-\dfrac{13\cdot\sqrt{6}}{6} = c$

#30: $r = \dfrac{13\cdot\sqrt{3}\cdot\sqrt{(\mathrm{SIN}(2\cdot\varphi))}}{6}$

The following commands are the base for the plot. It is important to switch between rectangular (#31, #32) and polar coordinates (#33).

It is one of Derive´s unique features that one is able to have both coordinates systems available in the same plot.

For plotting the family of lemniscates set $-\pi/2 \le \varphi \le \pi/2$.

#31: $\mathrm{FELD}\left(\dfrac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}, x, y, -3, -3, 1, -2, -2, 1\right)$

#32: $\left[\left[\left[\dfrac{9\cdot\sqrt{13}\cdot t}{169} - 3, -\dfrac{46\cdot\sqrt{13}\cdot t}{169} - 2\right]\right]\right]$

#33: $\mathrm{VECTOR}\left(\dfrac{\sqrt{2}\cdot c\cdot\sqrt{(\mathrm{SIN}(2\cdot\varphi))}}{2}, c, 0.5, 5, 0.5\right)$

#1: $\mathrm{HOMOGENEOUS}\left(\dfrac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}, x, y, -3, -2\right)$

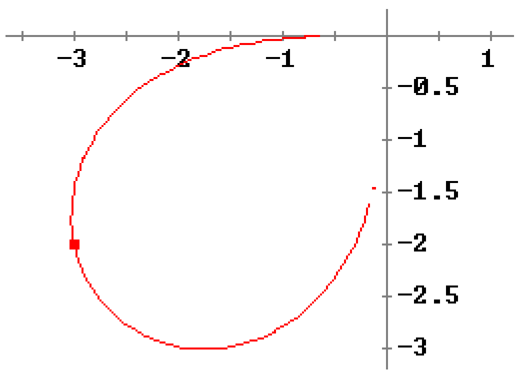
#2: $-\dfrac{\mathrm{LN}\left(\dfrac{54\cdot(x^2 + y^2)^2}{169\cdot x^3\cdot y}\right)}{2} = \mathrm{LN}\left(\dfrac{x}{3}\right) - \pi\cdot\hat{\imath}$

e^(#2):

#3: $\dfrac{13\cdot\sqrt{6}}{18\cdot(x^2 + y^2)\cdot\sqrt{\left(\dfrac{1}{x^3\cdot y}\right)}} = -\dfrac{x}{3}$

#4: [-3, -2]

This cannot be plotted in DERIVE 5, but interestingly enough it can be plotted in DERIVE 6. And here we find one part of the lemniscate.



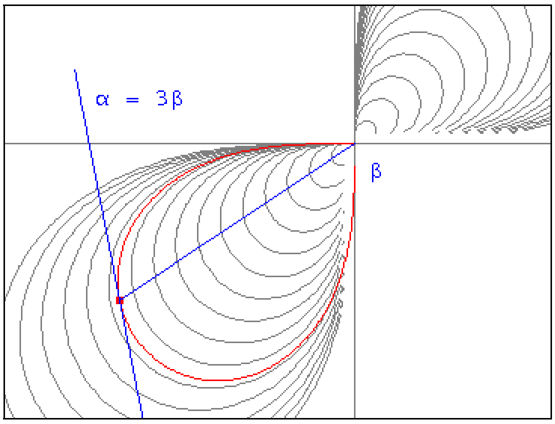
Finally we plot a family of the complete lemniscates together with the tangent in P(-3|-2) and the ray from 0 to P. We will show that the relation between the angles given in the task is true.

$$\beta := \pi + \text{ATAN}\left(-\frac{2}{-3}\right)$$

$$\alpha := \text{ATAN}\left(\text{SUBST}\left(\frac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}, [x, y], [-3, -2]\right)\right)$$

$$\alpha - 3\cdot\beta = -4\cdot\pi$$

$$3\cdot\beta - \alpha = 4\cdot\pi$$



Example 13: Show that the given differential equations are exact, give their general solutions and if there is a point given find also the respective special solution.

a) $2x^2 - y^2 + y - y'(2xy - x + 4y) = 0$; P(1/-1.5)

b) $2x + e^x \ln y + (e^x y')/y = 0$

c) $2xy - y'(2x^2 + y) = 0$; Q(-2|3)

DEs of form $p(x,y) + q(x,y)\,y' = 0$ may be exact. Exactness seldom is obvious, so `EXACT_TEST(p,q,x,y)` or `EXACT_IF_0(p,q,x,y)` in earlier versions will test this attribute.

EXACT_TEST is not included in recent ODE1.MTH. The test has been included into the EXACT()-functions for solving the DEs. For didactical reasons it might be useful to ask students to produce a selfmade EXACT_TEST-function. In the following I give the original EXACT_TEST from 1991 and an updated form for DERIVE 5 (and 6, of course).

$$\#246:\ \text{EXACT\_TEST}(p, q, x, y) := \frac{d}{dy}\,p - \frac{d}{dx}\,q$$

```
         EXACT_TEST_N(p, q, x, y) :=
           If DIF(p, y) - DIF(q, x) = 0
#247:         "exact"
              "not exact"
              "not exact"
```

$$\text{EXACT\_TEST}(2\cdot x^2 - y^2 + y, -(2\cdot x\cdot y - x + 4\cdot y)) = 0$$

$$\text{EXACT\_TEST\_N}(2\cdot x^2 - y^2 + y, -(2\cdot x\cdot y - x + 4\cdot y)) = \text{exact}$$

$$\text{EXACT\_TEST}\left(2\cdot x + \hat{e}^x\cdot \text{LN}(y), \frac{\hat{e}^x}{y}\right) = 0$$

$$\text{EXACT\_TEST\_N}\left(2\cdot x + \hat{e}^x\cdot \text{LN}(y), \frac{\hat{e}^x}{y}\right) = \text{exact}$$

$$\text{EXACT\_TEST}(2\cdot x\cdot y, -2\cdot x^2 - y) = 0$$

$$\text{EXACT\_TEST\_N}(2\cdot x\cdot y, -2\cdot x^2 - y) = \text{not exact}$$

Example 13 a

$$\#254:\ \mathrm{EXACT}(2\cdot x^2 - y^2 + y,\ -(2\cdot x\cdot y - x + 4\cdot y))$$

$$\#255:\ \frac{2\cdot x^3}{3} + x\cdot y\cdot(1 - y) - \frac{6\cdot y^2 + 2\cdot x0^3 + 3\cdot x0\cdot y0\cdot(1 - y0) - 6\cdot y0^2}{3} = 0$$

$$\#256:\ \mathrm{EXACT}\left(2\cdot x^2 - y^2 + y,\ -(2\cdot x\cdot y - x + 4\cdot y),\ x,\ y,\ 1,\ -\frac{3}{2}\right)$$

$$\#257:\ \frac{2\cdot x^3}{3} + x\cdot y\cdot(1 - y) - \frac{24\cdot y^2 - 91}{12} = 0$$

$$\#258:\ 8\cdot x^3 + 12\cdot x\cdot y\cdot(1 - y) - 24\cdot y^2 + 91 = 0$$

$$\#259:\ \mathrm{EXACT\_GEN}(2\cdot x^2 - y^2 + y,\ -(2\cdot x\cdot y - x + 4\cdot y))$$

$$\#260:\ \frac{2\cdot x^3}{3} + x\cdot y\cdot(1 - y) - 2\cdot y^2 = c$$

$$\#261:\ -\frac{91}{12} = c$$

leads to the same solution (#258)

Example 13 b

$$\#262:\ \mathrm{EXACT\_GEN}\left(2\cdot x + \hat{e}^x\cdot \mathrm{LN}(y),\ \frac{\hat{e}^x}{y}\right)$$

$$\#263:\ \hat{e}^x\cdot \mathrm{LN}(y) + x^2 = c$$

$$\#264:\ \mathrm{SOLVE}(\hat{e}^x\cdot \mathrm{LN}(y) + x^2 = c,\ y)$$

$$\#265:\ y = \hat{e}^{\hat{e}^{-x}\cdot(c - x^2)}$$

$$\#266:\ \mathrm{SOLVE}\left(\mathrm{EXACT}\left(2\cdot x + \hat{e}^x\cdot \mathrm{LN}(y),\ \frac{\hat{e}^x}{y}\right),\ y\right)$$

$$\#267:\ y = \mathrm{IF}\left(\hat{e}^{x0 - x} \le 1 \vee y0 > 0,\ \hat{e}^{\hat{e}^{-x}\cdot(\hat{e}^{x0}\cdot \mathrm{LN}(y0) - x^2 + x0^2)}\right)$$

Example 13 c

This equation is obviously not exact (see above). We assume, that we know an integrating factor $\mu = 1/y^3$. Now we are able to use `USE_INTEG_FCTR(μ,p,q,x,y,x0,y0)`. (*This function is not available in later Derive versions*.)

#268: USE_INTEG_FCTR(μ, p, q, x, y, x0, y0) := EXACT(μ·p, μ·q, x, y, x0, y0)

#269: $\text{USE\_INTEG\_FCTR}\left(\frac{1}{y^3}, 2\cdot x\cdot y, -2\cdot x^2 - y\right)$

#270: $\frac{y0^2\cdot x^2 - y\cdot(y\cdot(x0^2 + y0) - y0^2)}{y0^2\cdot y^2} = 0$

#271: $\text{USE\_INTEG\_FCTR}\left(\frac{1}{y^3}, 2\cdot x\cdot y, -2\cdot x^2 - y, x, y, -2, 3\right)$

#272: $\frac{9\cdot x^2 - y\cdot(7\cdot y - 9)}{9\cdot y^2} = 0$

#273: $\text{GEN\_INT\_FCT}\left(\frac{1}{y^3}, 2\cdot x\cdot y, -2\cdot x^2 - y\right)$

#274: $\frac{x^2}{y^2} + \frac{1}{y} = c$

#275: $\frac{7}{9} = c$

Solution: $\frac{x^2}{y^2} + \frac{1}{y} = \frac{7}{9}$

In times of Derive 5 and higher all the auxiliary functions (see later in example 14) are implemented in on function to solve immediately differential equations, which are supposed to be soluble by applying an integrating factor: I recommend to compare the recent ODE1.MTH file with the auxiliary procedures which where necessary in 1991. I can imagine that it might be useful for students studying differential equations to follow the old "recipes" (which can support understanding the solving process).

#277: $\text{INTEGRATING\_FACTOR}(2\cdot x\cdot y, -2\cdot x^2 - y, x, y, -2, 3)$

#278: $\frac{9\cdot x^2 - y\cdot(7\cdot y - 9)}{9\cdot y^2} = 0$

Compare with #275

File ODE1_1991.dfw is the original ODE1.MTH from Derive 2.05 together with ODE1_EXT.MTH from DNL#3.

Example 14: Try to find a solution of the DEs given below using the auxiliary functions of ODE1.MTH and ODE1_EXT.MTH concerning the application of integrating factors. Give the integrating factors.

a) $2xy - y'(2x^2 + y) = 0$

b) $(x^2 + y^2)(x\,dy - y\,dx) = (a + x)\,x^4\,dx$

c) $(x \cos y - y \sin y)dy + (x \sin y + y \cos y)dx = 0;\ P(1|\pi/2)$

d) $(2x^3y^2 - y)dx = (x - 2x^2y^3)dy$

The DEs must be brought into the form $p(x,y) + q(x,y)\,y' = 0$.
DERIVE can be helpful to find an integrating factor μ.

Preloaded is File ODE1.MTH, copyright (c) 1990 by Soft Warehouse, Inc.
& ODE1_EXT from Derive Newsletter #3

#41: $\text{FREE\_OF\_INDEPENDENT\_TEST}(2\cdot x\cdot y,\ -2\cdot x^2 - y) = -\dfrac{3}{y}$

simplifies to an expression free of x. Hence we get the solution by means of the next function:

#42: $\text{FREE\_OF\_INDEPENDENT}(2\cdot x\cdot y,\ -2\cdot x^2 - y)$

#43: $\dfrac{y0^2\cdot x^2 - y\cdot(y\cdot(x0^2 + y0) - y0^2)}{y0^2\cdot y^2} = 0$

#44: $\text{INT\_FCT}\left(-\dfrac{3}{y},\ y\right) = \dfrac{1}{y^3}$

Follow example 13c from above

Example 14 b

$$(x^2 + y^2)\,(x\,y' - y) = (a + x)x^4$$

The given equation must be rewritten as

$$-x^2 y - y^3 - a\,x^4 - x^5 + (x^3 + x\,y^2)\,y' = 0$$
$$x^2 y + y^3 + a\,x^4 + x^5 + (-x^3 - x\,y^2)\,y' = 0$$

#45: $\text{FREE\_OF\_INDEPENDENT\_TEST}(x^5 + a\cdot x^4 + x^2\cdot y + y^3,\ -x^3 - x\cdot y^2)$

#46: $-\dfrac{4\cdot(x^2 + y^2)}{x^5 + a\cdot x^4 + x^2\cdot y + y^3}$

which is not free of independent x, but

#47: $\text{FREE\_OF\_DEPENDENT\_TEST}(x^5 + a\cdot x^4 + x^2\cdot y + y^3,\ -x^3 - x\cdot y^2)$

#48: $-\dfrac{4}{x}$

which is free of dependent y. This leads to

#49: $\text{FREE\_OF\_DEPENDENT}(x^5 + a\cdot x^4 + x^2\cdot y + y^3,\ -x^3 - x\cdot y^2)$

#50: $\dfrac{x^2}{2} + a\cdot x - \dfrac{y}{x} - \dfrac{y^3}{3\cdot x^3} - a\cdot x0 - \dfrac{3\cdot x0^5 - 6\cdot x0^2\cdot y0 - 2\cdot y0^3}{6\cdot x0^3} = 0$

#51: $\text{INT\_FCT}\left(-\dfrac{4}{x},\ x\right) = \dfrac{1}{x^4}$ integrating factor!!

#52: $\text{GEN\_INT\_FCT}\left(\dfrac{1}{x^4},\ x^5 + a\cdot x^4 + x^2\cdot y + y^3,\ -x^3 - x\cdot y^2\right)$

#53: $\dfrac{x^2}{2} + a\cdot x - \dfrac{y}{x} - \dfrac{y^3}{3\cdot x^3} = c$

Now in DERIVE5:

$$\#279:\ \mathrm{INTEGRATING\_FACTOR}(x^5 + a\cdot x^4 + x^2\cdot y + y^3,\ - x^3 - x\cdot y^2)$$

$$\#280:\ \frac{x^2}{2} + a\cdot x - \frac{y}{x} - \frac{y^3}{3\cdot x^3} - a\cdot x0 - \frac{3\cdot x0^5 - 6\cdot x0^2\cdot y0 - 2\cdot y0^3}{6\cdot x0^3} = 0$$

Example 14 c

After rewriting we proceed:

$$\#54:\ \mathrm{FREE\_OF\_DEPENDENT\_TEST}(x\cdot \mathrm{SIN}(y) + y\cdot \mathrm{COS}(y),\ x\cdot \mathrm{COS}(y) - y\cdot \mathrm{SIN}(y))$$

$$\#55:\ 1$$

$$\#56:\ \mathrm{FREE\_OF\_DEPENDENT}\left(x\cdot \mathrm{SIN}(y) + y\cdot \mathrm{COS}(y),\ x\cdot \mathrm{COS}(y) - y\cdot \mathrm{SIN}(y),\ x,\ y,\ 1,\ \frac{\pi}{2}\right)$$

$$\#57:\ \hat{e}^{x}\cdot (y\cdot \mathrm{COS}(y) + (x - 1)\cdot \mathrm{SIN}(y)) = 0$$

$$\#58:\ \mathrm{INT\_FCT}(1,\ x) = \hat{e}^{x}$$

$$\#59:\ \mathrm{GEN\_INT\_FCT}(\hat{e}^{x},\ x\cdot \mathrm{SIN}(y) + y\cdot \mathrm{COS}(y),\ x\cdot \mathrm{COS}(y) - y\cdot \mathrm{SIN}(y))$$

$$\#60:\ \hat{e}^{x}\cdot (y\cdot \mathrm{COS}(y) + (x - 1)\cdot \mathrm{SIN}(y)) = c$$

substitute for x and v

$$\#61:\ 0 = c$$

Please check the result with the respective DERIVE5 function!

Example 14 d

$$\#62:\ \mathrm{FREE\_OF\_INDEPENDENT\_TEST}(2\cdot x^3\cdot y^2 - y,\ 2\cdot x^2\cdot y^3 - x) = \frac{4\cdot x\cdot (y^2 - x^2)}{2\cdot x^3\cdot y - 1}$$

$$\#63:\ \mathrm{FREE\_OF\_DEPENDENT\_TEST}(2\cdot x^3\cdot y^2 - y,\ 2\cdot x^2\cdot y^3 - x) = \frac{4\cdot y\cdot (x^2 - y^2)}{2\cdot x\cdot y^3 - 1}$$

neither free of x nor of y; we test other possibilities for the integrating factor:

We proceed in DNL#3 (1991) – style:

$$\#64:\ \mathrm{EXACT\_TEST}(x\cdot y\cdot (2\cdot x^3\cdot y^2 - y),\ x\cdot y\cdot (2\cdot x^2\cdot y^3 - x)) = 6\cdot x^2\cdot y^2\cdot (x^2 - y^2)$$

x y is no integrating factor

$$\#65:\ \mathrm{EXACT\_TEST}((x + y)\cdot(2\cdot x^3\cdot y^2 - y), (x + y)\cdot(2\cdot x^2\cdot y^3 - x)) = 4\cdot x^4\cdot y + 6\cdot x^3\cdot y^2 - 6\cdot x^2\cdot y^3 + x\cdot(1 - 4\cdot y^4) - y$$

no luck with (x + y)

$$\#66:\ \mathrm{EXACT\_TEST}(x^2\cdot y^2\cdot(2\cdot x^3\cdot y^2 - y), x^2\cdot y^2\cdot(2\cdot x^2\cdot y^3 - x)) = 6\cdot x^2\cdot y^2\cdot(x^2 - y^2)$$

no success with x^2 y^2

$$\#67:\ \mathrm{EXACT\_TEST}\left(\frac{1}{x^2\cdot y^2}\cdot(2\cdot x^3\cdot y^2 - y), \frac{1}{x^2\cdot y^2}\cdot(2\cdot x^2\cdot y^3 - x)\right) = 0$$

This was the right guess! Let´s finish the solution

$$\#68:\ \mathrm{GEN\_INT\_FCT}\left(\frac{1}{x^2\cdot y^2}, 2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x\right)$$

$$\#69:\ \frac{x^3\cdot y + x\cdot y^3 + 1}{x\cdot y} = c$$

with `MONOMIAL_TEST(p,q,x,y)` *DERIVE* can find an integrating factor of the form $x^m y^n$ if there exists one.

This is the way we do it now:

$$\#281:\ \mathrm{INTEGRATING\_FACTOR\_GEN}(2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x)$$

#282: inapplicable

$$\#283:\ \mathrm{MONOMIAL\_TEST}(2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x) = \frac{1}{x^2\cdot y^2}$$

$$\#284:\ \mathrm{INTEGRATING\_FACTOR\_GEN}\left(\frac{1}{x^2\cdot y^2}\cdot(2\cdot x^3\cdot y^2 - y), \frac{1}{x^2\cdot y^2}\cdot(2\cdot x^2\cdot y^3 - x)\right)$$

$$\#285:\ \frac{x^3\cdot y + x\cdot y^3 + 1}{x\cdot y} = c$$

```
Example 15: (x - 2y + 5)dx + (2x - y + 4)dy = 0
            a) Find the general solution using both (x0|y0) and constant c.
            b) Find the special solution with x = 1, y = 1.
            c) Sketch the integral curve and the direction field.
```

`LIN_FRAC(r, a, b, d, p, q, k, x, y, x0, y0)` simplifies to an implicit solution of a linear fractional equation $y' = r\cdot((a\cdot x+b\cdot y+d)/(p\cdot x+q\cdot y+k))$. However if q·a – p·b = 0, instead use `FUN_LIN_CCF` described above. (Online – Help).

We will tackle this problem in the DERIVE5/6 mode only and add the 1991 plot for c)

$$\#286:\ \mathrm{LIN\_FRAC}\left(\frac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, -1, 2, -5, 2, -1, 4\right)$$

$$\#287:\ \mathrm{LN}\left(\left|\frac{x + 1}{x0 + 1}\right|\cdot\sqrt{\left(-\frac{x - y + 3}{(x + y - 1)^3}\right)}\right) - \mathrm{LN}\left(\sqrt{\left(-\frac{x0 - y0 + 3}{(x0 + y0 - 1)^3}\right)}\right) = \mathrm{LN}(x + 1) - \mathrm{LN}(x0 + 1)$$

#288: $\text{LIN\_FRAC\_GEN}\left(\dfrac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, -1, 2, -5, 2, -1, 4\right)$

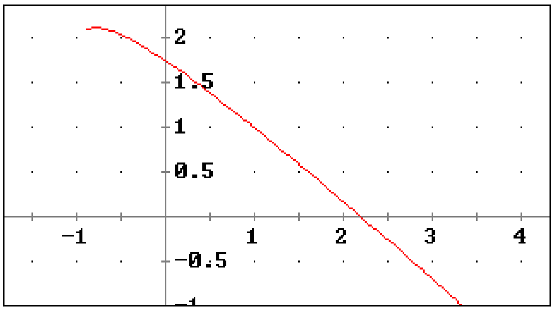
#289: $$\frac{\text{LN}\left(-\dfrac{(x + 1)^2\cdot(x - y + 3)}{(x + y - 1)^3}\right)}{2} = \text{LN}(x + 1) + c$$

#290: $$|x + 1|\cdot\sqrt{\left(-\frac{x - y + 3}{(x + y - 1)^3}\right)} = \hat{e}^c\cdot(x + 1)$$

#291: $\text{LIN\_FRAC}\left(\dfrac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, -1, 2, -5, 2, -1, 4, x, y, 1, 1\right)$

#292: $$\frac{\text{LN}\left(-\dfrac{(x + 1)^2\cdot(x - y + 3)}{12\cdot(x + y - 1)^3}\right)}{2} - \frac{\pi\cdot\hat{\imath}}{2} = \text{LN}\left(\frac{x + 1}{2}\right)$$

#293: $$\frac{\sqrt{3}\cdot|x + 1|\cdot\sqrt{\left(\dfrac{x - y + 3}{(x + y - 1)^3}\right)}\cdot\text{SIGN}((x + y - 1)\cdot(x - y + 3))}{6} = \frac{x + 1}{2}$$



This is the graph of #293. Building e^(2#292) leads to expression #294 which can be plotted together with the direction field. As you can see a discontinuity seems to appear at (-1|2).*

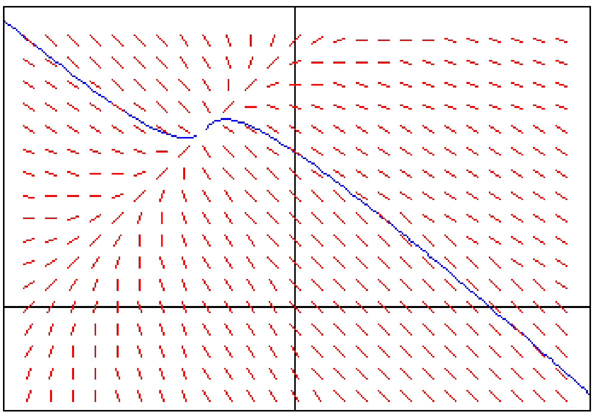
(Let students deduce this point from the task!)

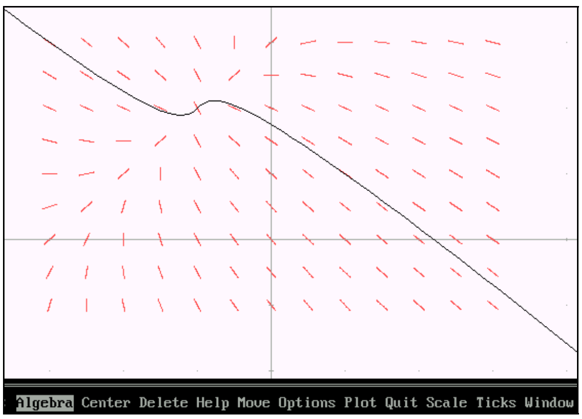
In 1991 we received the same solution – fortunately – but we couldn´t do the implicit plot. So I solved #294 for y and plotted the solutions together with the results of my FELD-function.

#294: $$\frac{x - y + 3}{3\cdot(x + y - 1)^3} = 1$$

#295: $\text{DIRECTION\_FIELD}\left(\dfrac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, x, -3, 3, 24, y, -2, 3, 20\right)$

The right graph is "Derive for DOS made".





Example 16: Clairaut equation $y = x\,y' + y' = y'^2$

Find the general solution, obtain the singular solution, plot the graph.

Give the special solution(s) containing P(2|1).

We bring the equation into the form $p(x\,v - y) = q(v)$ and then use `CLAIRAUT(p,q,x,y,v,c1)` taking v as a variable instead of y'.

#296: $y = x \cdot v + v - v^2$

#297: $x \cdot v - y = v^2 - v$

#298: $\text{CLAIRAUT}(x \cdot v - y,\ v^2 - v)$

#299: $\left[c \cdot x - y - c^2 + c = 0,\ x - 2 \cdot v + 1 = 0\right]$

In order to find the singular solution we solve the 2nd component of #299 for v and then substitute for v in the given differential equation (#296).

#300: $\text{SOLVE}(x - 2 \cdot v + 1 = 0,\ v) = \left[v = \dfrac{x + 1}{2}\right]$

#301: $\text{SUBST}\left(y = x \cdot v + v - v^2,\ v,\ \dfrac{x + 1}{2}\right)$

#302: $y = \dfrac{(x + 1)^2}{4}$

This parabola is the singular solution.

#303: $\text{SOLVE}(c \cdot x - y - c^2 + c = 0,\ y)$

#304: $y = c \cdot (x - c + 1)$

The 1st component of #299 gives the general solution.

#305: $\text{VECTOR}(y = c \cdot (x - c + 1),\ c,\ -3,\ 3,\ 0.5)$

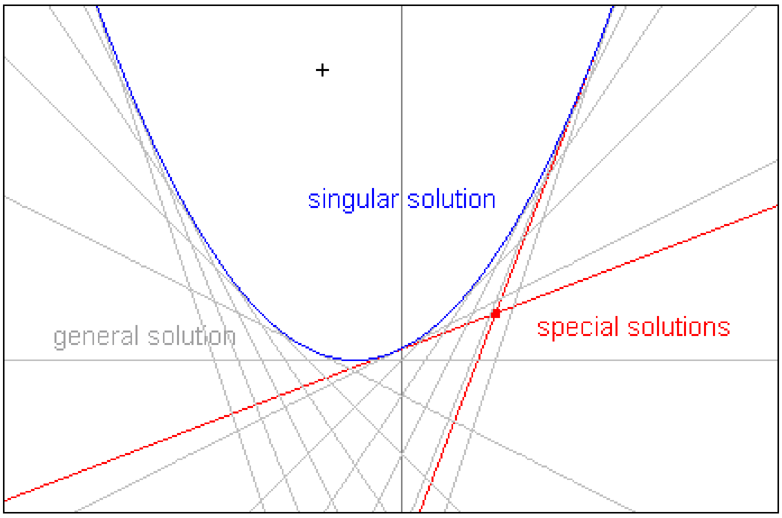
#306: $[2,\ 1]$

#307: $\text{SOLUTIONS}(\text{SUBST}(y = c \cdot (x - c + 1),\ [x,\ y],\ [2,\ 1]),\ c)$

#308: $\left[\dfrac{\sqrt{5}}{2} + \dfrac{3}{2},\ \dfrac{3}{2} - \dfrac{\sqrt{5}}{2}\right]$

#309: $\text{VECTOR}\left(y = c \cdot (x - c + 1),\ c,\ \left[\dfrac{\sqrt{5}}{2} + \dfrac{3}{2},\ \dfrac{3}{2} - \dfrac{\sqrt{5}}{2}\right]\right)$

#310: $\left[y = \dfrac{(\sqrt{5} + 3) \cdot (2 \cdot x - \sqrt{5} - 1)}{4},\ y = \dfrac{(3 - \sqrt{5}) \cdot (2 \cdot x + \sqrt{5} - 1)}{4}\right]$



The special solution(s) are the two tangents from point P to the singular solution.

Example 17: Find a curve with its tangent´s intercept between the axes having constant length a = 2.

Give both general and singular solutions.

The problem leads to the differential equation $y = x\,y' \pm \frac{2y'}{\sqrt{(1+y'^2)}}$, which is of Clairaut-form.

#311: $c1 := x\cdot v - y = \frac{2\cdot v}{\sqrt{(1 + v^2)}}$

#312: $\text{CLAIRAUT}\left(x\cdot v - y, \frac{2\cdot v}{\sqrt{(1 + v^2)}}\right)$

#313: $\left[\frac{c\cdot x\cdot\sqrt{(c^2 + 1)} - y\cdot\sqrt{(c^2 + 1)} - 2\cdot c}{\sqrt{(c^2 + 1)}} = 0,\ x - \frac{2}{(v^2 + 1)^{3/2}} = 0\right]$

#314: $\text{SOLUTIONS}\left(x - \frac{2}{(v^2 + 1)^{3/2}} = 0,\ v\right)$

#315: $\left[\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)},\ -\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)}\right]$

#316: $\text{VECTOR}\left(c1,\ v,\ \left[\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)},\ -\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)}\right]\right)$

#317: $\left[x\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)} - y = 2^{2/3}\cdot x^{1/3}\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)},\ -x\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)} - y = -2^{2/3}\cdot x^{1/3}\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)}\right]$

#318: $y^2 = (2^{2/3} - x^{2/3})^3$

DERIVE2 was unable to solve #5 for v (see below), so I tried to find another way:

#5: $\frac{x\cdot(v^2 + 1)^{3/2} - 2}{(v^2 + 1)^{3/2}} = 0$

Substitute for v in the second component and in the equation: v = tan(φ).
Solve for x and y to obtain a parameter representation of the singular solution.

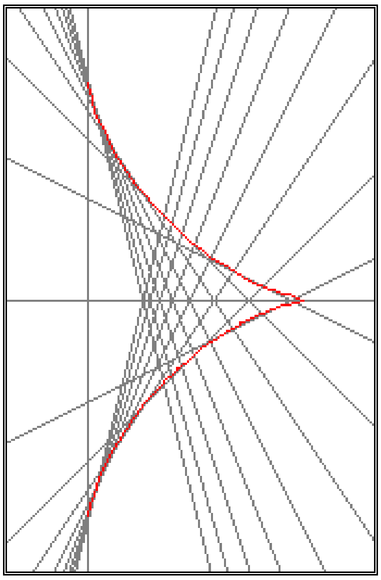
#6: $x - 2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)| = 0$

#7: $x = 2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)|$

#8: $2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)|\cdot\text{TAN}(\varphi) - y = \frac{2\cdot\text{TAN}(\varphi)}{\sqrt{(1 + \text{TAN}(\varphi)^2)}}$

#9: $y = -2\cdot\text{SIN}(\varphi)^3\cdot\text{SIGN}(\text{COS}(\varphi))$

#10: $[2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)|,\ -2\cdot\text{SIN}(\varphi)^3\cdot\text{SIGN}(\text{COS}(\varphi))]$



#318 (*DERIVE*5) and #10 (*DERIVE*2) give an astroid and the general solution is a family of lines (tangents of the astroid).

$$\#319:\ \text{SOLVE}\left(\frac{c\cdot x\cdot\sqrt{c^2+1} - y\cdot\sqrt{c^2+1} - 2\cdot c}{\sqrt{c^2+1}} = 0,\ y\right)$$

$$\#320:\ y = \frac{c\cdot(x\cdot\sqrt{c^2+1} - 2)}{\sqrt{c^2+1}}$$

$$\#321:\ \text{VECTOR}\left(y = \frac{c\cdot(x\cdot\sqrt{c^2+1} - 2)}{\sqrt{c^2+1}},\ c,\ -4,\ 4,\ 0.5\right)$$

Example 18: The following equations of form y = x p(y') + q(y') are called LAGRANGE differential equations.
Use LAGR(p,q,x,y,v,c) to solve the following equations:

a) $y = x\,(1 + y') + y'^2$

b) $y'^3 - 3y' = y - x$

c) $(x\,y' + y)^2 = y^2\,y'$

`LAGR(p,q,x,y,v,c)` with v instead of y' simplifies to a vector of four components with the last two of them giving the parameter form of the general solution. If one is able to solve the 2nd component for $c1$ one will obtain the singular solution by substituting for $c1$ in the 1st component. `LAGR()` is not part of ODE1.MTH but of ODE1_EXT.MTH. Look at the examples:

Example 18a

$$\#323:\ \left[y = x\cdot(c1 + 1) + c1^2,\ c1 + 1 = c1,\ x = c\cdot\hat{e}^{-v} - 2\cdot(v - 1),\ y = c\cdot\hat{e}^{-v}\cdot(v + 1) - v^2 + 2\right]$$

No singular solution because of `c1 + 1 = c1.`

Integral curves + direction field:

$$\#324:\ \left[c\cdot\hat{e}^{-v} - 2\cdot(v - 1),\ c\cdot\hat{e}^{-v}\cdot(v + 1) - v^2 + 2\right]$$

$$\#325:\ \text{VECTOR}(\left[c\cdot\hat{e}^{-v} - 2\cdot(v - 1),\ c\cdot\hat{e}^{-v}\cdot(v + 1) - v^2 + 2\right],\ c,\ -5,\ 5,\ 0.5)$$

$$\#326:\ \text{SOLVE}(y = x\cdot(1 + v) + v^2,\ v) = \left(v = \frac{\sqrt{x^2 - 4\cdot x + 4\cdot y} - x}{2} \lor v = -\frac{\sqrt{x^2 - 4\cdot x + 4\cdot y} + x}{2}\right)$$

$$\#327:\ \text{DIRECTION\_FIELD}\left(\frac{\sqrt{x^2 - 4\cdot x + 4\cdot y} - x}{2},\ x,\ -5,\ 5,\ 10,\ y,\ -4,\ 4,\ 8\right)$$

$$\#328:\ \text{DIRECTION\_FIELD}\left(\frac{\sqrt{x^2 - 4\cdot x + 4\cdot y} + x}{2},\ x,\ -5,\ 5,\ 10,\ y,\ -4,\ 4,\ 8\right)$$

Example 18b

$$\#329:\ \mathrm{LAGR}(1,\ v^3 - 3\cdot v)$$

$$\#330:\ \left[y = x + c1^3 - 3\cdot c1,\ 1 = c1,\ x = c + \frac{3\cdot v\cdot(v + 2)}{2},\ y = c + \frac{v^2\cdot(2\cdot v + 3)}{2}\right]$$

$$\#331:\ \mathrm{SUBST}(y = x + c1^3 - 3\cdot c1,\ c1,\ 1) = (y = x - 2)$$

singular solution!!

$$\#332:\ \mathrm{VECTOR}\left(\left[c + \frac{3\cdot v\cdot(v + 2)}{2},\ c + \frac{v^2\cdot(2\cdot v + 3)}{2}\right],\ c,\ -3,\ 3,\ 0.5\right)$$

Example 18c

$$\#333:\ \mathrm{SOLVE}((x\cdot v + y)^2 = y^2\cdot v,\ y) = \left(y = \frac{v\cdot x}{\sqrt{v} - 1} \vee y = -\frac{v\cdot x}{\sqrt{v} + 1}\right)$$

Let´s take the first solution:

$$\#334:\ \mathrm{LAGR}\left(\frac{v}{\sqrt{v} - 1},\ 0\right)$$

$$\#335:\ \left[y = \frac{c1\cdot x}{\sqrt{c1} - 1},\ \frac{c1}{\sqrt{c1} - 1} = c1,\ x = \frac{c\cdot(\sqrt{v} - 1)}{\sqrt{v}},\ y = c\cdot\sqrt{v}\right]$$

$$\#336:\ x = \frac{c\cdot\left(\frac{y}{c} - 1\right)}{\frac{y}{c}}$$

$$\#337:\ \mathrm{SOLVE}\left(x = \frac{c\cdot\left(\frac{y}{c} - 1\right)}{\frac{y}{c}},\ y\right)$$

Example 18c

After eliminating parameter *v* we find a family of hyperbolas as general solution.

We have two singular solutions (lines).

The result of the LAGRANGE DE derived from the second solution for *y* in #333 can be found on the next page.

$$\#338:\ y = \frac{c^2}{c - x}$$

$$\#339:\ \mathrm{VECTOR}\left(y = \frac{c^2}{c - x},\ c,\ -3,\ 3,\ 0.5\right)$$

$$\#340:\ \mathrm{SOLUTIONS}\left(\frac{c1}{\sqrt{c1} - 1} = c1,\ c1\right) = [0,\ 4]$$

$$\#341:\ \mathrm{VECTOR}\left(y = \frac{c1\cdot x}{\sqrt{c1} - 1},\ c1,\ [0,\ 4]\right)$$

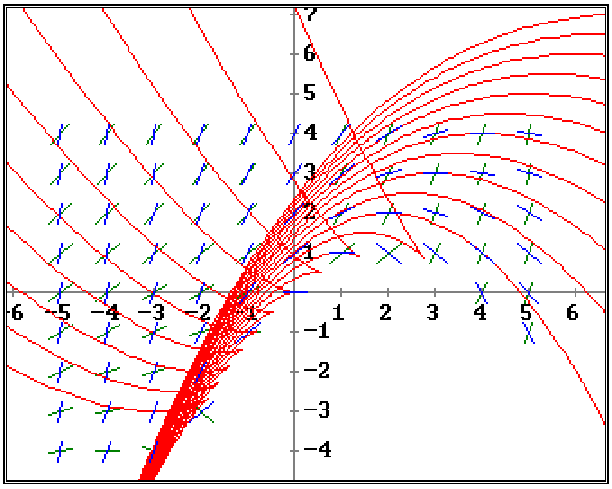
$$\#342:\ [y = 0,\ y = 4\cdot x]$$

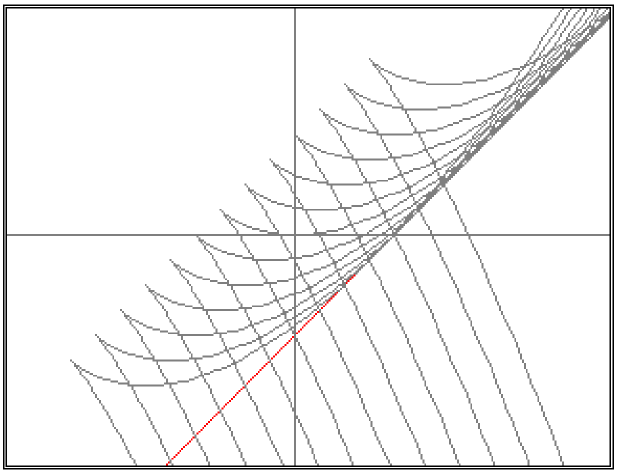
$$\#343:\ \mathrm{LAGR}\left(\frac{v}{\sqrt{v} + 1},\ 0\right)$$

$$\#344:\ \left[y = \frac{c1\cdot x}{\sqrt{c1} + 1},\ \frac{c1}{\sqrt{c1} + 1} = c1,\ x = \frac{c\cdot\hat{e}^{-2/\sqrt{v}}\cdot(\sqrt{v} + 1)}{\sqrt{v}},\ y = c\cdot\sqrt{v}\cdot\hat{e}^{-2/\sqrt{v}}\right]$$

Example 18a

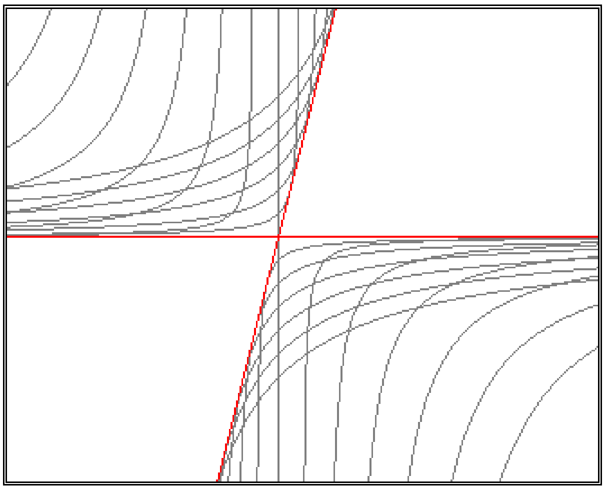
Yo see a part of the direction field and a family of integral curves

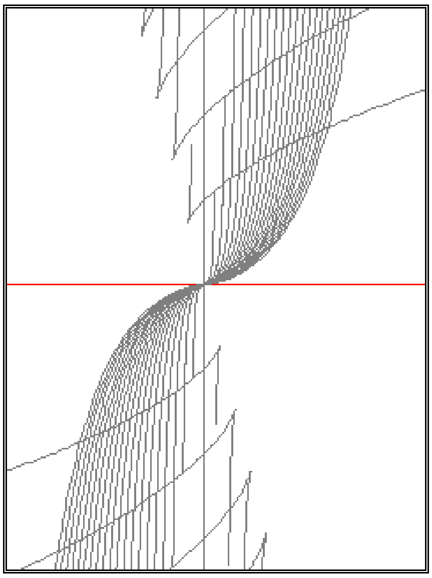




Example 18 b (left)

Example 18 c (below)
the hyperbolas and the other family of solution curves.





Differentialgleichungen mit dem TI-89/TI-92+/Voyage 200
Differential Equations with TI-89/TI-92+ and V200

A Workshop

Hier soll nicht die Theorie der DGL und ihre Lösungsmechanismen besprochen, sondern an einigen ausgewählten Aufgaben die Vorgangsweise auf den TI-CAS-Rechnern beschrieben werden. Der TI-89/TI-92PLUS und der Voyage 200 verfügen über ein gehöriges Potential an graphischen und analytischen Werkzeugen zur Behandlung der Differentialgleichungen.

We will not deal with theory of DEs, but show the solving procedure using the TI-CAS calculators.

Beispiel 1/Example 1

$y' = y(x+1)$

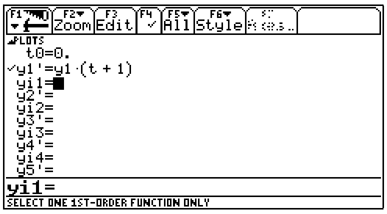
a) Zeichne das Richtungsfeld. Plot the direction field.

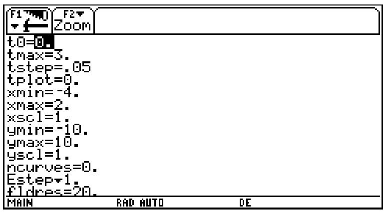
b) Zeichne spezielle Lösungen der DGL für y(0) = 3, y(0) = -3 und y(-1) = 4

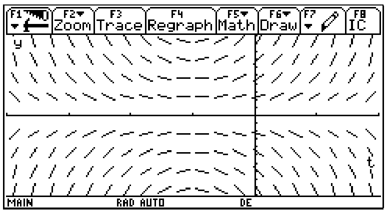
c) Bestimme die allgemeine Lösung und die oben angeführten speziellen Lösungen mit Hilfe des Rechners indem du die händische Vorgangsweise nachvollziehst.
Find the general solution reproducing the manual solving procedure.

d) Bestimme diese Lösung mit allen vom CAS zur Verfügung gestellten Hilfsmitteln. Die Lösung aus b) könnte mit der entsprechenden nun gefundenen Lösung überlagert werden.
Use all means of CAS to find the solutions.

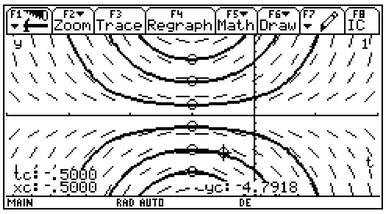
a) Der Rechner wird im MODE auf Graph DIFF EQUATIONS eingestellt. Die Differentialgleichung wird in den [Y=]-Editor geschrieben. Aber Achtung: die Rolle des *x* wird vom Parameter *t* übernommen! (*x* must be replaced by *t*!)

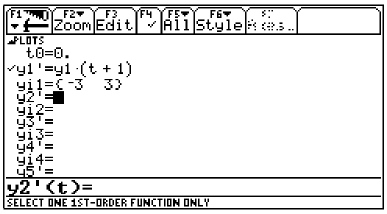


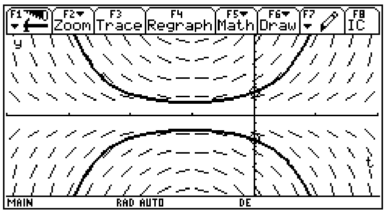




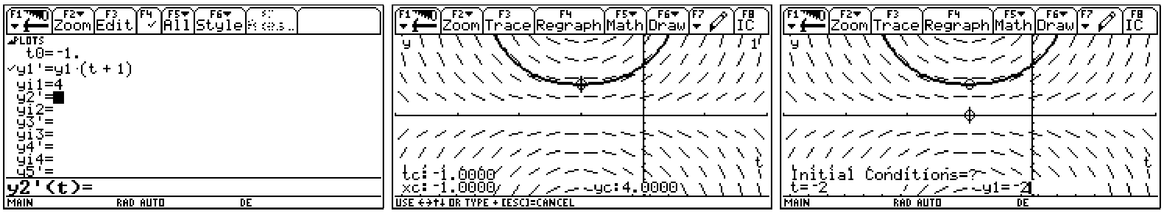
Wenn Du in den [WINDOW]-Werten für ncurves etwa den Wert 6 einsetzt, dann werden gleichmäßig über den *y*-Bereich 6 Lösungskurven eingezeichnet. Über F3 Trace können die Kurven abgetastet werden.



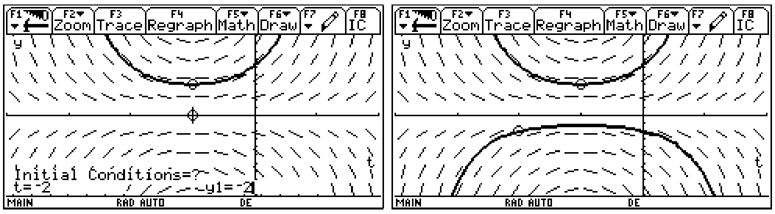




b) Um die beiden Lösungskurven zu den gegebenen Anfangsbedingungen zu sehen, gibt man die beiden Funktionswerte zu $x = t = 0$ in einer Liste zu yi1 an. Für die dritte gesuchte Lösungskurve muss der Wert für t0 auf –1 und yi1 auf 4 gesetzt werden.

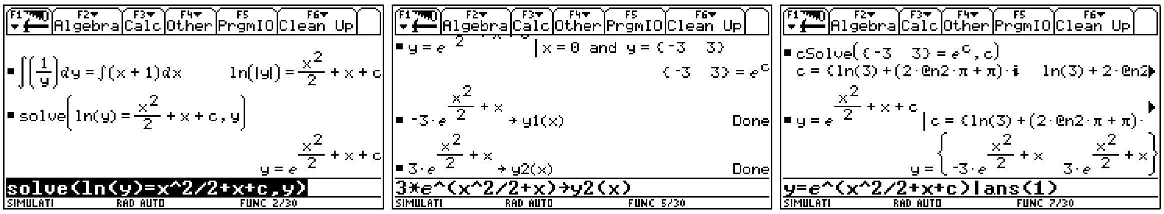


Es gibt noch eine weitere – interaktive – Möglichkeit, Lösungskurven durch bestimmte Punkte zu legen. Über F8 IC können Anfangsbedingungen (Initial Conditions) direkt eingegeben werden. So wollen wir zB die Kurve durch P(-2|-2) legen.

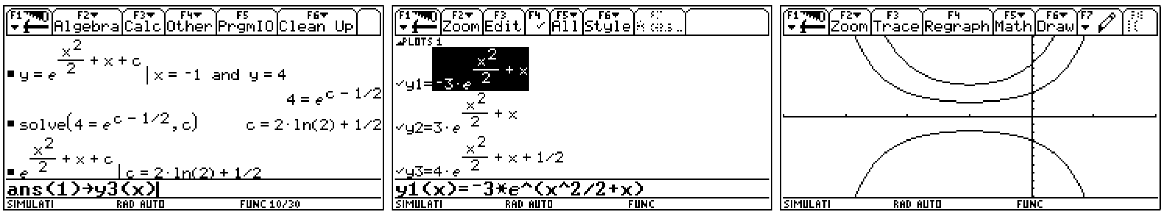


c) Als Lösungsmethode kommt hier die *Methode der Trennung der Variablen* in Frage.
We apply Separation of Variables.

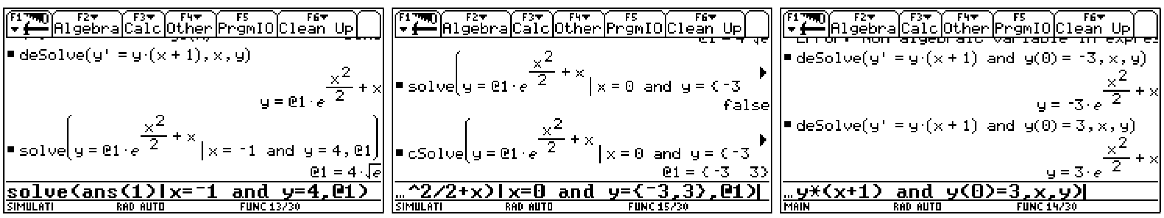
$$\frac{dy}{dx} = y(x+1) \rightarrow \frac{dy}{y} = (x+1)\,dx$$



Alle drei speziellen Lösungen werden gefunden und über den Funktioneneditor gezeichnet. Beachte den Einsatz der cSolve-Funktion (wenn man nicht direkt für e^c substituiert).
We find all special solutions and plot them. Notice application of cSolve if not substituting directly for e^c.



d) Mit der implementierten **deSolve**-Funktion lässt sich die DGL direkt lösen. Auch hier führt cSolve zum gewünschten Ergebnis. Die @- Zeichen stehen für freie Parameter.
The @- characters are free parameters.



Auch die Randbedingungen (oder andere gegebene Punkte der Lösungskurve) können sofort mit angegeben werden. We can include points of the integral curve into the **deSolve-**function.

Beispiel 2/**Example 2**

$$y' = \frac{2y}{x}\ln\left(\frac{3y}{x}\right)$$

a) Bestimme die Lösung mit Hilfe des Rechners, ohne vorerst desolve() einzusetzen.

Tipp: Substituiere $z = \frac{y}{x}$. Solve without using deSolve() – substitution!!.

b) Wie lautet die spezielle Lösung für $y(1) = \sqrt[3]{e}$? Special solution for $y(1) = \sqrt[3]{e}$

c) Zeichne das Richtungsfeld und lege die gefundene Lösungskurve drüber. Direction field.

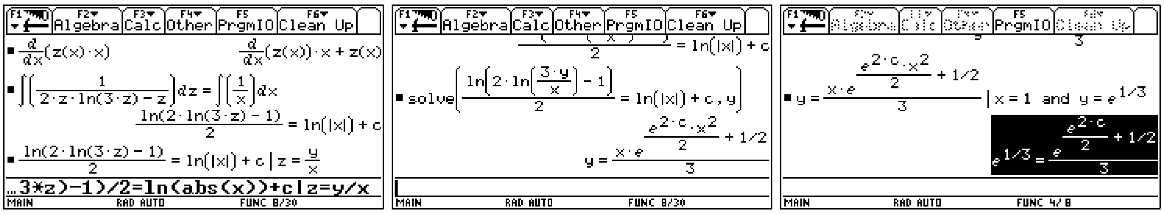
d) Bestimme diese Lösung mit allen vom CAS zur Verfügung gestellten Hilfsmitteln.

e) Fixiere die Lösungen von a), b) und d) in einem Script (Textfile). Produce a script.

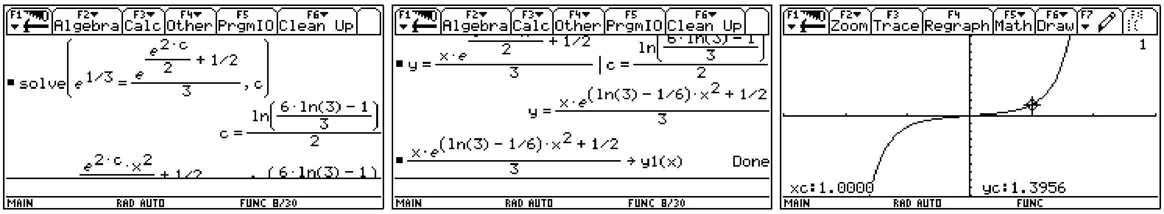
a) $z = \frac{y}{x} \rightarrow y = z \cdot x \rightarrow y' = z' \cdot x + z$

Daher wird aus der DGL: $z' \cdot x + z = 2z \cdot \ln(3z) \rightarrow \frac{dz}{dx} \cdot x = 2z \cdot \ln(3z) - z \rightarrow \frac{dz}{2z \cdot \ln(3z) - z} = \frac{dx}{x}$.

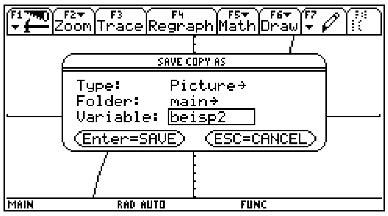
Die Trennung der Variablen ist geglückt. Es fragt sich nur, ob die Integrale geschlossen ausgewertet werden können? Separation was possible, is integration possible, too?



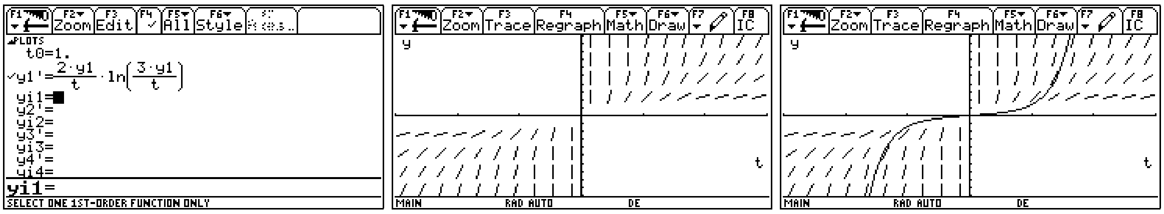
Das Integral lässt sich auswerten. Welche Integrationsregel(n) führen zum Ziel? [1]
Which integration rules are leading us to success? [page 12]



b) Es gibt eine allgemeine und die gesuchte spezielle Lösung. Der Graph der Lösungskurve wird als Bild F1 2:Save Copy As unter dem Namen beisp2 abgespeichert – um ihn später über das Richtungsfeld zu legen. We save the graph for later combining it with the direction field.

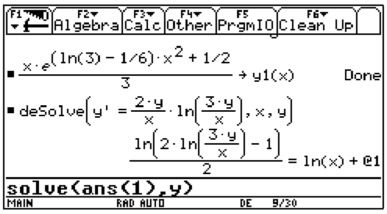


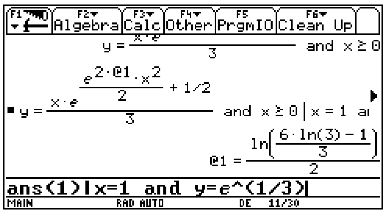
c) Über MODE wird in den DIFF EQUATIONS Modus gewechselt. Die [WINDOW]-Werte sollen mit denen von vorhin identisch sein.

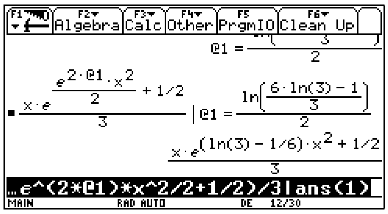


Über F1 1:Open wird beisp2 über das Richtungsfeld gelegt – und alles passt.

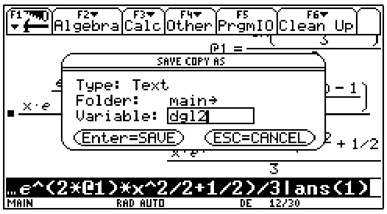
d) Wenn es "nur" um die Lösung geht, versucht man gleich desolve().

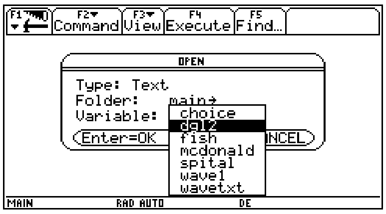


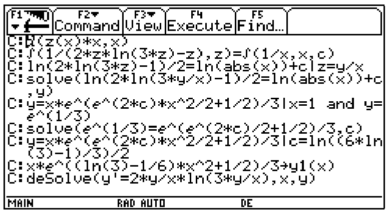




e) Der komplette Prozess wird in einer Textdatei gespeichert (F1 2:Save Copy As).

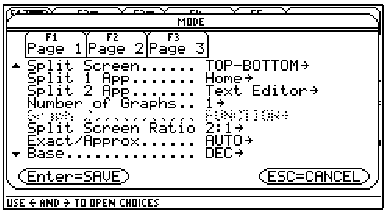


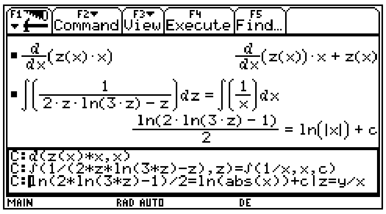


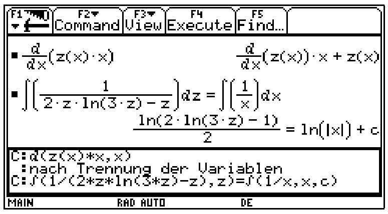


Im Texteditor wird diese Datei wieder geöffnet. Jede mit **C:** eingeleitete Zeile ist ein ausführbares Kommando, das mit **F4** "exekutiert" werden kann. Es empfiehlt sich, den Schirm laut Muster zu teilen.

We open the file in the text editor. Each line showing a **C:** can be "executed" in the Home Screen by pressing **F4**. I recommend to split the screen according to the pictures.







Diese Frage stellen wir *DERIVE* 6. Der eingebaute "schrittweise Simplifier" verrät uns die angewendeten – elementaren – Rechenregeln. So können wir von einem CAS nicht nur die Ergebnisse rascher erhalten, sondern gewinnen auch Kenntnisse in den Regeln – oder wir werden wieder an sie erinnert.

DERIVE 6 gives the answer. Stepwise simplification shows us the secrets how to find the closed form of the integral.
So the CAS does not only help us finding the results very quick, but also helps us to gain new knowledge or reminds us on possibly forgotten knowledge.

#1: $\int \frac{1}{2\cdot z\cdot LN(3\cdot z) - z}\, dz$

If x>0,

$LN(x\cdot z) \to LN(x) + LN(z)$

#2: $\int \frac{1}{z\cdot(2\cdot(LN(3) + LN(z)) - 1)}\, dz$

$\int \frac{F(LN(a\cdot x))}{x}\, dx \to SUBST(\int F(x)\, dx,\ x,\ LN(a\cdot x))$

#3: $SUBST\left(\int \frac{1}{2\cdot z + 2\cdot LN(3) - 1}\, dz,\ z,\ LN(z)\right)$

$\int \frac{1}{a + b\cdot x}\, dx \to \frac{LN(a + b\cdot x)}{b}$

#4: $SUBST\left(\frac{LN(2\cdot LN(3) + 2\cdot z - 1)}{2},\ z,\ LN(z)\right)$

$SUBST(F(x),\ x,\ a) \to F(a)$

#5: $\frac{LN(2\cdot LN(3) + 2\cdot LN(z) - 1)}{2}$

#6: $\frac{LN(2\cdot LN(3\cdot z) - 1)}{2}$

Beispiel 3 / Example 3

$$\left(3x^2y+8xy-y+2\right)dx+\left(x^3+4x^2-x+\frac{y}{2}\right)dy=0$$

a) Zeige, dass hier eine "exakte DGL" vorliegt, d.h., dass die Integrabilitätsbedingung erfüllt ist.

$P(x,y)dx+Q(x,y)dy=0$ ist exakt, wenn gilt: $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$. Show that the DE is exact.

b) Löse die DGL auf traditionelle Weise. Erst allgemein, dann für $y(0) = 2$.
Solve in the traditional way, general and special for y(0)=2.

c) Erzeuge für die Lösung dieses Typs von DGL ein Programm.
Produce a program for solving DEs of this kind.

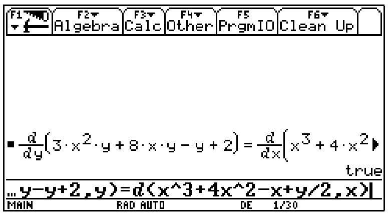
d) Erfinde eine eigene *exakte Differentialgleichung* (die Winkelfunktionen enthält).
Find an own exact DE containing trig-functions.

e) Suche die Lösungen von a) und b) auf direktem Weg mit dem CAS.
Find the solutions for a) and b) directly using CAS.

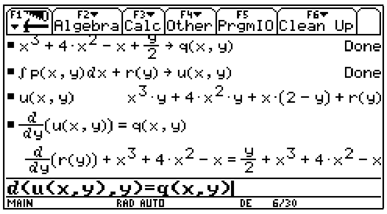
f) Löse auch *deine* Aufgabe mit deSolve und mit dem Programm.
Solve your problem (d) with deSolve and with the program.

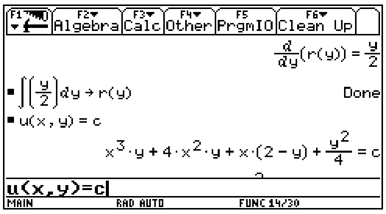
g) Zeichne das Richtungsfeld mit der speziellen Lösung aus b).
Plot the direction field including the special solutions from b).

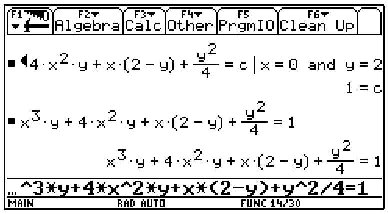
a)



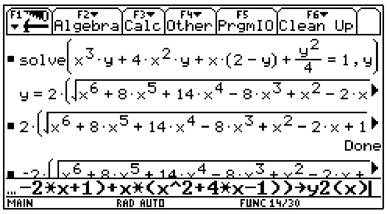
b) Der allgemeine Ansatz für die Lösung $u(x,y)$ lautet: $u(x,y)=\int P(x,y)dx+r(y)$. Aus dem so gewonnenen $u(x,y)$ wird nach Ableitung nach y ein Ausdruck für $r'(y)$ bestimmt, der dann nach y integriert zu $r(y)$ führt.

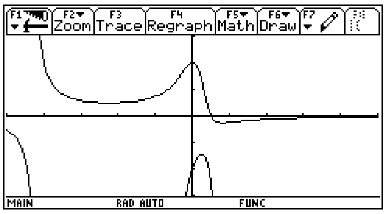






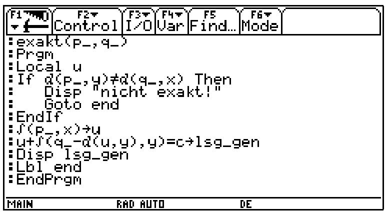
Von den beiden Funktionsgraphen gilt natürlich nur der obere!

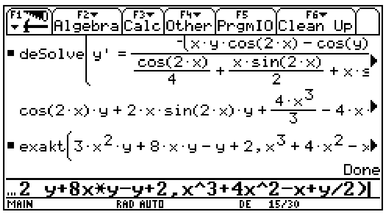


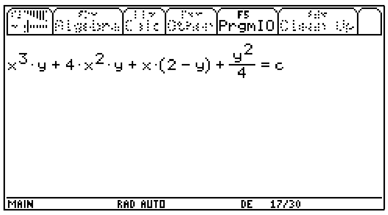


Only the upper graph is valid.

c) Ein Programmvorschlag/**Proposal for a program** (following an idea of Günter Redl).







d) Ein Vorschlag ist, mit mit Kern $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = s(x, y)$ zu beginnen. Dieser Term muss sowohl nach x als auch nach y integrierbar sein und ergibt dann die Terme $P(x,y)$ und $Q(x,y)$, die noch mit jeweiligen Integrationskonstanten ergänzt werden dürfen.

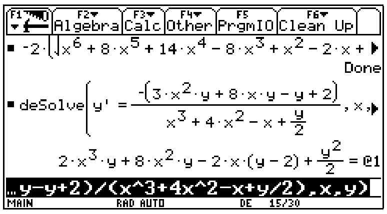
$$s(x, y) = x \cdot \cos(2x) + \sin y$$

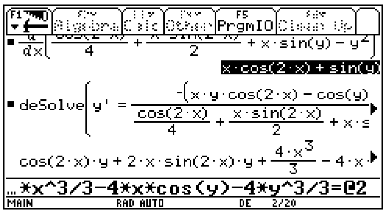
$$P(x, y) = \int s(x, y)dy + konst = x \cdot y \cdot \cos(2x) - \cos y + x^2$$

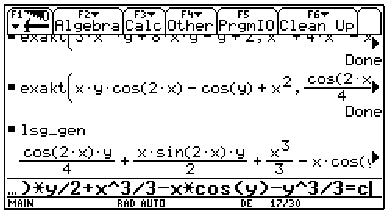
z.B. $$Q(x, y) = \int s(x, y)dx + konst = \frac{\cos(2x)}{4} + \frac{x \cdot \sin(2x)}{2} + x \cdot \sin y - y^2$$

$$\left(x \cdot y \cdot \cos(2x) - \cos y + x^2\right) \cdot dx + \left(\frac{\cos(2x)}{4} + \frac{x \cdot \sin(2x)}{2} + x \cdot \sin y - y^2\right) \cdot dy = 0$$

e) und f) Die CAS-Lösungen:



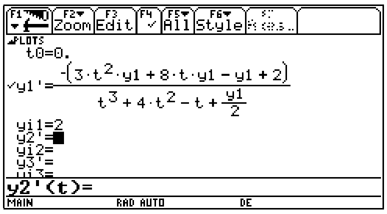


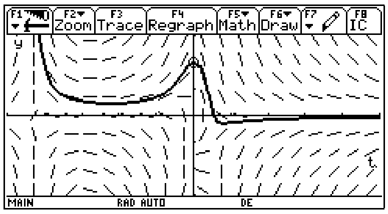


Im dritten Bild wird die Programmlösung der selbst erstellten DGL gezeigt. Sind die Ergebnisse mit **deSolve()** und dem Programm äquivalent?

Are the deSolve()- and the program solutions equivalent?

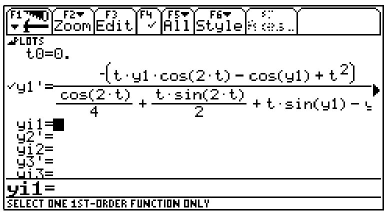
g) Das Richtungsfeld mit der speziellen Lösung.

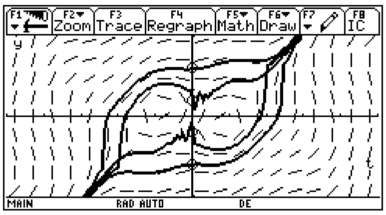


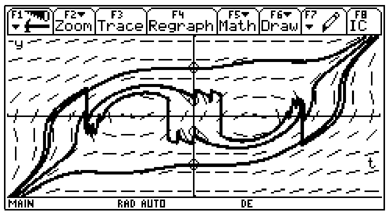


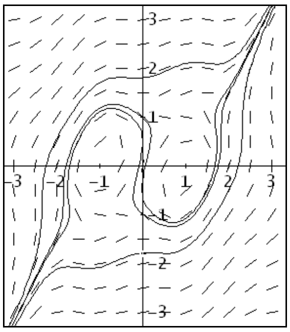
Für „unsere“ DGL (d) enstehen recht barocke Lösungskurven (ncurves = 4). Leider kann man am TI keine implizit gegebenen Relationen (Funktionen) zeichnen lassen – mit Umwegen über den R^3 ist dies recht mühsam möglich – siehe später. Über den Funktionseditor wird entweder das EULER- oder das RUNGE-KUTTA-Verfahren für eine numerische Approximation eingesetzt. Ich vergleiche die Ausgabe am TI mit der Ausgabe von *DERIVE*.

"My personal DE" from (d) results in very baroque solution curves. Unfortunately we cannot do implicit plots on the TI in an easy way – we can do it via R^3 representation as shown later. I compare theTI-plot with the respective Derive plot.









Eine Verbesserung der Darstellung lässt sich durch Verkleinerung des **tstep** -Werts bei gleichzeitiger Verkleinerung des Bildausschnitts erreichen. Dass man auch übertreiben kann, zeigt die Abbildung rechts oben. (Zooming in does not necessarily improve the graphic representation.)

Im linken Bild sieht man die *DERIVE*-Darstellung der implizit gegebenen Lösungskurven mit den Anfangsbedingungen $y(0) = -$

Beispiel 4/Example 4

$$2y'' + 3y' - 27y = 0; \qquad y(0) = -2,\ y'(0) = \frac{1}{2}$$

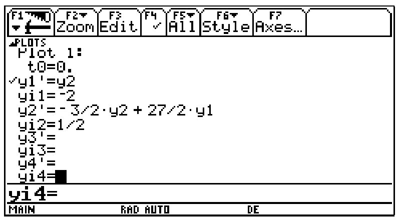
a) Bestimme die graphische Lösung am TI. Graphic solution on the TI.
b) Löse die DGL auf traditionelle Weise – CAS-unterstützt, aber ohne desolve().
c) Vergleiche mit der reinen CAS-Lösung. Compare the traditional with the CAS-solution.
d) Interpretiere die Werte in der TI-Tabelle ([TABLE]). Discuss the [TABLE]-values.
e) Ein Richtungsfeld ergibt sich über F9 Format Fields DIRFLD im [GRAPH]-Fenster. Versuche, auch die Lösungskurve zu zeichnen und vergleiche mit den früher gefundenen Graphen. Plot the direction field together with the solution curve from above.

a) Für die Eintragung im Funktionseditor muss ein kleiner Trick angewendet werden. Die vorliegende DGL muss zuerst durch geeignete Substitution in ein System von zwei DGL umgewandelt werden. The given DE must be transformed into a system of two DEs.

Man setzt: y1 = y, y2 = y', y3 = y" usw. Durch Differenzieren ergibt sich dann:
y1' = y', y2' = y", y3' = y"' usw.

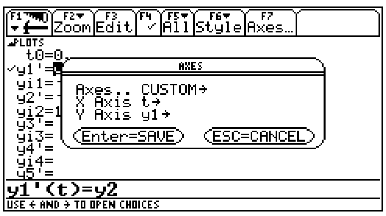
Bei der vorliegenden Aufgabe bedeutet dies:

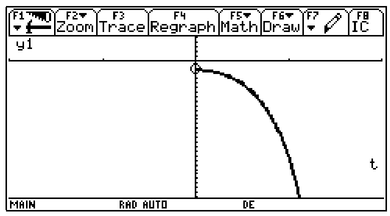
$$y'' = -\frac{3}{2}y' + \frac{27}{2}y$$



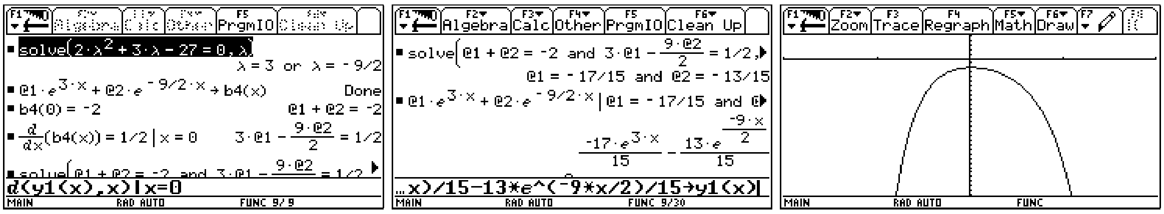
Für die Zeichnung der Lösungskurven müssen über F7 die Axes nach der Abbildung eingestellt werden.

Format Fields FLDOFF



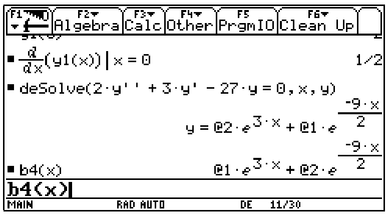


b) Wir lösen die charakteristische Gleichung, die zwei reelle getrennte Lösungen liefert. Die allgemeine Lösung wird als Summe von zwei Exponentialfunktionen mit unbekannten Koeffizienten angesetzt. Diese werden leicht aus einem linearen GLS berechnet, das aus den Anfangsbedingungen gewonnen wird. Mit geeigneten [WINDOW]-Werten lässt sich die Lösungskurve schön darstellen. Solve the characteristical equation and find the general solution as usual.

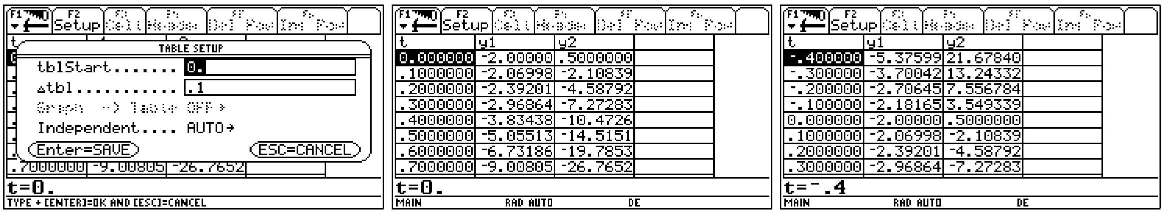


c) Die reine CAS-Lösung ergibt sich sofort in allgemeiner Form. Zur speziellen Lösung muss man vorgehen wie in b) gezeigt.

We obtain the pure CAS-solution in one single step.



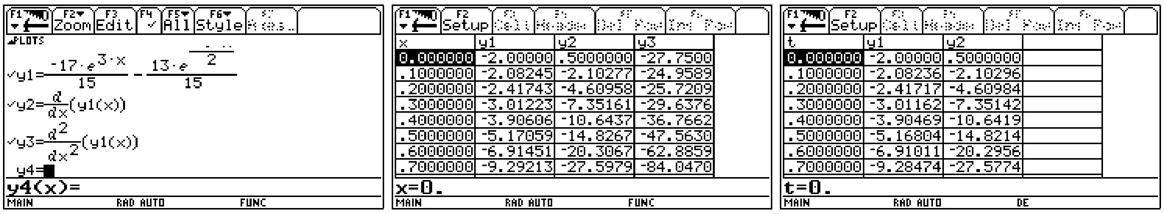
d) Die Tabelle wird zuerst eingerichtet, dann kann sie betrachtet werden. Unter y1 und y2 werden die Werte für $y(x)$ und $y'(x)$ ausgewiesen.



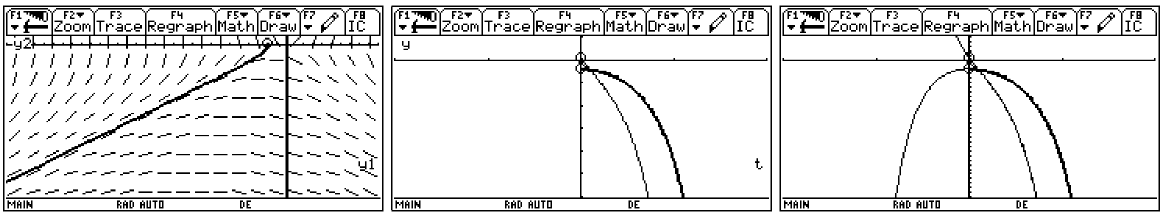
Setup the table and inspect the values of y(x) and y'(x) as y1 and y2.

Wenn man auch die Werte der zweiten Ableitung sehen will, dann muss man wieder in den FUNCTION-Modus zurückwechseln. Der Vergleich zeigt den „Fehler“ bei der hier angewendeten EULER-Methode. Wenn man im DE-Modus auf die RUNGE-KUTTA-Methode umstellt, dann fällt der Fehler deutlich geringer aus.

For finding the values of the 2nd derivative one has to switch back to FUNCTION-Mode. Comparison shows the "error" applying EULER-method. Applying RUNGE-KUTTA results in smaller errors.



e) Im Richtungsfeld wird der Zusammenhang zwischen y1 (= Lösungskurve) und y2 (= erste Ableitung) dargestellt. Stellt man die Achsen auf TIME um und aktiviert y1' und y2', dann werden Funktion und erste Ableitung gezeichnet. (Hier wurde die Funktion mit dem Style Thick ausgezeichnet. Anschließend habe ich die Graphen der exakten Lösung und deren erste Ableitung über die numerische Approximation gelegt (RUNGE-KUTTA).



Wird die EULER-Methode eingesetzt, kann man leichte Abweichungen auch optisch erkennen.

Beispiel 5/Example 5

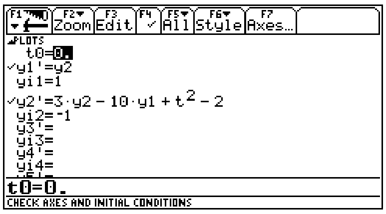
$y'' - 3y' + 10y = x^2 - 2; \qquad y(0) = 1,\ y'(0) = -1$

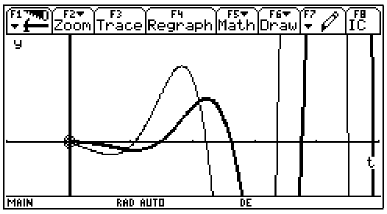
$\qquad\qquad\qquad\qquad\qquad\quad y(1) = 1,\ \ y'(1) = 2$

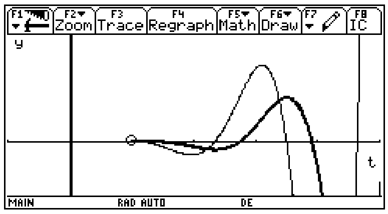
a) Bestimme die graphische Lösung am TI. Graphic solution on the TI.
b) Löse die DGL auf traditionelle Weise – CAS-unterstützt, aber ohne deSolve().
c) Führe die "Probe" durch. First without and then check with deSolve().
d) Vergleiche mit der reinen CAS-Lösung. Compare with the pure CAS solution.

e) Die beiden Lösungskurven werden leicht gefunden. Für die zweite sind die Anfangswerte zu verändern. Diese können auch interaktiv über F8 im Grafikfenster eingestellt werden.

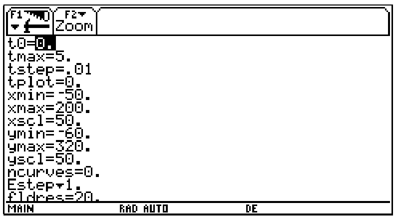
a)

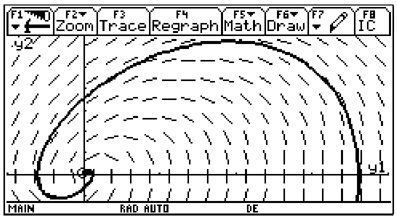




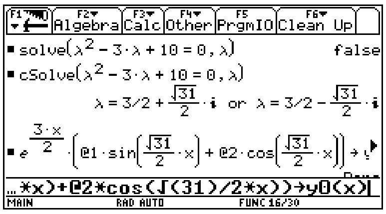


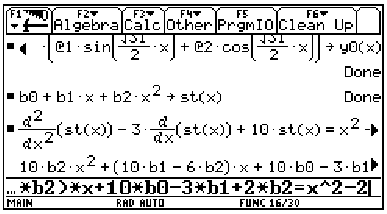
Besonders interessant wird hier das DIRFLD für Funktion und Ableitung. DIRFLD for y1 and y2.

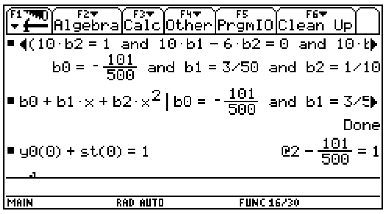




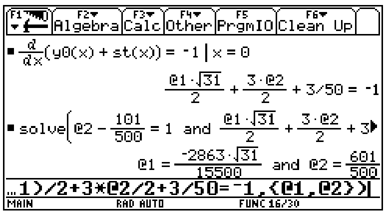
b) Die traditionelle Lösungsmethode beginnt wieder mit der charakteristischen Gleichung, die hier zwei konjugiert komplexe Lösungen ergibt. Der Realteil wird für die Exponentialfunktion und der Imaginärteil für den trigonometrischen Lösungsansatz verwendet.

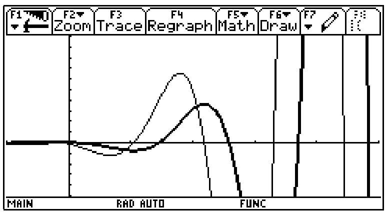






Dabei bleiben zwei Parameter. Für das Störglied $x^2 - 2$ wird ein unbestimmtes quadratischen
Polynom st(x) angesetzt, dessen Koeffizienten nach Einsetzen von st(x) in die DGL durch Koeffizientenvergleich gefunden werden (b0, b1 und b2). Durch Einsetzen der Anfangsbedingungen erhält man schließlich die noch offenen Parameter. The traditional way supported by the TI.



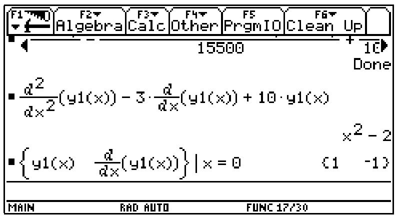


Das ist die Lösung für das erste Paar von Anfangsbedingungen – in „Schönschrift“.

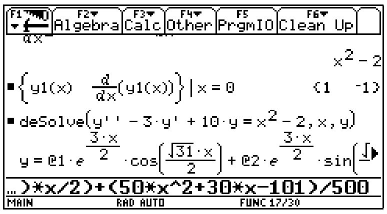
$$y(x)=\frac{601e^{\frac{3x}{2}}\cos\frac{\sqrt{31}x}{2}}{500}-\frac{2863\sqrt{31}\,e^{\frac{3x}{2}}\sin\frac{\sqrt{31}x}{2}}{15500}+\frac{x^2}{10}+\frac{3x}{50}-\frac{101}{500}$$

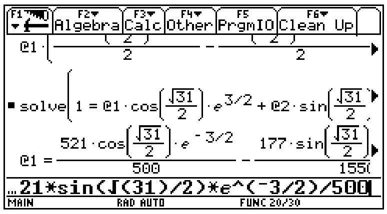
The solution for the first pair of initial conditions. In c) we check the result.

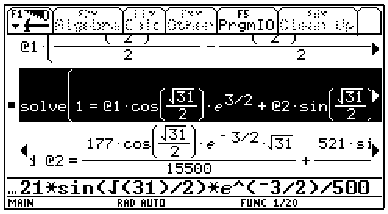
c) Die Probe bestätigt unser Ergebnis.



d) Wir lösen die Aufgabe mit desolve() und verwenden die allgemeine Lösung zur Berechnung der speziellen Lösung für das zweite Paar von Bedingungen.







Die komplette zweite spezielle Lösung kann mit einem mit *DERIVE* produzierten Ausdruck verglichen werden: Compare the 2nd special solution with the respective Derive result.

DSOLVE2_IV(−3, 10, x² − 2, x, 1, 1, 2)

$$e^{3\cdot x/2}\cdot\left(\cos\left(\frac{\sqrt{31}\cdot x}{2}\right)\cdot\left(\frac{521\cdot e^{-3/2}\cdot\cos\left(\frac{\sqrt{31}}{2}\right)}{500}-\frac{177\cdot\sqrt{31}\cdot e^{-3/2}\cdot\sin\left(\frac{\sqrt{31}}{2}\right)}{15500}\right)+\sin\left(\frac{\sqrt{31}\cdot x}{2}\right)\cdot\left(\frac{177\cdot\sqrt{31}\cdot e^{-3/2}\cdot\cos\left(\frac{\sqrt{31}}{2}\right)}{15500}+\frac{521\cdot e^{-3/2}\cdot\sin\left(\frac{\sqrt{31}}{2}\right)}{500}\right)\right)+\frac{50\cdot x^2+30\cdot x-101}{500}$$

Beispiel 6/Example 6

Eine Anwendung: Schwingungsgleichung Vibration Equation

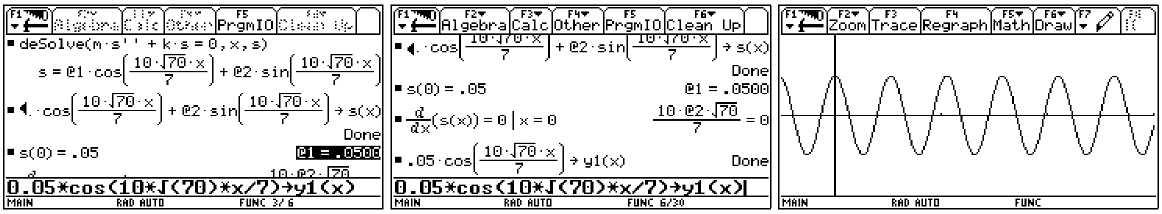
Die Feder eines PKW wird durch das Gewicht des Fahrzeugs (ca. 3500 N pro Rad) verformt. Die Feder wird um weitere 5 cm verformt und dann sich selbst überlassen. Die Schwingungsgleichung $s(t)$ ist aufzustellen und der Graph (Weg-Zeit-Diagramm) zu erzeugen.

Dabei ist m die Masse (m = 350 kg), k die Federkonstante (z.B. 50000 kg/s^2) und l der Dämpfungsfaktor. Verschiedene Werte für l führen zu verschiedenen Fällen. Untersuche die folgenden Fälle und interpretiere die entsprechenden Graphen:

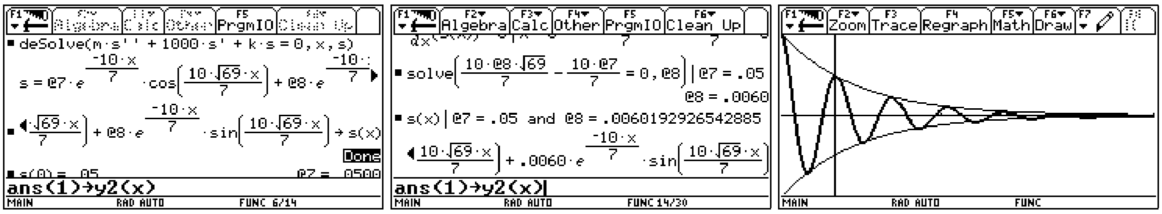
a) Dämpfungsfaktor $l = 0$ dvarious damping factors
b) Dämpfungsfaktor l = 1000 kg/s
c) Dämpfungsfaktor l = 10000 kg/s
d) Welcher Dämpfungsfaktor ergibt eine gerade ausreichende Dämpfung?
Which damping factor is necessary for sufficient damping?

Alle Aufgaben werden mit größtmöglicher CAS-Unterstützung gelöst.
This example was presented by Günter Redl in a T3-ACDCA-Seminar. Many thanks, Günter.

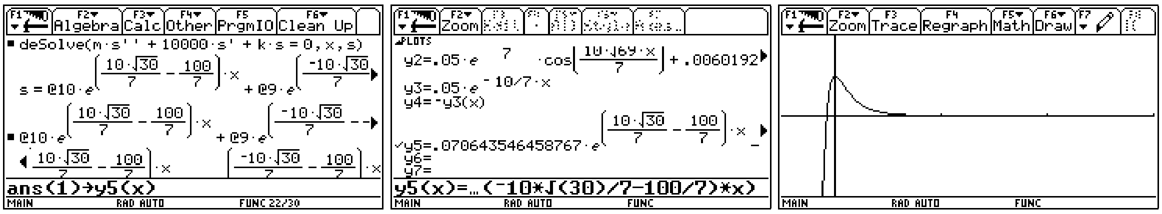
a) Es ergibt sich das Bild einer ungedämpften Schwingung. undamped oscillation



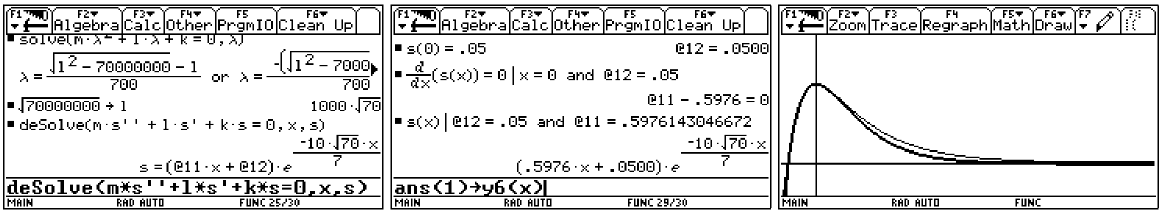
b) Hier sehen wir eine zu geringe Dämpfung – eine gedämpfte Schwingung. Die Funktion $s(t)$ gemeinsam mit $\pm\, 0{,}05e^{-(10/7)x}$. too less damping



c) Mehr als ausreichende Dämpfung – Kriechfall, aperiodische Bewegung. too much damping



d) Eine gerade ausreichende Dämpfung ergibt sich, wenn der Dämpfungsfaktor l so gewählt wird, dass die charakteristische Gleichung der DGL eine reelle Doppellösung hat.



Choose damping factor l so that the characteristical equation must have a real double solution.

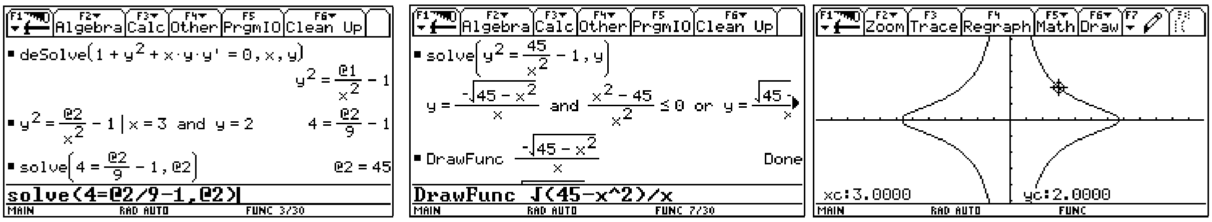
On the next pages you can find problems from DERIVE Newsletter #2 and #3 (1991). Then we demonstrated the use of the MATH-file ODE1.MTH. Some of the then useful auxiliary functions are missed in recent ODE1.MTH, because programming made it possible to integrate them into one procedure.

I recommend to compare with the updated contributions given in DNL#2 and DNL#3. Josef

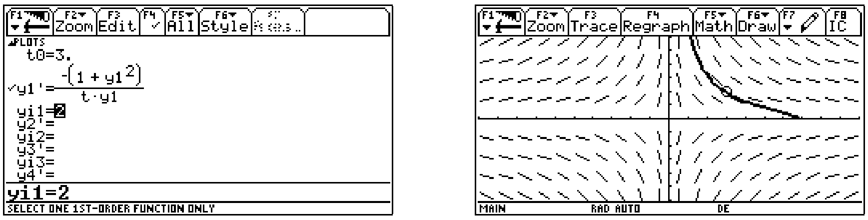
Examples from DNL#3 worked out on the TI-92+/V 200.
Compare with the DERIVE procedures!

Ex 1: $(1+y^2)\,dx + x\,y\,dy = 0$
Give the general solution and give the special solution containing P(3|2).
Sketch the direction field, some integral curves and the graph of the special solution from above.

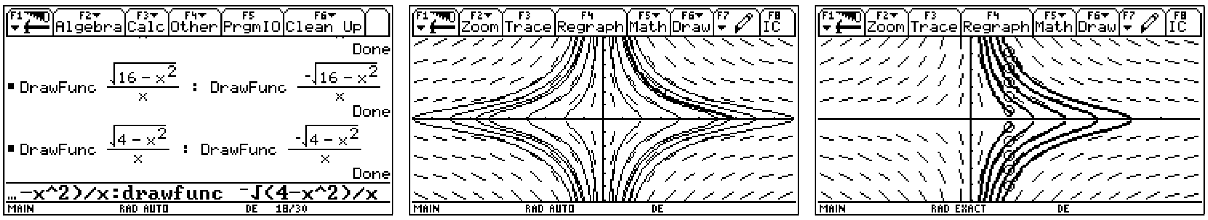
$(1+y^2)\,dx + x\,y\,dy = 0 \;\rightarrow\; 1+y^2+x\;y\;y' = 0$



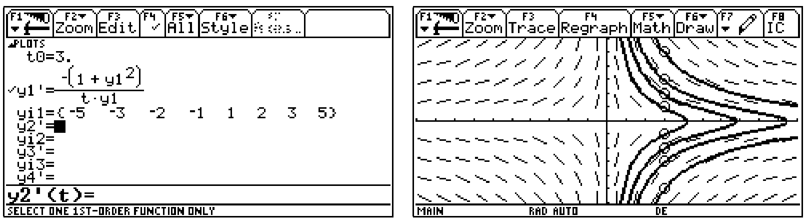
General solution is $y^2 = \dfrac{C - x^2}{x^2}$. We plot the direction field together with the integral curve passing the given point.



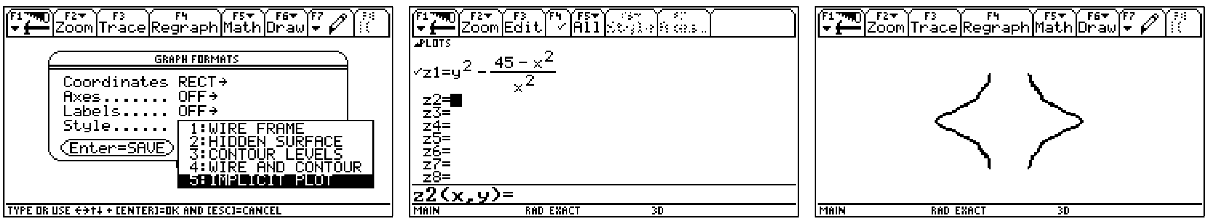
ncurves from the [WINDOW]-settings does not work – (t0 is temporarily set at the middle of the screen and initial conditions are distributed evenly along the y-axis). As there are no defined values on the y-axis ncurves must fail!! We can add solution curves via DrawFunc in the [GRAPH]-window or we change xmax and xmin to shift the y-axis out of the center (ncurves = 10)



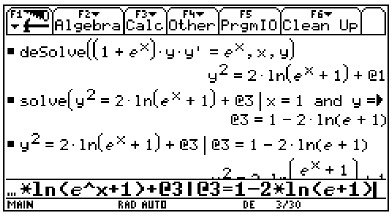
The [Y=]-settings allow plotting several solution curves.

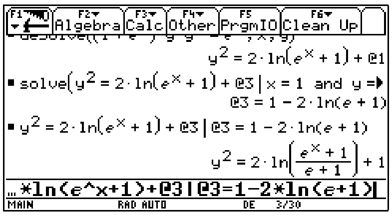


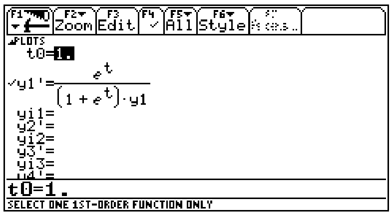
Find below the special solution produced as an implicit plot.

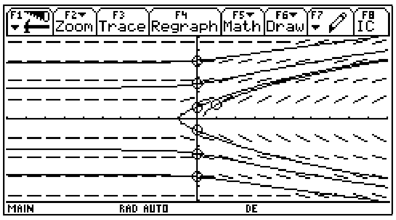


Ex 2: $(1+e^x)\,y\,y' = e^x$; $P(1|1)$







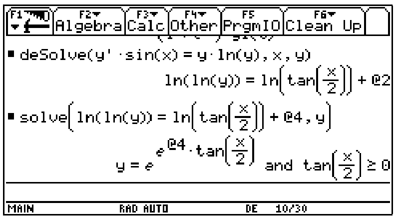


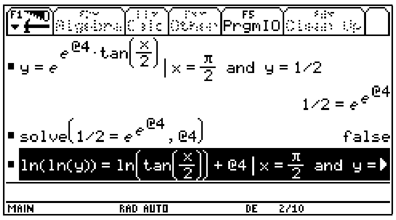
Solutions:

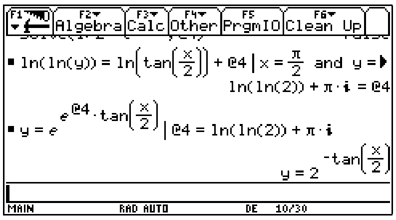
$$y^2 = \ln(e^x + 1) + C$$

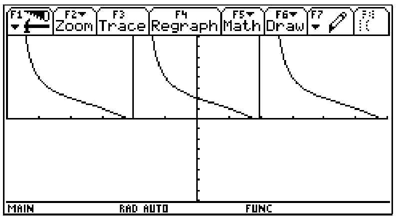
$$y^2 = 2\ln\left(\frac{e^x + 1}{e + 1}\right) + 1$$

Ex 3: $y' \sin x = y \ln y$; $y\left(\frac{\pi}{2}\right) = \frac{1}{2}$







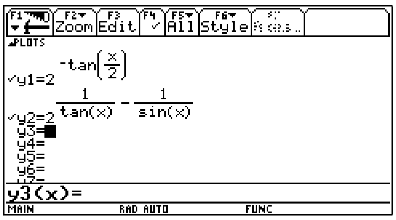


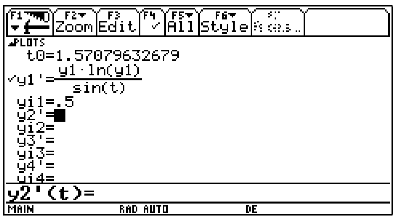
Solutions:

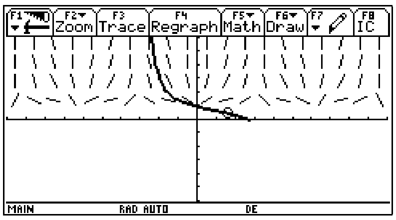
$$y = e^{e^C \tan\left(\frac{x}{2}\right)}$$

$$y = 2^{-\tan\left(\frac{x}{2}\right)}$$

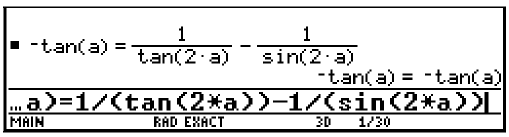
Derive version 2 showed the solution: `y = 2`$^{\texttt{COT(x)-1/SIN(x)}}$





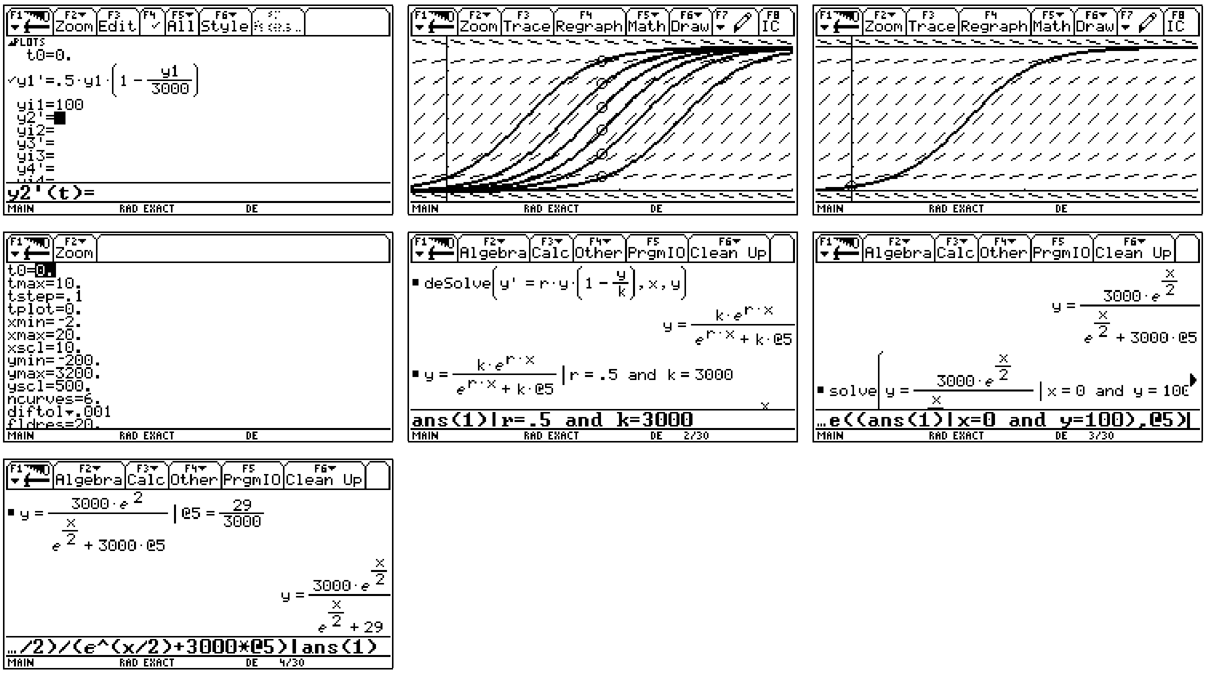


Plotting both functions y1 and y2 on the same axes shows the identity of the solutions. Then we see the direction field together with one branch of the special solution.



Ex 4: The logistic growth curve

$$dy = dx \cdot r \cdot y \cdot \left(1 - \frac{y}{k}\right); \quad r = 0.5; \; k = 3000; \; y(0) = 100$$



Ex 5: The general equation expressing the relation between electromotive force U and current I in a circuit containing the resistance R and the inductance L is:

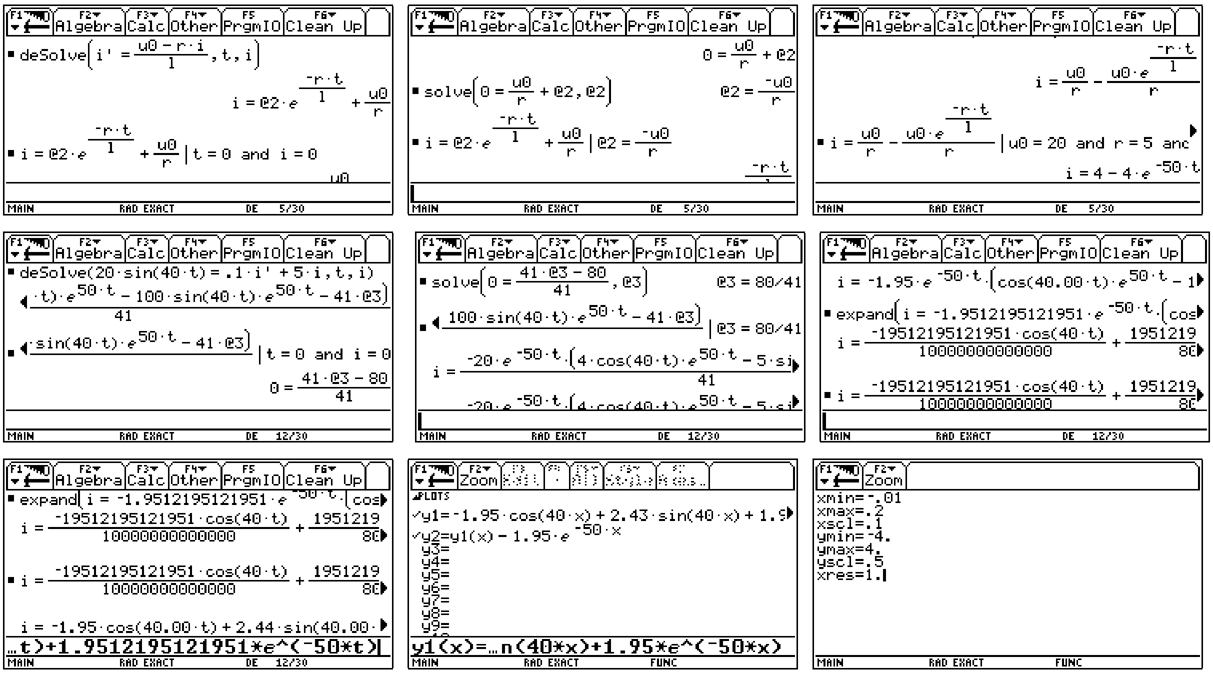
$$U = L \cdot I' + R \cdot I$$

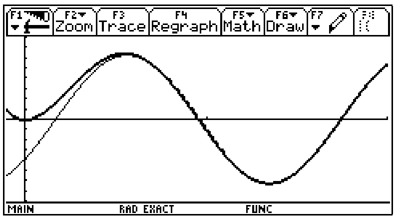
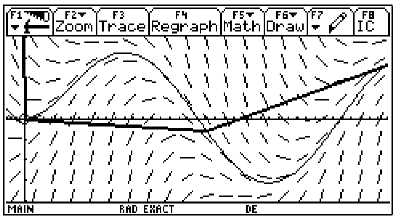
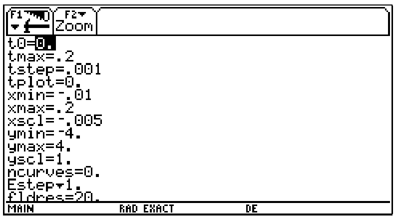
Solve this equation first for a constant $U = U_0$.

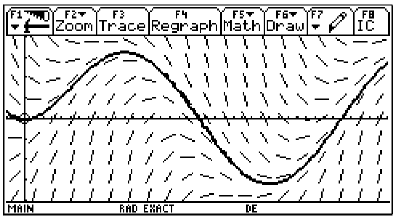
$I(t=0) = 0$

Then for $U_0 = 20V$, $R = 5\Omega$, $L = 0.1$ Henry.

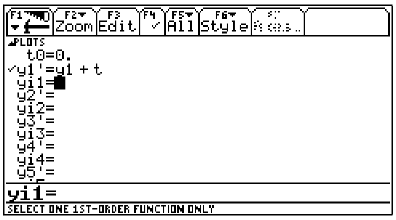
Solve this equation, when $U = U_0 \sin(\omega t)$! $(\omega = 40)$

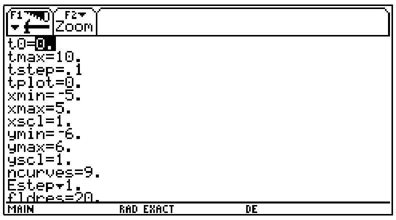


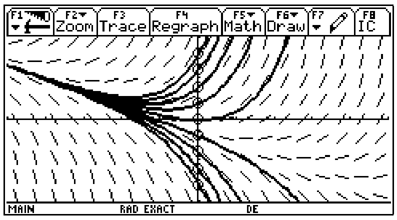






Ex 6: The DE $y' = x + y$ defines a nice direction field. Let the Voyage 200 plot this field together with a family of integral curves.





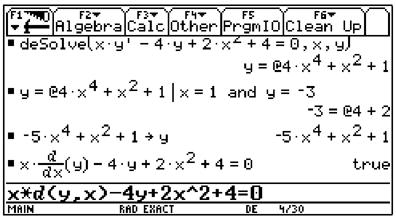


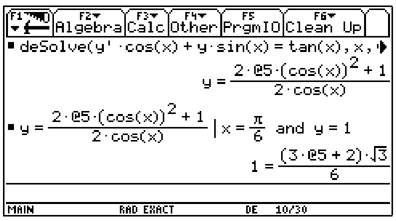
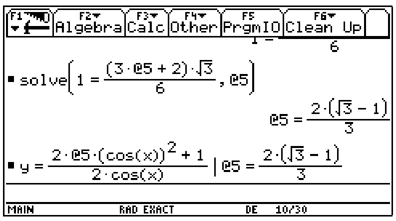
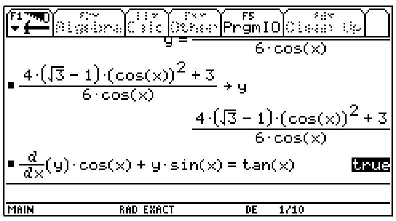
Ex 7: Find the general solutions of the given DEs and verify them. Find also the special solutions containing the given points.

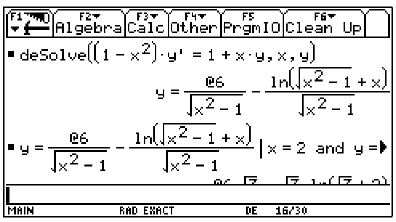
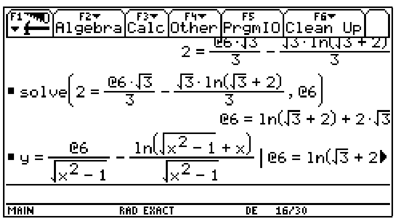
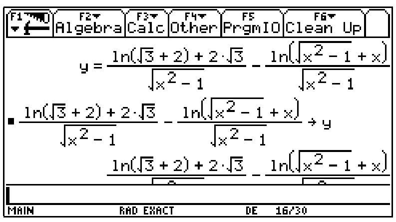
a) $x\,y' - 4y + 2x^2 + 4 = 0;\quad P(1\,|\,-3)$

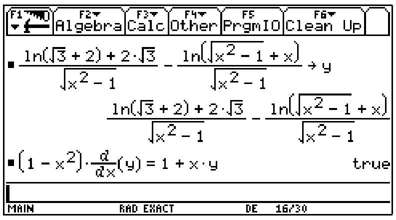
b) $y'\cos x + y\sin x = \tan x;\quad P\left(\frac{\pi}{6}\,|\,1\right)$

c) $(1 - x^2)\,y' = 1 + x\,y;\quad P(2\,|\,2)$



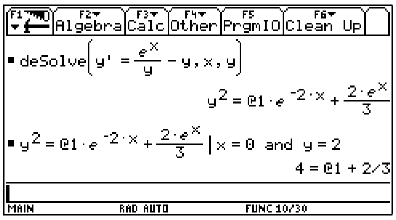


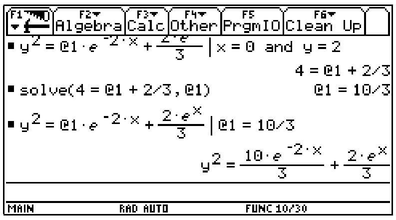
Ex 8: Both of the following equations are of the Bernoulli-type:

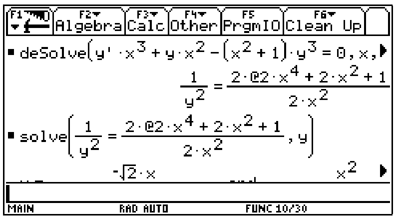
$y' + p(x)\cdot y = q(x)\cdot y^k \quad (k = \text{const.})$

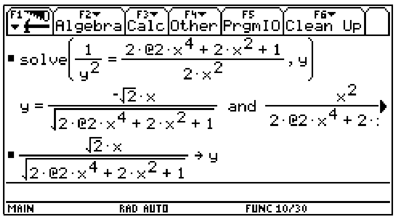
a) $y' = \dfrac{e^x}{y} - y; \quad P(0\,|\,2)$

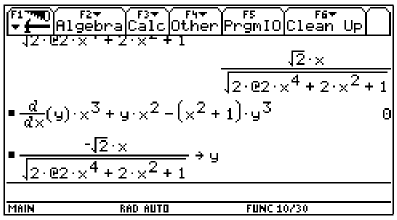
b) $y'x^3 + y\,x^2 - (x^2+1)\,y^3 = 0;$ Verify!

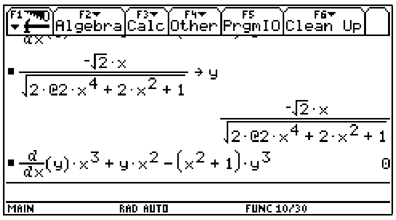










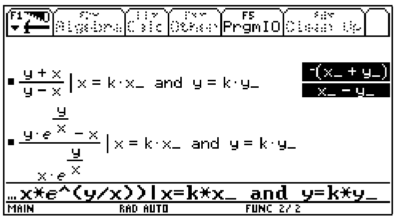


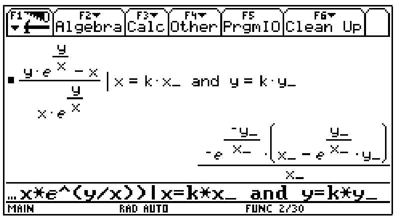
Ex 9: Test the following DEs on homogeneity. Then solve the equations.

a) $y' = \dfrac{y+x}{y-x}; \quad P(1\,|\,2)$

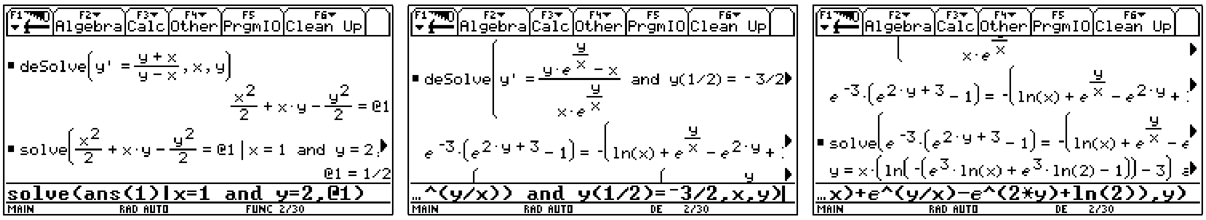
b) $y' = \dfrac{y\,e^{\frac{y}{x}} - x}{x\,e^{\frac{y}{x}}}; \quad P\left(\dfrac{1}{2}\middle|-\dfrac{3}{2}\right)$, verify the solution!

The test:





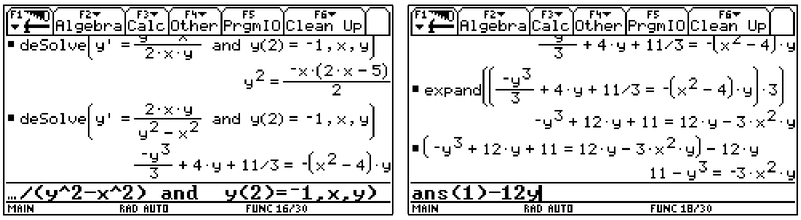
and the solutions:



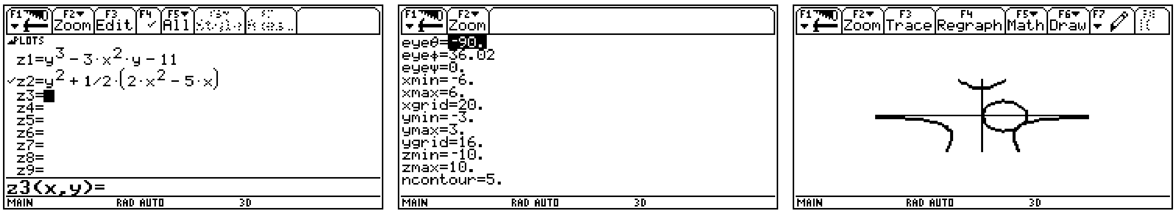
Ex 10: Compare the integral curves of

$$y' = \frac{y^2 - x^2}{2\,x\,y} \quad \text{and} \quad y' = \frac{2\,x\,y}{y^2 - x^2} \quad \text{intersecting in } P(2|-1).$$

Plot both curves together with the tangents in P.



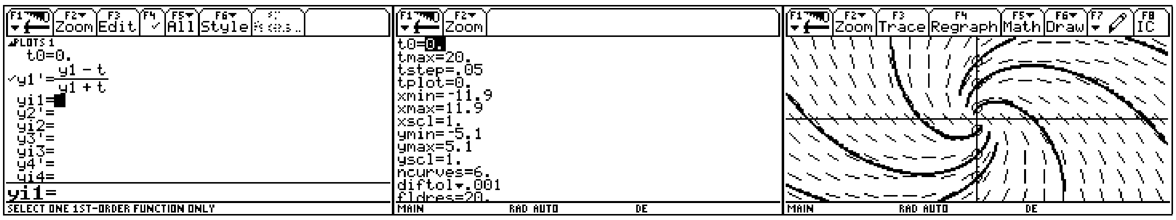
I tried to superimpose two implicit plots using the 3D mode. On the TI I cannot plot the tangents.

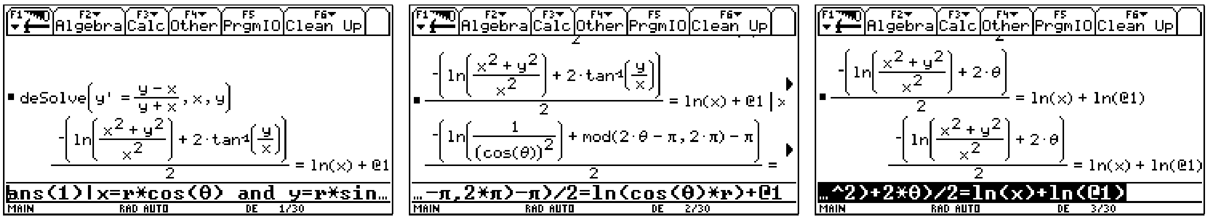


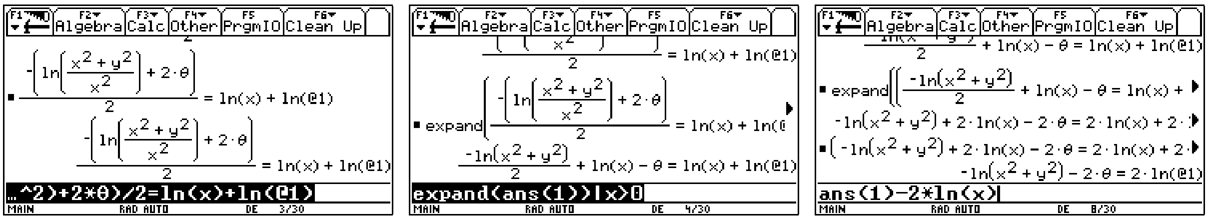
Ex 11: Find the curves which intersect the circles $x^2 + y^2 = r^2$ at angles of $\pi/4$.

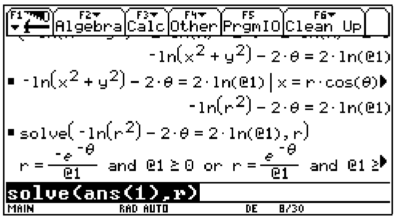
Plot some circles together with a family of curves representing the general solution of the underlying differential equation.

According to [2] the problem leads to the homogeneous DE $y' = \dfrac{y - x}{y + x}$.

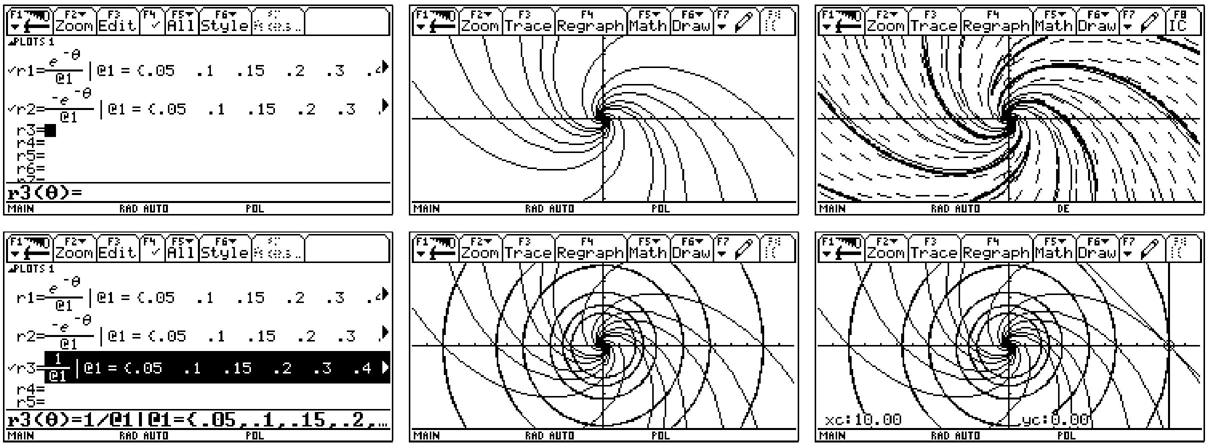






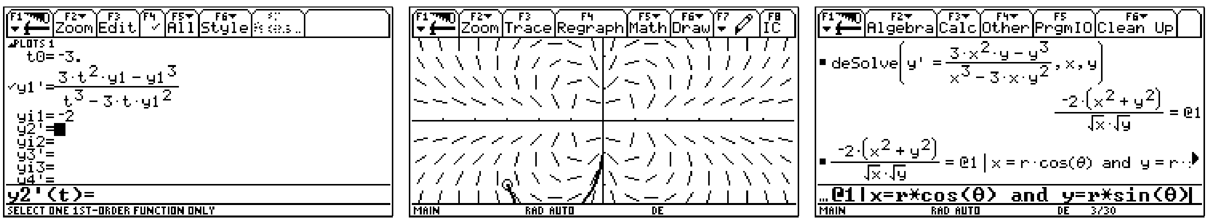


We find a family of logarithmic spirals intersecting the circles. As you can see we switch to polar coordinates. Performing this task requires a lot of mathematical and technological competence.

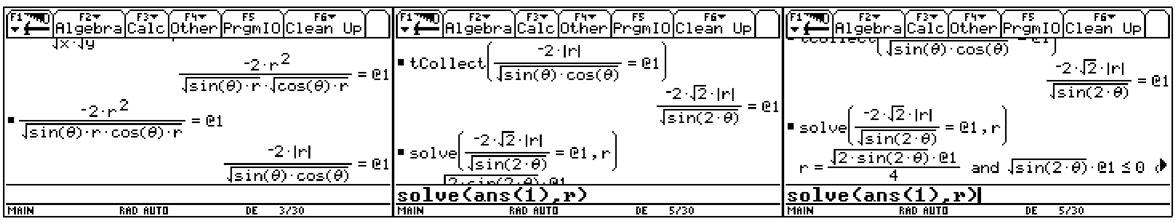


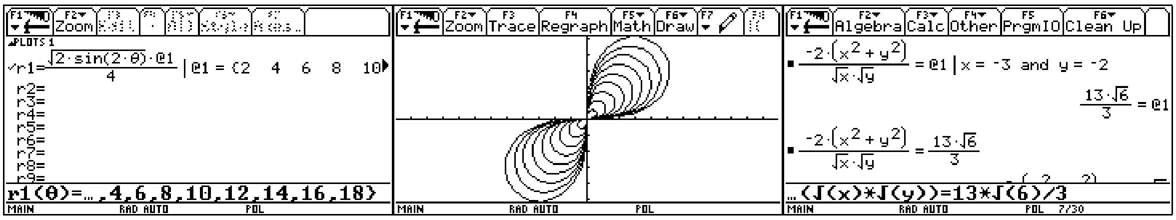
Ex 12: A point moves on a curve in the x-y-plane in such a way that the angle formed by the tangent of the curve with the x-axis is three times the angle between the radius vector and the x-axis. What is the Cartesian equation of the family of curves satisfying this condition.

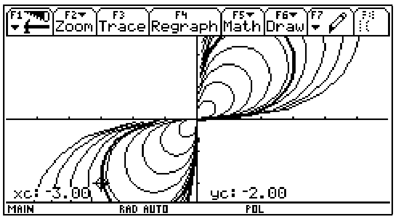
Plot a family of solution curves, especially this one which contains $P(-3|-2)$.



The first – graphical – approach is not really satisfying. Again application of polar coordinates proves to be a successful means to obtain better results. More explanations can be found in DNL#3.







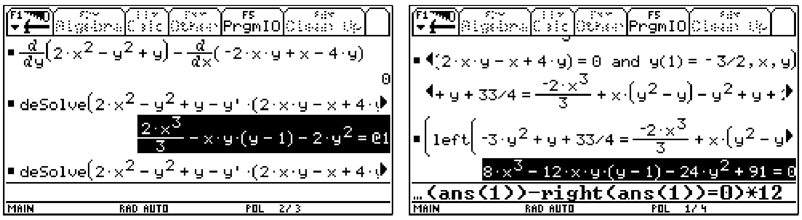
The solution curves are lemniscates.

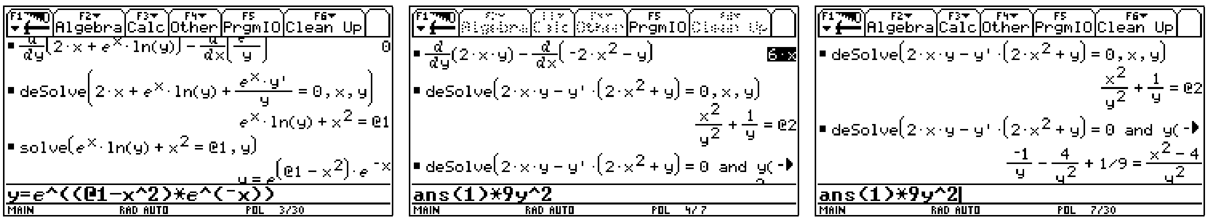
Ex 13: Show that the given DEs are of exact form. Give their general solutions and to each one the special solution if there is a point given:

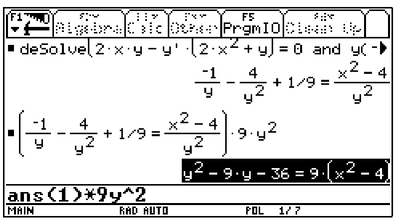
a) $2x^2 - y^2 + y - y'\cdot(2xy - x - 4y) = 0; \quad P(1\,|\,-1.5)$

b) $2x + e^x \ln y + \dfrac{e^x y'}{y} = 0$

c) $2xy - (2x^2 + y)y' = 0; \quad Q(-2\,|\,3)$







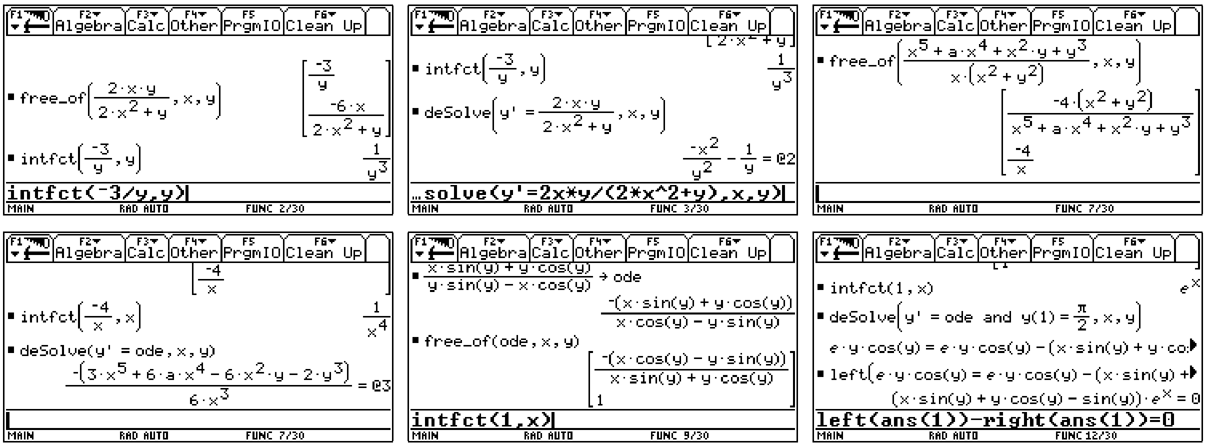
Ex 14: Try to find a solution of the differential equations given below. Manually they are solved by means of an integrating factor. Try to find this integrating factor and prove it.

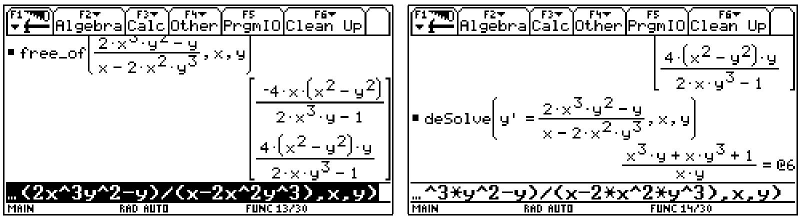
a) $2x\,y-(2x^2+y)\,y'=0$

b) $(x^2+y^2)\,(x\,dy-y\,dx)=x^4\,(a+x)\,dx$

c) $(x\cos y-y\sin y)\,dy+(x\sin y+y\cos y)\,dx=0;\quad P\left(1\middle|\frac{\pi}{2}\right)$

d) $(2x^3y^2-y)\,dx=(x-2x^2y^3)\,dy$





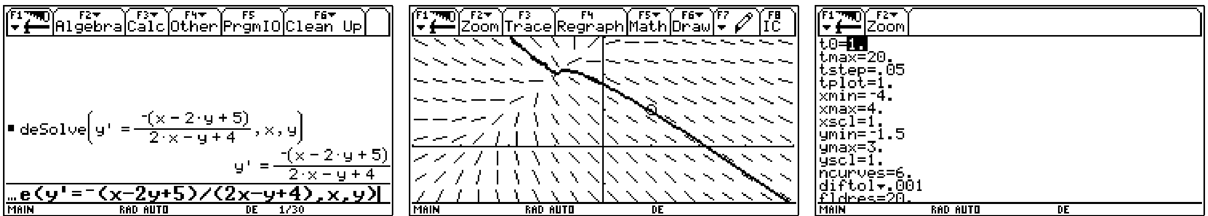
free_of and **intfct** are functions according to the functions which could be found in earlier Derive´s ODE1.MTH.

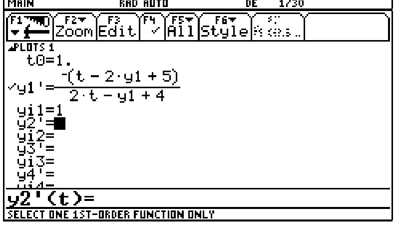
Ex 15: $(x-2y+5)\,dx+(2x-y+4)\,dy=0$

a) Find the general solution using $(x_0|y_0)$ and using a parameter c.

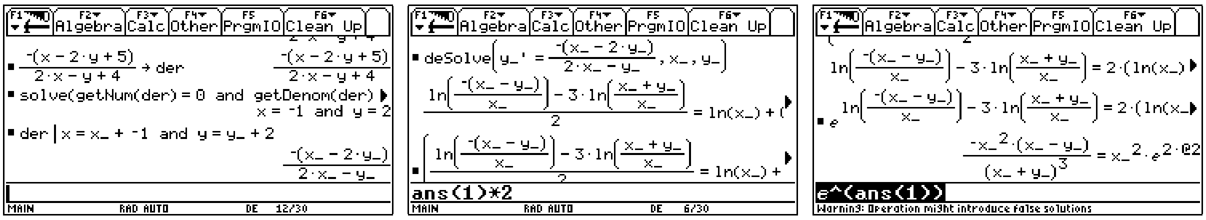
b) Find the special solution with $y(1)=1$.

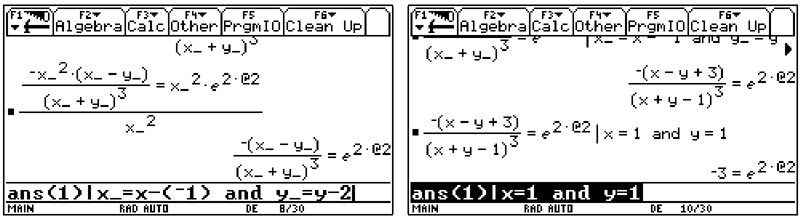
c) Sketch the integral curve and the direction field.

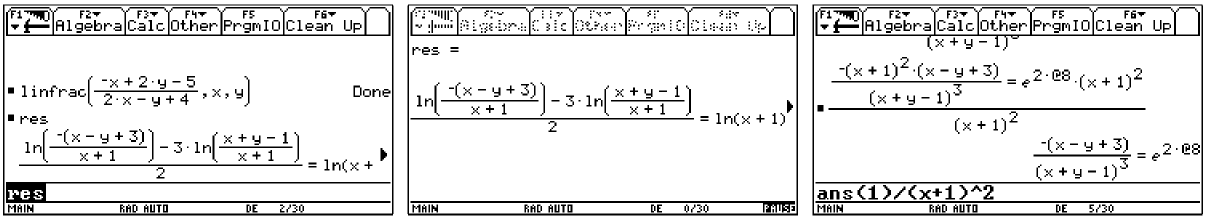


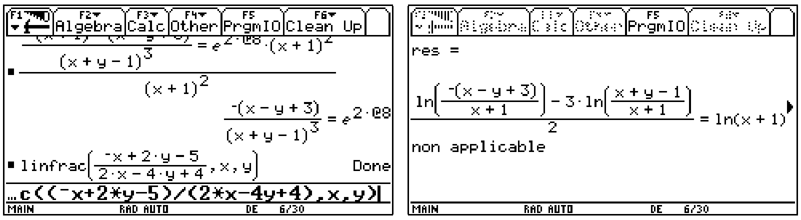


deSolve does not work. I investigated how Derive does and first solved the equation step by step to pack later the solving procedure it into one function **linfrac**. (download).









$\text{LIN\_FRAC}\left(\frac{-x+2\cdot y-5}{2\cdot x-4\cdot y+4}, -1, 2, -5, 2, -4, 4, x, y, 1, 1\right)$

?

$\text{FUN\_LIN\_CCF\_GEN}\left(\frac{-x+2\cdot y-5}{2\cdot x-4\cdot y+4}, 2, -4, 4\right)$

$\frac{x-2\cdot(y-1)}{2}-\frac{3\cdot \text{LN}(2\cdot x-4\cdot y+7)}{4}=x+c$

$\left(\frac{x-2\cdot(y-1)}{2}-\frac{3\cdot \text{LN}(2\cdot x-4\cdot y+7)}{4}=x+c\right)\cdot 8$

$4\cdot(x-2\cdot(y-1))-6\cdot \text{LN}(2\cdot x-4\cdot y+7)=8\cdot(x+c)$

$-6\cdot \text{LN}(2\cdot x-4\cdot y+7)+4\cdot x-8\cdot y+8=8\cdot x+8\cdot c$

$(-6\cdot \text{LN}(2\cdot x-4\cdot y+7)+4\cdot x-8\cdot y+8=8\cdot x+8\cdot c)-8\cdot x$

The final solution:

$-6\cdot \text{LN}(2\cdot x-4\cdot y+7)-4\cdot x-8\cdot y+8=8\cdot c$

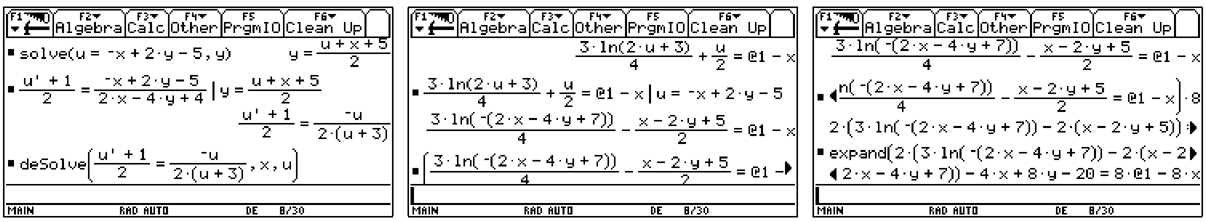
Special case if "numerator line" and "denominator line" don´t intersect.

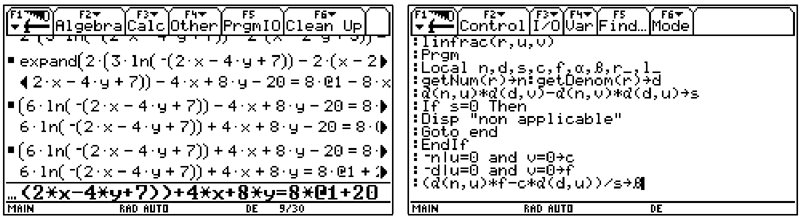
Then substituting u = numerator leadds to a DE which can be treated by separation of variables. You can follow this below.

You might extend the TI-program **linfrac** in such a way that this special case is included instead of answering "non applicable".

(download)

Derive provides a special utility function for this case.

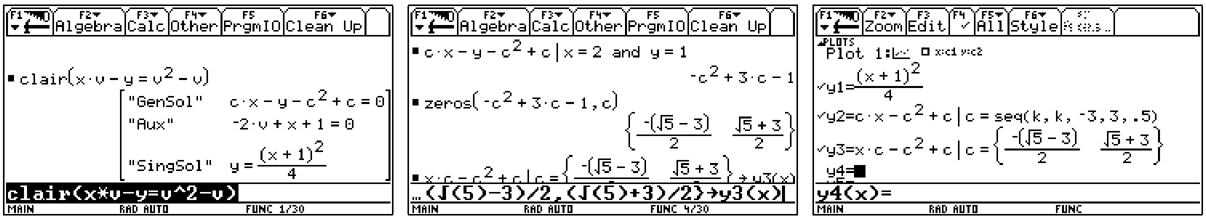


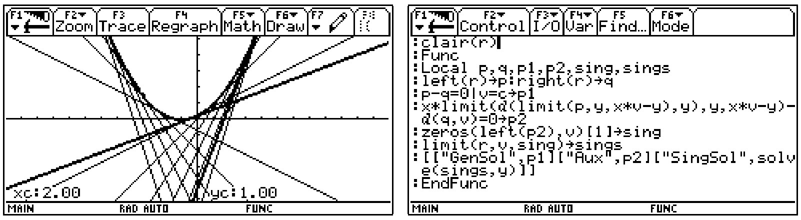


Ex 16: Clairaut equation $y = x\,y' + y' - y'^2$

Find the general solution, give the singular solution and plot the respective graph.

Find the special solution(s) through $P(2|1)$.

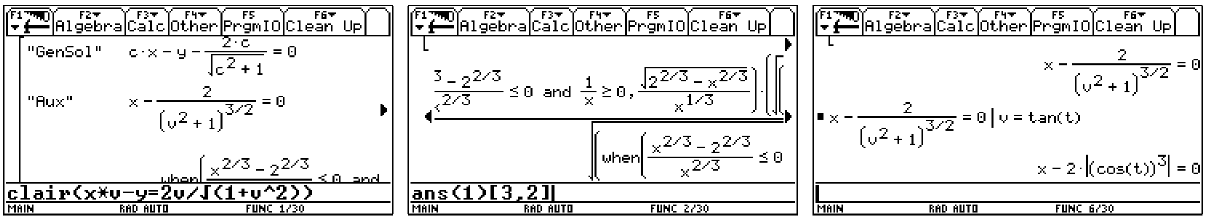


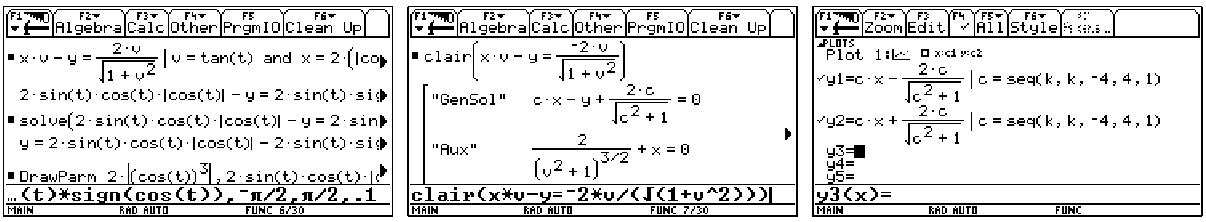


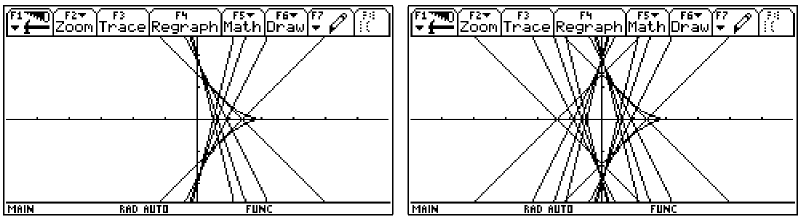
deSolve cannot be applied. Help yourself and write a function **clair**. (Replace y' by v.)

Ex 17: Which is the curve with its tangent´s segment between the axes having constant length $a = 2$?

Give both general and singular solutions.







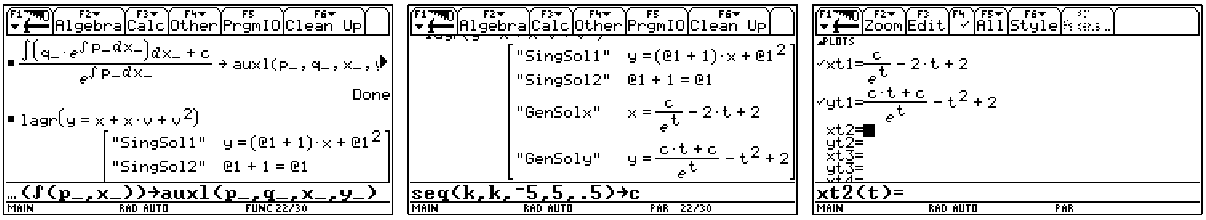
More explanations can be found in DNL#3.

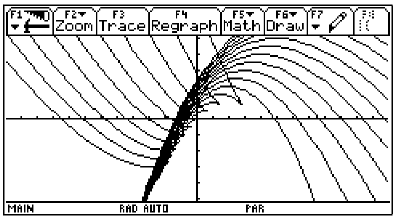
Ex 18: The following equations of form $y = x \cdot p(y') + q(y')$ are called LAGRANGE DEs.

a) $y = x \cdot (1 + y') + y'^2$

b) $y'^3 - 3y' = y - x$

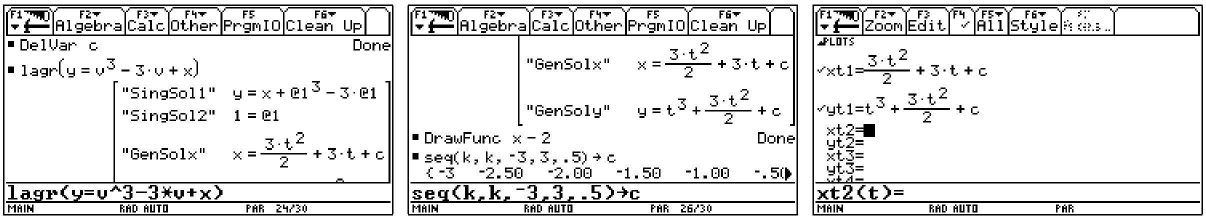
c) $(x\,y' + y)^2 = y^2 \cdot y'$

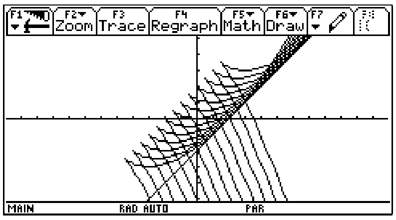




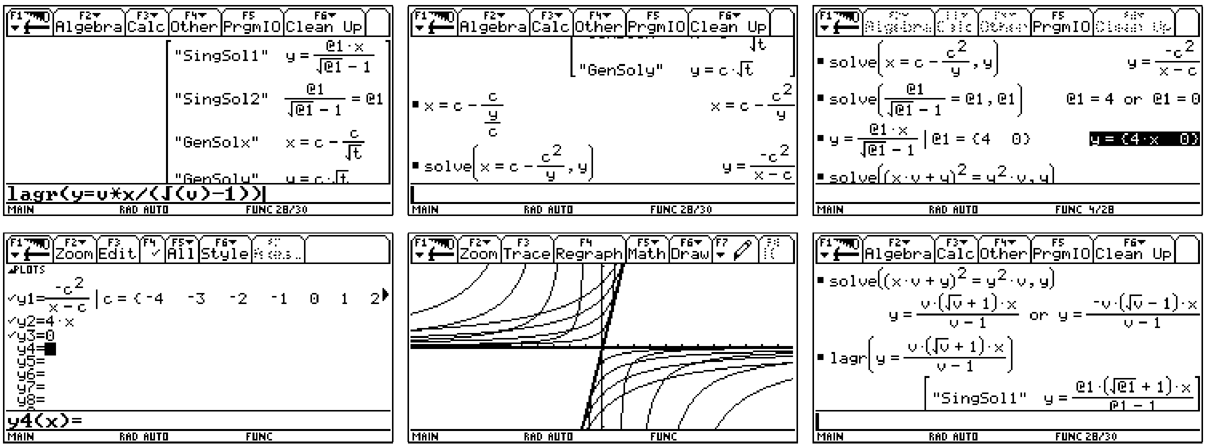
lagr(equ) with v for y' follows the Derive function from ODE1.MTH. You find the explanation in DNL#3. (download)

No singular solution, a family of special soultions.





Singular solution $y = x - 2$ together with a family of special solutions.



[1] Günter-Kusmin, Aufgabensammlung zur Höheren Mathematik, Berlin 1964

[2] The Calculus Problem Solver, Research and Education Association 1985

[3] Büktas, Aufgabensammlung, Diesterweg 1978

[4] Günter Redl, T^3-Course Materials

[5] Derive Newsletter 2, 3, 54