How to Make Traditional Tasks Technology Compatible

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One of the MFAQ (= Most Frequent Asked Questions) after workshops or lectures on the use of modern technologies like the TI-family, *DERIVE*, MS-EXCEL or others for maths education in Secondary Schools is how to find problems and tasks for exercises and assessment. Many of the more experienced teachers have a collection of problems of their own and it is very understandable that they don´t want to throw all their treasures away and start again like at the very begin of their career as a teacher.

I was and still am in the same situation. My standard answer is: you don´t have to change everything, take your problem collection and try to achieve another point of view. Then in many cases you will be able to give the problems a spin into the right direction and thus they will adopt a new quality, focus on another aim as usual, open new insights or just let it read a bit uncommon.

Please don´t expect results of a deep scientific research on that topic within the next short hour and don´t expect any classification of how to give "special spins". I was an ordinary school teacher on a Secondary College for Business Administration in St.Pölten, the capital town of Lower Austria. I also have been busy in in- and pre-service courses for teachers and give a lecture on the use of modern technologies in maths education on the Vienna Technical University. So you can imagine that I am very deep involved in the discussion mentioned above.

You can find many smaller examples of how to make traditional tasks technology compatible in my paper "Give it a Spin" which i included on the Conference CD. (You will find that I used the same introductional paragraphs – because this lecture deals with the same problem).

Today I´d like to demonstrate my ideas using some more extended problems in my talk and you will have the opportunity to work on some tasks in the workshops. I will show a couple of endexamination (= "Matura-") problems. They all emerged from traditional "Matura" problems from the eighties – i.e. from times without any symbolic calculator (and in Austria we didn´t use any graphic calculator, too – at least at most of the schools).

We will not see specially TI- or *DERIVE*-designed examples. There are a lot of publications containing wonderful problems together with the solutions. I always wonder how much time would be necessary to prepare the students to solve this or that very special problem.

I´ll start with a problem which was part of an endexamination in the late eighties:

Make New from Old

The original version reads:

All graphs of a family of functions $f_k(x)$ of degree 4 which are symmetric with respect to the y-axis intersect the graphs of another family of curves $g_k(x) = \dfrac{1}{16k}x - \dfrac{1}{32k}x^2$ orthogonal in P(2/0) and also have the origin in common.

a) Show that $f_k(x) = k\,x^4 - 4\,k\,x^2$.

b) Sketch the graphs of both functions for $k = 1/4$ including the zeros, extremal values and inflection points.

c) The graphs of f_k and g_k form an area for $k \geq 0$. Show that this area is given by $A(k) = \dfrac{1}{16k}x + \dfrac{64k}{15}$.

d) For which positive k will we receive the minimum area A?

This is my proposal for a new version:

Given is a family of funktions $g_k(x) = \dfrac{1}{16k}x - \dfrac{1}{32k}x^2$.

a) Which is the form of all curves of the family?
b) What are the common properties of all graphs? Give reasons!
c) What is the influence of parameter k on form and position of the graphs. Present your findings using an appropriate survey.
d) Find a "partner family" $f_k(x)$ so that each g_k is intersecting the corresponding f_k orthogonally.
e) Show for any k that condition d) is fulfilled.
f) What is the common area enclosed by two "partners"? Shade this area on your device for any appropriate k. Explain how you can achieve this.
g) Which value of k makes this area extremal? Is it a Maximum or a Minimum?

Proposed Solution:

a) It is a family of parabolas. Plot some members of the family with positive k and expect downward opening parabolas which all pass (0/0) and (2/0) and are symmetric wrt $x = 1$. See b)

#1:   VECTOR$\left[\dfrac{x}{16 \cdot k} - \dfrac{x^2}{32 \cdot k},\ k,\ 1,\ 5\right]$

#2:   VECTOR$\left[\dfrac{x}{16 \cdot k} - \dfrac{x^2}{32 \cdot k},\ k,\ 0.1,\ 1,\ 0.1\right]$
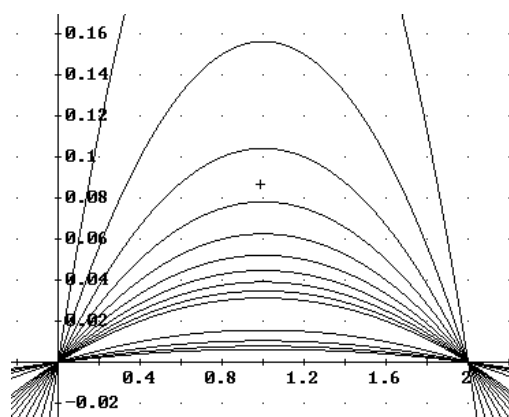
b) Rewrite the expression in $g_k(x) = \dfrac{1}{32k}(-x^2 + 2x)$. It follows that

- the zeros are independent of the choice of k in $x_1 = 0$ and $x_2 = 2$. Hence the vertex and axis of symmetry is in $x = 1$,
- $k \neq 0$, otherwise special case (two vertical lines $x = 0$ and $x = 2$),
- $k < 0$, parabolas opening upwards (coefficient of the quadratic expression is positive),
- $k > 0$, parabolas opening downwards,
- large $|k|$ gives flat parabolas and small $|k|$ gives steep parabolas.

c) Has already been answered

d) As four conditions must be satisfied – which results in four equations – we expect a cubic. The solution shows, that family f_k again consists of parabolas. This might be caused by the symmetry, because a cubic is not symmetric wrt a line.

```
#3:    g(x) :=   x      x²
               ─────  ─ ─────
               16·k     32·k

#4:    f(x) := a·x³ + b·x² + c·x + d

#5:    [f(0) = 0, f(2) = 0, f'(0) = - ─────── , f'(2) = - ───────]
                                       g'(0)               g'(2)

#6:           [d = 0, 8·a + 4·b + 2·c + d = 0, c = - 16·k, 12·a + 4·b + c = 16·k]

#7:    SOLVE([d = 0, 8·a + 4·b + 2·c + d = 0, c = - 16·k, 12·a + 4·b + c = 16·k], [a, b, c,
         d])

#8:                   [a = 0 ∧ b = 8·k ∧ c = - 16·k ∧ d = 0]
```

$$f_k(x) = 8\,k\,x^2 - 16\,k\,x$$

e) Choose $k = 1/4$:

$$\text{SUBST}\left[[f'(0)\cdot g'(0),\ f'(2)\cdot g'(2)],\ k = \frac{1}{4}\right]$$

$$[-1,\ -1]$$

f)

$$\int_0^2 (f(x) - g(x))\,dx = -\frac{256\cdot k^2 + 1}{24\cdot k}$$

$$y \geq 2\cdot x^2 - 4\cdot x \ \wedge\ y \leq \frac{x}{4} - \frac{x^2}{8} \ \wedge\ x \geq 0 \ \wedge\ x \leq 2$$
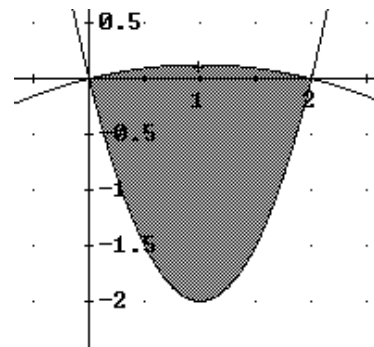
g)

$$f1(k) := \frac{256\cdot k^2 + 1}{24\cdot k}$$

$$\text{SOLVE}(f1'(k) = 0,\ [k]) = \left[k = -\frac{1}{16} \ \vee\ k = \frac{1}{16}\right]$$

$$\left[f1''\left(-\frac{1}{16}\right),\ f1''\left(\frac{1}{16}\right)\right] = \left[-\frac{1024}{3},\ \frac{1024}{3}\right]$$

$$\left[f1\left(\frac{1}{16}\right),\ f1\left(-\frac{1}{16}\right)\right] = \left[\frac{4}{3},\ -\frac{4}{3}\right]$$

For both cases we find the minimum for $k = \pm 1/16$. The reason for the different signs is the sign for the area. It depends on the position of the graphs – which one is the upper one? The mimum area is 4/3.

Check if it is really the minimum using two other values for k:

$$\left[f1\left(\frac{1}{2} \right), \ f1\left(-\frac{1}{2} \right) \right] = \left[\frac{65}{12}, \ -\frac{65}{12} \right]$$

$$\left[f1\left(\frac{1}{20} \right), \ f1\left(-\frac{1}{20} \right) \right] = \left[\frac{41}{30}, \ -\frac{41}{30} \right]$$

Comparing the two versions you will find that – maybe surprisingly for you – I do without the quartic in the technology version. Nevertheless it is my strong feeling that the problem has reached some new qualities including reasoning, presenting results, working with graphs, working with logic expressions.

The old version is one or two pages full of operations and the student might get lost in his/her calculations. It is no problem for me to disclaim some operational skills instead of these new and thrilling possibilities.

Again an Extremal Value Problem!

A window has the form of a rectangle together with a semicircle (see the details in picture at the right).

(CU = Currency Units)

All straight connections cost 22 CU/dm, and the curved parts cost 25 CU/dm kosten. Which measurements keep the production cost minimal?

a) Use the Dynamic Geometry model [1] to obtain an appropriate estimation applying an approximating function of your choice. Check first the correctness of the model. Explain carfully your strategy (table, sketch, ...).
    Additional Question: What is the reason that a polynomial function cannot fit exactly?

b) Solve the problem analytically using means of Calculus.

c) Show that the graph of the costfunction found in b) passes the data points produced in a) and answer the following question dependent on the choice of the independent variable (b or h):

   c1) If b was chosen: What is the end behaviour of the curve? Find the approximating curve for the end behaviour and interpret this curve in connection with the problem.

   c2) If h was chosen: Interpret the intercept of the cost function on the vertical coordinate axis..

d) How does the result change if one has to consider the cost for the glassing (8 CU/dm^2), too? Only the analytical treatment is necessary.

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[1] The Cabri-Model can be downloaded and via *GraphLink* transferred to your TI. The students receive this model at the begin of the examination They also have the self produced script `extrem` at their disposal. A rich collection of other Cabri-objects can be found on the diskette accompanying the bk-teachware booklet *Optimzation graphically, numerically and analytically*. This Cabri-model is on the CD.

Proposed Solution:

a)

Presentation of the cost dependent on b.
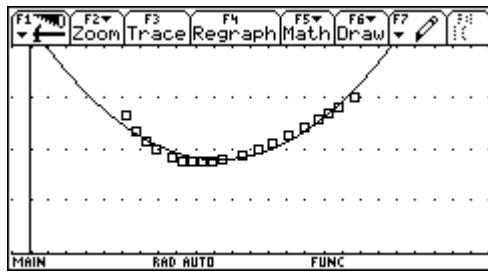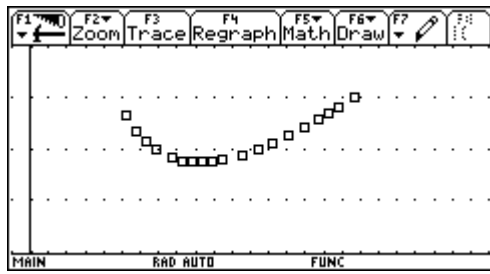
Quadratic regression seems to be not very fine. (But if the datapoints collected are gathering around the expected minimum a parabola could be sufficient for a reasonable first approximation.
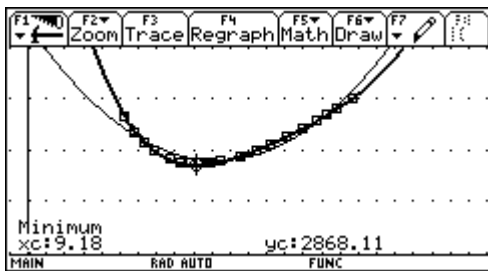

A quartic regression line fits very well and one obatins an estimation for the minimal cost with 2868 CU.

This is in full accordance with our impression from the data table.

Additional question: if b becomes smaller and smaller then h becomes greater and tends to infinity, so we expect a pole for $b = 0 \rightarrow$ no polynomial function.

b) For solving the extremal value task in the traditional way the students can work as they like. Maybe that they are using a self-provided generalized script (see below). We go on using b as the remaining variable in the main condition.
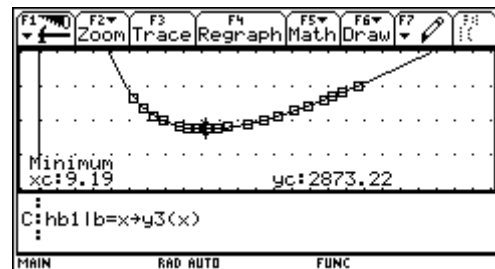
the exact modelfunction together with the minimum

One could add the investigation of the 2nd derivative to confirm that it is a minimum?

Working with *DERIVE* and using an appropriate program the problem can be reduced to set up main- and side condition. (You can find it on the CD.)

$$\text{extrem}\left[22\cdot\left(3\cdot h + \frac{13\cdot b}{2}\right) + \frac{25\cdot b\cdot\pi}{2},\ b\cdot h + \frac{\left(\frac{b}{2}\right)^2\cdot\pi}{2} = 200,\ b,\ h\right]$$

b	h	Optimum	Art
$\dfrac{40\cdot\sqrt{33}}{\sqrt{17\cdot\pi + 572}}$	$\dfrac{20\cdot\sqrt{33}\cdot(143 - 4\cdot\pi)}{33\cdot\sqrt{17\cdot\pi + 572}}$	$20\cdot\sqrt{33}\cdot\sqrt{17\cdot\pi + 572}$	lokales Minimum

b	h	Optimum	Art
9.188308456	18.15855181	2873.216558	lokales Minimum

Extracting *b* from the side condition the process becomes more difficult because of the quadratic equation. The CAS masters this difficulty easily, if the "Master of the CAS" selects the appropriate solution of the equation.

c)

The approximating curve is a line passing the origin $y = \left(143 + \dfrac{17\pi}{4}\right)x$. If *b* becomes larger then height *h* will become so small that it can be neglected in the cost function $K = 66h + 143b + 12.5\pi b$ and only the cost for the – coinciding - segments $6{,}5b \times 22 = 143b$ and the semicircle with radius $b/2$, $b\pi/2 \times 25$, are remaining. This only does not give the slope of the line. One has to take in account also the expression for $h = \dfrac{1600 - b^2\,\pi}{8\,b} = \dfrac{200}{b} - \dfrac{b\,\pi}{8}$. And here

one finds that a correcting contribution of $-\dfrac{b\pi}{8}$ for each *h* must be considered. This leads to

$$143b + \frac{25b\pi}{2} - 66\cdot\frac{b\pi}{8} = b\left(\frac{17\pi}{4} + 143\right).$$

If h tends to 0, then the window consists of the – mostly coinciding – segments with total length of $6{,}5b$ and the semicircle. To fill the 2m² the diameter of the semicircle is $b = \dfrac{40}{\sqrt{\pi}}$.

The cost are $b\,(6{,}5 \times 22 + \pi/2 \times 25) = 4113{,}39$.
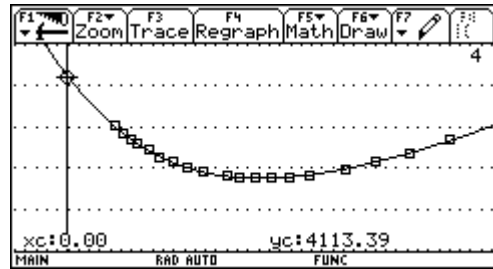
d) The result will not show any changes in the measurements, because the goal function is increased by the constant value of $200 \times 8 = 1600$. The constant has no effect on the 1st derivative. The minimal cost are now 4473,22 GE.

Finally I show the "general script" to work on maximum-minimum-value problems.

```
:Main Condition and save as hb
:
:Side Condition and save as nb
:
:Solve nb for one variable
:
:solve(nb,variable)
:
:and substitute in hb
:
:hb|ans(1) (if only one solution)
:and store as hbi

:hb|ans(1) →hb1
:
:differentiate wrt variable and
:find the zeros
: zeros(d(hb1,Variable),Variable)
:
:Check if it is really an extremal
:value

: d(hb1,Variable,2)|Variable=solution
: must be ≠ Øn

:the solution found is substituted
:into the expression for variable 2
: and into hb1.
```

On left hand side you can find a possibly solution for the "Window-Problem" performed using a German script.

```
:Hauptbedingung aufstellen
:und unter hb speichern
:
C:22*(3h+13b/2)+b/2*π*25→hb
:
:Nebenbedingung aufstellen
:und unter nb speichern
:
C:b*h+(b/2)^2*π/2=2ØØ→nb
:
:nb nach einer Variablen
:aufloesen:
:solve(nb,Variable)
:
C:solve(nb,h)
:
:und dafuer in hb einsetzen
:hb|ans(1) (wenn nur eine Loesung)
:und unter hb1 speichern
:
C:hb|ans(1)→hb1
:
C:hb1|b=x→y3(x)
:
:nach Variable diff. und Ø setzen
:zeros(d(hb1,Variable),Variable)
:
C:zeros(d(hb1,b),b)
:
C:ans(1)[2]→bopt
:
:Test ob's wirklich ein Extremwert
:ist
:d(hb1,Variable,2)|Variable = Loe
 sung muss ≠ Ø sein
:
C:d(hb1,b,2)|b=bopt
:
:die geeignete Loesung in den Term
:fuer die  zweite Variable und in
:die hb1     einsetzen
:
C:¯(b^2*π-16ØØ)/(8*b)|b=bopt→hopt
:
C:approx({bopt,hopt})
:
C:approx(hb1)|b=bopt
:
:oder
:
C:approx(hb)|b=bopt and h=hopt
```

I present this example perfomed on the TI. I can imagine to work with CABRI and DERIVE on the PC, too.

You see that I left the important part – setting up main- and side condition – but the operational part, which was and still is the core part in traditional exam questions is left to the CAS: I know and surely you know that the main problem for the students is not the calculus part, but solving the various equations, substituting for the variables and working with fractions etc.

Here I force them to use not only analytical means. At the other hand they have to show very analytical competences to answer question c) and also question d).

Financial Mathematics

What can we do for our Environment?

I was teacher on a Handelsakademie (= College for Business Administration). Our curriculum contains among others financial mathematics. Many colleagues complain that financial mathematics doesn´t leave any room for "real" mathematics. It is only working with percentages, interests, annuities, present value, future value, etc.
The following example was in its original version a traditional finance math problem – calculating annuities, number of payments and in those times the "high end" calculating the internal rate of return working with a decimal search – using the geometric series and an old pocket calculator.

See the change:

A company has to build an air filter system to avoid air pollution with total cost of 170 000 EURO. The company made savings for this investment during the last three years: at the end of each months it made a deposit of 1300 EURO, which gained 6.4% interests annually.

a)   How much could be saved for this important investment during the three years?
     How much outside capital – rounded to 100 EURO – is still necessary for this investment?

The company can choose between two financial models:

1.Model   The company receives a subsidy from the government in an amount of 15% of the total
             cost. The remaining rest must be financed by a bank loan with a life of 15 years and an
             annual interest rate of 7.5%.

b)   How much is the loan and how much are the monthly payments (due at the end of each period)?

c)   What is the internal rate of return of this financial model considering a credit tax of 0.8%, a
     charge of 3% and the subsidy (accurate to 0.1%).
     (Tax and charges are subtracted from the loan immediately).

2.Model: The company takes a loan which is subsidized by the government. The loan must be
             paid back within 10 years at a nominal annual interest rate is 6% with 4 compounding
             periods/year. The payments are due at the begin of each quarter and they consist of the
             constant amortization rate and the interests of the outstanding loan.

d)   Give the first four and the last three rows of the amortization table, showing the number of the
     period, the outstanding loan, the amortization rate, the interests and the full quarterly payment.

e)   Find a formula for the payment in any period k and compare with the respective values in the
     amortization table.

f)   Subsidy of the government is paying 15% of the amortization rate. Complete the table by the
     values of the reduced payments. What is the formula for the reduced payments?

g)   What is the effective interest rate of this loan considering the subsidy, a tax of 0.8% and credit
     charges of 1% (accurate to 0.1%)?

h)   Use CAS to find a closed formula for the present value of these variable payments for a general
     "Abzinsungsfaktor" v? ($v = (1+i)^{-1}$ with i = interest rate).

Prposed Solution:

a) The value of the saving is the future value of a monthly payment which is due 36 times at the end of each period (in German "Endwert einer nachschuessigen Rente → `ewrn(...)`).

So we have 51 305.25 € and we need a credit with a sum of 118 700 € .

The syntax of `ewrn()`-function is in detail:

```
ewrn(payment,number of payments,payment periods/year,
            increasing factor,compounding periods/year)
```

b) Subsidy is 25 500€, so a loan of 93 200€ is needed. Monthly payments are 850.99€.

In fact we have to reduce the loan by 3541.60€ (tax + charges).

c) For calculation of the effective rate of interest one has to consider that 180 times 850.99€ must be paid. The company receives 118700 – 3541.60€. This results in an interest rate of 4.06%.

d) For setting up an amortization table we can use the `Data/Matrix-Editor`. If `CellSheet` is at disposal it can be worked like using a spreadsheet program.

The headers for the columns are as follows:

```
c1: seq(i,i,1,40)
c2: seq(118700 - 118700/40*i,i,1,40)
c3: seq((118700 - 118700/40*(i - 1))*0.015,i,1,40)
c4: seq(118700/40,i,1,40)
c5: c3+c4
c6:c3 + 0.85*c4            (for question f) )
```

The first four and the last three rows can be read off now.

The CellSheet performing:

Formulae:
A2:   A1 + 1
B2:   B1 - D2
C2:   B1*0.015
D2: B1 / 40
E2:   D2+C2
F2:   C2 + 0.85*D2

e) and f)

The payments are stored as z(k) and zz(k).

Above you can find the accomplished table.

A check: The present value of all payments using 1.5% quarterly interest rate results in the loan.

Comparing the payments for the 7th quarter we can confirm the correctness of the formulae.
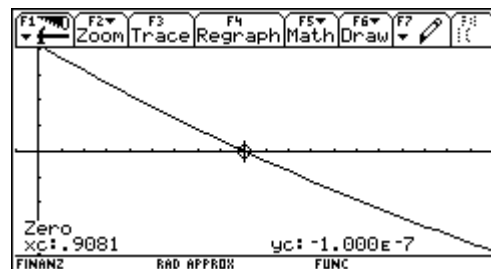
g) The loan is effectively 116 563.40€ (118 700 * 0.982). So we look for an interest rate making the present value of all payments zz(k) equal 116 563.40.

One can solve this equation numerically, graphically or using a table:

The fastest way is working with the table – having defined a function. Numerical solution and plotting the function followed by investigating the zero needs a lot of time.

h)

Obviously this doesn´t work! No closed form is given but only a long sum of power expressions.

Interestingly enough for the students – and not for them alone - we receive a closed formula generalizing the sum using a general number *n* of elements. (Now CAS cannot divide the expression).

The *TI* refuses again collecting the expressions having substituted *n* by 40:

The tiny – caused by rounding errors – difference of 2.16€ shows, that the formula for the sum should be ok, but we have to collect the expression. Gather all your skills from pre-CAS times and do it by hands using paper and pencil.

$$-\frac{1187v^{41}\left((120-293)v-120+290\right)}{80\,(v-1)^2}-\frac{1187v(293v-290)}{80(v-1)^2}=$$

$$=\ -\frac{1187v}{80(v-1)^2}\left(v^{40}(-173v+170)+293v-290\right)\ =$$

$$=\ \frac{1187v(173v^{41}-170v^{40}-293v+290)}{80(1-v)^2}$$

If we have not lost all our basic skills then we can confirm the result numerically.

DERIVE also divides the expression and gives a sum of power expressions. But the substitution can be done using the program (which now requires some "basic *DERIVE* skills").

Comparing with the traditional version which was restricted on the use of the formulae to calculate the annuities we can find some new qualities in the new form:

- Working with a self produced package of functions to solve basic problems from financial mathematics,

- Setting up the amortization table needs a deeper understanding of the process and a lot of abstraction to define the formulae,

- Changing between methods to solve equations approximatively,

- Showing the competence working with a CAS and recognizing its restrictions. Applying traditional "handicrafts",

- Checking the results by various means.

End Exam 2001 in St. Pölten

As many of the non Austrain readers might know, we don´t have national exams in Austria. Each school - and sometimes each teacher - sets up his/her tasks for the end exams (= "*Matura*") on the various forms of Secondary Schools. In my school - the "Bundeshandels-akademie St. Pölten - we always set up a common exam. This year we were three - Tania Koller, Günter Schraik and I - to teach in the last forms and we decided to give three tasks: one from statistics with a bit probability theory, one from financial mathematics (investment calculation) and one from analysis. The working time is 4 hours.

It was on me to set the analysis problem. Inspired by Carl Leinbach´s lecture in Gösing and our following discussion and a by a problem set by Sibylle Stachniss-Carp I posed the problem as follows:

1cm = 600m

In the region north of River Danube a golf course is planned. Its northern border is formed by a newly to build street which shall near *Winkl* (point W) continue the straight - also to be constructed - street S_1. Then it shall pass point O to finally near *Absdorf* (point A) run into the - also to be constructed - road S_2.

a) In first approximation find the line of construction of the northern border, which shall be described by a polynomial function of lowest degree. The connections between this road and S_1 and S_2 must be smooth.
   The data must be read off from the map as accurate as possible. What is the function term?
   Give reasons for your choice of the polynomial degree.

b) The connection between roads should not be only smooth - i.e. differentiable, but one has to avoid sudden changes of curvature, too. In order to reach this goal the curves must have the second derivatives in common.
Which polynomial function will describe the run of the road now. Use this function for the next tasks.

c) Draw by hands the new road on the map using at least 6 additional points.
Define the line S_1 - new road - S_2 as a piecewise defined function. Write down the correct syntax.

d) What is the difference in the lengths of both models?

e) Which area is available for the course if the southern border is formed by an existing road S (connecting road between W and A)? In 1st approximation you can use the airline between W and A.

f) What do you think? Is the real area greater or less that the value calculated? Try to improve your result in a not too difficult way. Describe shortly your choice of improvement.

Proposal for a solution

a) Most of the students chose the origin in W with x - axis horizontal and y-axis vertical, but there were also some of them who chose line WA as x-axis with W as origin. Let´s follow the majority:

We find by measuring W(0 | 0), O(3.9 | 2.7) and A(7.8 | 0.8), the slope of S_1 is either found by rise/run ≈ 0.6125 or tan(31°). To model the smooth connection in A one has to draw the tangent on S_2 using a ruler and read off the slope as ≈-1, So we obtain material to set up 5 equations - to determine 5 variables, which leads to the conclusion to set up a polynomial of degree 4: $y = a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e$.

$y(0) = 0$ $\rightarrow$ $e = 0$
$y'(0) = 0.6125$ $\rightarrow$ $d = 0.6125$
$y(3.9) = 2.7$ $\rightarrow$ $3.9^4\,a + ... = 2.7$
$y(7.8) = 0.8$ $\rightarrow$ $7.8^4\,a + ... = 0.8$
$y'(7.8) = -1$ $\rightarrow$ $4 \cdot 7.8^3 + ... = -1$

The solution of the system of linear equations was obtained by using the `rref()` command for producing the row reduced echelon form:
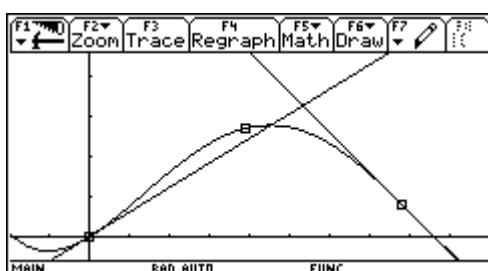

$$y = 0.0032460\,x^4 - 0.0588186\,x^3 + 0.2020053\,x^2 + 0.6125\,x$$

b) Two additional conditions give to more equations → two more parameters → polynomial of degree 6.

$y''(0) = 0$

To find an approximation for the 2nd derivative we need a quadratic approximation for S_2 in point A. The students read off three points on S_2 close by and including A: A(7.8 | 0.8), B(8.3 | 0.45), C(8.9 | 0.5).

Using the regression tool we find the parabola passing A, B and C: $y = 0.7121\,x^2 - 12.1652\,x + 52.3627$

Hence $y''(7.8) = 1.4242$ and we improve $y'(7.8) = -1.056$.

The 7×8 matrix must be filled and using `rref()` again we find the solutions for the parameters a through g in order to define the polynomial of degree 6:

$$y = 0.000309\,x^6 - 0.005205\,x^5 + 0.0245\,x^4 - 0.02947\,x^3 + 0.6125\,x. \quad \text{(stored as } \texttt{pol6(x)}\text{)}$$
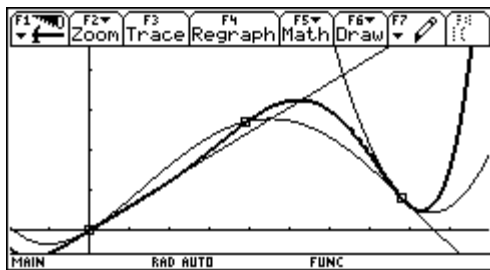
You can see both polynomials together with the graphic representation of S_1 and the approximating parabola of S_2.

```
when(x<Ø,Ø.6x,when(x<7.8,y2(x),y3(x))
)
```

Line WA: $y = \dfrac{0.8}{7.8}\,x$ (stored as `ger(x)`)

c) From the table we easily can obtain the coordinates of points of the curve to tranfer them onto the map:

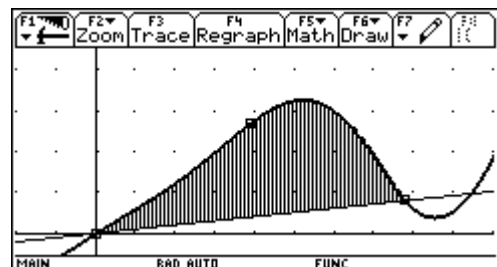d) The arc lengths can be found in the Graph Screen ([F5]) or in the Home Screen:

The lengths are 5.50km and 5.95km, so there is a difference of 450m.

e) The area is the integral shown above. But - not surprising for all of you who are teachers - the most difficult problem for the students was the conversion of the 11.89 cm^2 into km^2!!!

Area $\approx 11.89 \times (600\text{ m})^2 = 11.89 \times (0.6\text{ km})^2 = 4.3\text{ km}^2$.

f) The real area is larger.

We can approximate the additional area by a triangle Δ: $8 \times 0.6 / 2 = 2.4$ cm^2 $\rightarrow$ 0.93 km^2 (total 5.23 km^2).
or we find another parabola passing W, A and point D(4.7 | -0.1) and then we use the TI to calculate the area between pol6(x) and this parabola giving a total sum of 5.17 km^2.

The background of this problem is finding a polynomial function by some given properties. This version makes the problem more open because of the given data. The students must fix their own system of coordinates to have data.

The simultaneous system of equations are not to handle without technology.

The very useful and important concept of piecewise defined function is used. My experience was that the students found a lot of ways to estimate the additional area (f) and demonstrated their understanding of the beasics of integrating (summing up smaller areas).

A nice story is the following: This example was one out of three – working time was 4 hours – and some days after when the students asked for thier results one of them came to us and asked: "May I have a copy of this street-example, please. It liked it so much and I enjoyed the examination with especially this problem. I´d like to keep it as a memory on my maths-endexamination". I believe that this experience doesn´t need any more comment.
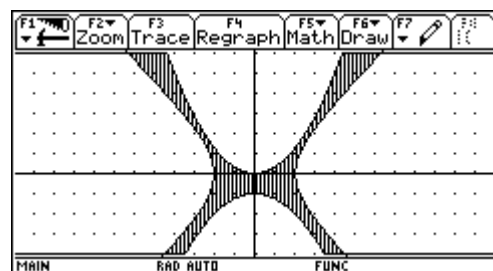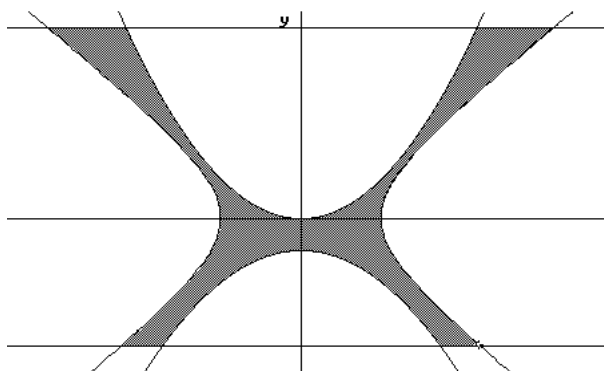
Finally I´d like to present two of my favorites how to include new technologies into old problems:

A cup of brass is produced by rotation of the hyperbola h: $x^2 - y^2 = 4$ and the parabolas:

$p_1 : y = \dfrac{5x^2}{16}$ and $p_2 : y = -a\,x^2 - 1$ for $-4 \le y \le 6$ around the y-axis.

a)   which is the complete equation of p_2, if the circular ring serving as base has a thickness of 1?

..................
....

You know all the tasks connected with problems like this: volume, area of intersection .....

But here: ask for producing the shaded intersection figure! Ask for the minimum thickness of the cup! We can do this in the workshop.
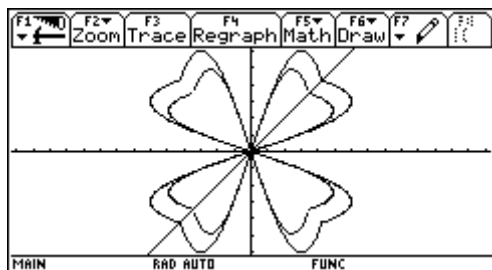
Another problem:

Find a an odd cubic by a given point and int 2nd derivative. Then use symmetries with respect to the axes to receive finally the following figure:
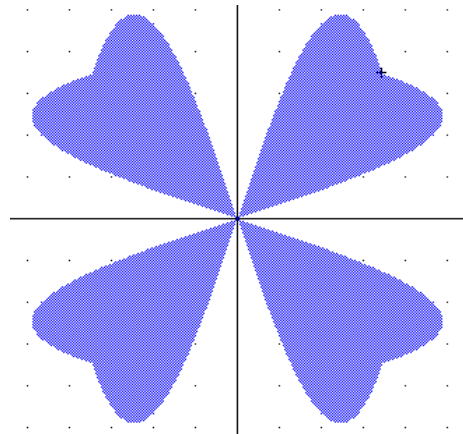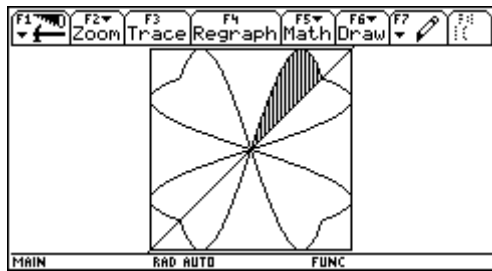
It is not difficult to shade a part of the leaf, but it is difficult to shade the whole figure.

What about this question:

Change the given figure by applying a compression with its center in the origin, that the area of the four-leaved trifoil leaf is a third of the area of the circumscribed square.

This is my summary how to make problems technology compatible:

- Include the graphing capabilities for additional tasks - forcing visualization.

- Encourage and ask for numerical and graphical solutions, ask for more than one way of solution, accept heuristic methods, encourage and ask for "Try and Error Methods".

- Let the pupils produce self made tools and give opportunities to apply them.

- Provoke functional thinking.

- Find the problem to presented solutions.

- Let the pupils set up conjectures - but also ask sometimes for the proofs.

- Include the various ways of the CAS´ output into the task, interpret the output and compare with calculation by hands, make the CAS built-in simplification rules to a mathematical subject.

- Use the CAS-capabilities for additional questions (and keep staying within the curriculum!!)

- But over all take care that you don´t give problems which set the calculator in the center of the problem, we should remain guided by mathematics and not by the manual.

Let me close with two warnings: don´t try all this the first time in the frame of an assessment. The students must be given many opportunities to acquire and to demonstrate their additional compentencies and abilities.
And: Marking the tests will be more difficult. Sometimes you will get differing solutions and very differing ways to find solutions. Marking verbal answers is not easy and sometimes it is not easy to explain the marks to the students.