The Use of Matlab/Maple in Solving Interval Hull Of A System of Linear Interval Equations.

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Abstract

Using Matlab/Maple for solving a system of linear interval equations Ax = b , where A, x and b are interval matrices. It is necessary to know about interval algebra first. rather than using Matlab/Maple straightly. It is because those software do not support interval computation. In reality , most practical work all quantities within an interval. So interval arithmetic's become an elegant tool in computing.
In this paper we want to compute the interval hull of the system of linear interval equations.
Many methods could be apply for this propose, but we used The interval Gauss elimination's and will be support by Matlab/Maple. In the end of this paper we want to compare the results.

Introductions

We assume that we know about the elementary properties of the interval arithmetic. But we can remind some definitions and properties with the terminology Newmaier [3]

A real interval is a set of the form

$$x \equiv [\underline{x}, \overline{x}] := \{ \tilde{x} \ \varepsilon \ R / \underline{x} \le \tilde{x} \le \overline{x} \}$$ closed and bounded in subset R

The open interval

$$]\underline{x}, \overline{x}[= \text{int}(x) ;= \{ \tilde{x} \ \varepsilon \ R / \underline{x} < \tilde{x} < \overline{x} \}$$

The set of all intervals is denoted by IR

$\underline{x} \equiv \inf(x)$; $\overline{x} \equiv \sup(x)$; $\hat{x}$ = mid(x) := ($\overline{x}$ + $\underline{x}$)/2 ; rad (x) := ($\overline{x}$ - $\underline{x}$)/2

$\tilde{x} \varepsilon x \Leftrightarrow | \tilde{x} - \hat{x} | \leq \text{rad}(x)$

$|x| \equiv \text{mag}(x) \equiv \max \{ |\overline{x}|, |\underline{x}| \}$; $< x > \equiv \min \{ |\underline{x}|, |\overline{x}| \}$

If S is a non-empty bounded subset of R we denote by

S : = [inf (S), sup(S)], the hull of S

The order relations are antisymetris and transitive,$\leq$ and $\geq$ are reflexive. But two interval need not be comparable since, e.g, [1,3] < [2,4] < [1,3]

Elementary operations $\circ \, \varepsilon \, \Omega$:= {+ , - , * , / , **} are defined on the set of interval by putting

$$x \circ y := \{ \tilde{x} \circ \tilde{y} / \tilde{x} \varepsilon x , \tilde{y} \varepsilon y \} = \{ \tilde{x} \circ \tilde{y} / \tilde{x} \varepsilon x , \tilde{y} \varepsilon y \}$$

For all x, y ε IR such that $\tilde{x} \circ \tilde{y}$ is defined for all $\tilde{x} \varepsilon x$, $\tilde{y} \varepsilon y$

This restricts the definition of the division x/y, to interval y with $0 \notin y$. Similarly, the exponential x**y is restricted to one of the cases: (I) $\underline{x} > 0$, (ii) $\underline{x} \geq 0$, $\underline{y} > 0$ (iii) $0 \notin x$, y an integer ≤ 0 or (iv) y a positive integer

If A ε IR^{nxn} is a regular interval matrix, we define a matrix inverse of A by :

$$A^{-1} = \{ A^{-1} : A \, \varepsilon \, A \} \qquad (1)$$

That is A^{-1} is the smallest interva[l] matrix that contains the set { A^{-1} : A ε A }

We call an interval matrix A inverse positive if A is regular and A-1 ≥ 0

An nxn interval matrix A is called an M-matrix if and only if $A_{ij} \leq 0$ for all $i \neq j$, and A**u** > 0 for some positive vector **u** ε IR^n

We use $<A>$ to denote the Ostrowski's comparison matrix of A, with entries $<A>_{ii} = <A_{ii}>$, $<A>_{ik} = -|A_{ik}|$ for $i \neq k$, thus $<A>$ has non negative diagonal elementsand nonpositive off-diagonal elements. We call A an H-matrix if and only if its comparasion matrix $<A>$ in an M-matrix [4]

From Kuttler [4] we get :

Proposition 1.

Let $A = [\underline{A} , \overline{A}] \varepsilon IR^{nxn}$. If $\underline{A}$, and $\overline{A}$ are regular and $\underline{A}^{-1} \geq 0$, $\overline{A}^{-1} \geq 0$ then A is regular and

$$A^{-1} = [\overline{A}^{-1} , \underline{A}^{-1}] \geq 0 \qquad (1)$$

It is known that every M-matrix is inverse positive, that why we can compute A^{-1}, if A is an M-matrix.

Suppose that $A \ \varepsilon\ IR^n$ is a regular interval matrix and $b \ \varepsilon\ IR^n$. So the solution set of interval system Ax = b ,

$$\sum (A,b) = \{ x \ \varepsilon\ IR^n : Ax = b \text{ for some } A \ \varepsilon\ A , b \ \varepsilon\ b \} \qquad (2)$$

In Neumaeir [4] , it is known that the hull A^H b of (2), the smallest interval vector that contains $\sum(A,b)$, satisfies :

$$A^H b = \{ x_D : |D| = I, \inf(D(Ax_D - b)) = 0 \} \qquad (3)$$

From now on the quantity A^H b = the hull (3) requires solution of $\inf (D(A x_D - b)) =0$ (4)

where D ranges over all D with $|D| = I$

S. Ning and R.B Kear Folt [4] have a method to compute the exact hull of Ax = b , when A is centered about a diagonal matrix. And also how to compute those hull when A is inverse positive. The base of their computing is :

THEOREM 1 (Beeck [4])

Let A ε IR^{nxn} be inverse positive then

$A^H b = [\ (A^{(1)})^{-1}\underline{b}\ ,\ (A^{(2)}\ \overline{b}\ ] = [\ \underline{x},\ \ \overline{x}]$

where $A^{(1)}$, $A^{(2)}$ are defined by

$$A_{ik}^{(1)} = \overline{A}_{ik} \text{ if } \underline{x}_{ik} \geq 0 \text{ and } A_{ik}^{(1)} = \underline{A}_{ik} \text{ otherwise,}$$
$$A_{ik}^{(2)} = \overline{A}_{ik} \text{ if } \overline{x}_{ik} \leq 0 \text{ and } A_{ik}^{(2)} = \underline{A}_{ik} \text{ otherwise.} \tag{4}$$

In particular

$$A^H b = A^{-1} B = \begin{cases} [\overline{A}^{-1}\underline{b}, \underline{A}^{-1}\overline{b}] & if \quad b \geq 0 \\ [\underline{A}^{-1}\underline{b}, \underline{A}^{-1}\overline{b} & if \quad 0 \in b \\ [\underline{A}^{-1}\underline{b}, \overline{A}^{-1}\overline{b} & if \quad b \leq 0 \end{cases}$$

In [2], Hansen proposed a scheme to estimate the hull of a preconditions linear system .
Suppose that :

A = $\hat{A}$ + [-1, 1] rad (A) ε IR^{nxn} and b = [$\underline{b}, \overline{b}$] ε IR^n

Consider the equation : Ax = b (5)

$\hat{A}$ = mid (A)

Multiply on the left by $\hat{A}^{-1}$ we obtain the preconditioned equation

$\hat{A}^{-1}$A x = $\hat{A}^{-1}$ b ⇒ Mx = r (6)

Theorem Hansen [2]

Suppose M = [$\underline{M}, \overline{M}$] of (60 satisfies $\underline{M}^{-1} \geq$ (7)

Let $$s^{(i)} = \begin{cases} \overline{x_i} & forj = 1 \\ \max\{-\underline{r}_j, \overline{r_j}\} & for \quad j \neq i, \quad j = 1,2,..n \end{cases} \tag{8}$$

$$t^{(i)} = \begin{cases} \underline{r_i} & for \quad j = i \\ \min\{\underline{r_j}, -\overline{r}_j\} & for \quad j \neq i, \quad j = 1,2,..n \end{cases} \tag{9}$$

$$c_I = \frac{1}{2(\underline{M}^{-1})_{ii} - 1} \tag{10}$$

Then the hull of (6) is

$$M^H r = [\underline{x}, \overline{x}] \tag{11}$$

for

$$\underline{x}_i = \begin{cases} c_i e_i \underline{M}^{-1} t^{(i)} & for \quad \underline{x_i} \geq 0 \\ e_i^T \underline{M}^{-1} t^{(i)} & for \quad \underline{x_i < 0} \end{cases} \tag{12}$$

for i = 1,2, . . . , n when e_I^T is unit vector whose i^{th} coordinate is one and zero else where.

By Rohn [4] there is a

THEOREM

Assume $\underline{M}$ is inverse positive. Then hull of (6) is $M^H \gamma = [\underline{x}, \quad \overline{x}]$ where

$$\underline{x_i} = \min\left\{ \underset{\sim}{x_i}, c_i \underset{\sim}{x_i} \right\} \tag{13}$$

$$\overline{x_i} = \max\left\{ \tilde{x}_i, c_i \tilde{x}_i \right\} \tag{14}$$

where $$\underset{\sim}{x_i} = -x_i * + \left(\underline{M}^{-1}\right)_{ii} \left(\tilde{r} + | \tilde{r} | \right)_i , \tag{15}$$

$$\tilde{x}_i = x_i * + \left(\underline{M}^{-1}\right)_{ii} \left(\tilde{r} - | \tilde{r} | \right)_i , \tag{16}$$

$$x_i * = \left(\underline{M}^{-1} \left(\tilde{r} + \frac{\overline{r} - \underline{r}}{2} \right) \right)_i \tag{17}$$

$$c_i = \frac{1}{2\left(\underline{M}^{-1}\right)_{ii} - 1} \in (0,1) \tag{18}$$

where $\tilde{r}$, the mid point vector, is that vector whose i^{th} component is $\dfrac{r_i + \tilde{r}_i}{\underset{\sim}{2}}$.

THE THEORY

In Gauss elimination the main concept for solving the e.g. $Ax = b$ is to decompose $A = LU$. By Fiedler and Pta'k [4] , there is a

LEMMA

Suppose that $A \in R^n$ is an M matrix and $A = LU$ where L is lower triangular and U is upper triangular, and $L_{ii} = 1 \; for \; i = 1,2,\ldots,n.$ Then

$$U_{ij} = A_{ij} - \sum_{k=1}^{i-1} L_{ik} U_{kj} \qquad forj \geq i \tag{19}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj}}{U_{jj}} \qquad fori > j \tag{20}$$

and $U_{ii} > 0, \; U_{ji} \leq 0, \; L_{ij} \leq 0 \qquad fori \geq j$. That is the triangular factors of an M-matrix are M-matrices themselves.

THEOREM (from Bartk & Beck [4])

Suppose that $A = [\underline{A}, \;\; \overline{A}] \in \boldsymbol{R}^{nXn}$ is an interval M-matrix. $L = [\underline{L}, \;\; \overline{L}]$ be that lower triangular interval matrix with $L_{ii} = [1,1] \; for \; i = 1,2,\ldots, \; n$ and let $U = [\underline{U}, \;\; \overline{U}]$ be that upper triangular interval matrix defined by

$$U_{ij} = A_{ij} - \sum_{k=1}^{i-1} L_{ik} U_{kj} \qquad forj \geq i \tag{21}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj}}{U_{jj}} \qquad for i > j$$

Then $A \subseteq LU,\ A = LU$ and $\overline{A} = \overline{L}\ \overline{U};\ L\ \&\ U$ are interval M-matrices. $A^{-1} = L^{-1}U^{-1}$, that is, $[\ \overline{A}^{-1},\ \overline{A}^{-1}] = [\ \overline{U}^{-1}, \underline{U}^{-1}]\ [\ \overline{L}^{-1},\ \underline{L}^{-1}]$. Moreover

$$A^H b \subseteq \overline{U}^{-1}(L^{-1}b) = [\overline{A}^{-1}, \underline{A}^{-1}]b \tag{22}$$

This following theorem helps us to obtain the interval hull easily when A is inverse positive.

THEOREM 2

Suppose $A \in \boldsymbol{R}^{nXn}$ is inverse positive and suppose that $b,\ x^{(0)} \in \boldsymbol{R}^n$ and $A^H b \subseteq x^{(0)}$. *For* $i,\ k = 1,2,\ldots,\ n,$ define $A^{(1)},\ A^{(2)} \in A$ by

$$A_{ik}^{(1)} = \overline{A}_{ik} \quad if \quad \underline{x}_k^{(0)} \geq 0 \quad and \quad A_{ik}^{(1)} = \underline{A}_{ik} \quad otherwise$$

$$A_{ik}^{2} = \overline{A}_{ik} \quad if \quad \overline{x}_k^{(0)} \leq 0 \quad and \quad A_{ik}^{(2)} = \underline{A}_{ik} \quad otherwise$$

Define

$$x = \{(A^{(1)})^{-1}b,\ (A^{(2)})^{-1}\ \overline{b}\}$$
$$= ([x_1,\ \overline{x_1}],\ldots,[x_n,\ \overline{x_n}])^T$$

where {v,w} is the interval hull of the vector v and w, i.e., the smallest vector that contains the set {v, w}. Then $x \subseteq A^H b \subseteq x^{(0)}$.

In particular, if $\underline{x}_k \underline{x}_k^{(0)} \geq 0$ and $\overline{x}_k\ x_k^{(0)} \geq 0$ for $k = 1,2,\ \ldots,\ n$ then

$$x = \left\{ \left(A^{(1)}\right)^{-1} \underline{b}, \left(A^{(2)}\right)^{-1} \overline{b} \right\} = A^H b$$

Proof in [4].

Implementation

The following examples illustrate the cases covered in the theory of the preceding sections and examine several cases outside this theory.
the computations in the examples were programmed in Matlab and Maple. This arithmetic was accessed with a modification in Maple and Matlab. The end points of interval in the results are rounded using the default conversion routines in printing function Maple and Matlab.

We have 4 examples , which were computed by Gaussian Elimination's and Hansen technique supported by Maple and Matlab

If we see the results below, we can examine that the results supporting by Maple/Maple almost the same.
But the results with different method not quite the same , sometimes Gauss Elimination's gives sharper bounds on the hull, while in other cases Hansen's technique does. In some problems, the intersection of the results of Gauss Elimination's and Hansen's in narrower than either result takes separately.

Matrix	Gauss Elimination		Hansen Technique	
	Maple	**Matlab**	**Maple**	**MatLab**
$A=\begin{bmatrix}[0.7,1.3] & [-0.3,0.3] & [-0.3,0.3] \\ [-0.3,0.3] & [0.7,1.3] & [-0.3,0.3] \\ [-0.3,0.3] & [-0.3,0.3] & [0.7,1.3]\end{bmatrix}$ $b=\begin{bmatrix}[-14,-7] \\ [9,12] \\ [-3,3]\end{bmatrix}$	[-70.10576926, 49.91346155] [-44.81250002, 70.52884616] [-63.00000003, 58.50000003]	[-70.10576923, 49.91346154] [-44.81250000, 70.52884615] [-63.00000000, 58.50000000]	[-101.0000000, 17.00000000] [-15.00000000, 99.00000000] [-90.00000000, 90.00000000]	[-101 , 17] [-15 , 99] [-90 , 90]
$A=\begin{bmatrix}[3.7,4.3] & [-1.5,0.5] & [0,0] \\ [-1.5,-0.5] & [3.7,4.3] & [-1.5,0.5] \\ [0,0] & [-1.5,-0.5] & [3.7,4.3]\end{bmatrix}$ $b=\begin{bmatrix}[-14,14] \\ [-9,9] \\ [-3,3]\end{bmatrix}$	[-6.377672560, 6.377672560] [-6.398258981, 6.398258981] [-3.404699587, 3.404699587]	[-6.377672558, 6.377672558] [-6.398258977, 6.398258977] [-3.404699585, 3.404699585]	[-6.377672562, 6.377672562] [-6.398258979, 6.398258979] [-3.404699587, 3.404699587]	[-6.377672558 , 6.377672558] [-6.398258977 , 6.398258977] [-3.404699585 , 3.404699585]
$A=\begin{bmatrix}[15,17] & [-3,3.01] & [-3,3.01] & [-3,3.01] \\ [-3.3,3.01] & [15,17] & [-3,2.99] & [-3,2.99] \\ [-3.2,2.99] & [-3,2.99] & [15,17] & [-3,3.01] \\ [-3,3.01] & [-3,3.01] & [-3,2.99] & [15,17]\end{bmatrix}$ $b=\begin{bmatrix}[-6,-2] \\ [-4,5] \\ [-2,4] \\ [8,10]\end{bmatrix}$	[-.6852706317, .5003751456] [-.5629445672, .9298723089] [-.4505833838, .8504468415] [.2485362122, 1.160108185]	[-0.685270632 , 0.500375146] [-0.562944567 , 0.929872309] [-0.450583384 , 0.850446842] [0.248536212 , 1.160108185]	[-1.066738815, .3918622847] [-.9017754418, .9854864361] [-.7615187127, .9299308809] [.2441198166 , 1.263264214]	[-1.066738815 , .3918622848] [-.9017754421 , .9854864363] [-.7615187126 , .9299308808] [.2441198169 , 1.2632642141]
$A=\begin{bmatrix}[4,6] & [-1,1] & [-1,1] & [-1,1] \\ [-1,1] & [-6,4] & [-1,1] & [-1,1] \\ [-1,1] & [-1,1] & [9,11] & [-1,1] \\ [-1,1] & [-1,1] & [-1,1] & [-11,9]\end{bmatrix}$ $b=\begin{bmatrix}[-2,4] \\ [1,8] \\ [-4,10] \\ [2,12]\end{bmatrix}$	[-1.306353150, 1.239818594] [-2.002473053, 2.762874889] [-.8336113576, 1.186575964] [-1.241632653, 1.601895735]	[-1.306353150, 1.239818594] [-2.002473052, 2.762874889] [-0.833611358, 1.186575964] [-1.241632653, 1.601895735]	[.0776978414 , .3913669070] [1.985611510 , .1510791365] [-.1438848919, .7956834529] [1.244604317 , .3165467626]	[.0776978417 , .3913669065] [1.9856115108 , .1510791367] [-.1438848921 , .7956834532] [1.2446043165 , .3165467626]

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