EXPONENTIAL GROWTH

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It is commonly known that people usually do not have a well-developed feeling of exponential growth. If it were not so, how could various organizers of Earn Money Fast schemes be so successful.
Let's have a look at the introduction of exponential function in a typical high school book (in Slovenia it is part of the second year grammar school curriculum). It goes somewhat as follows:

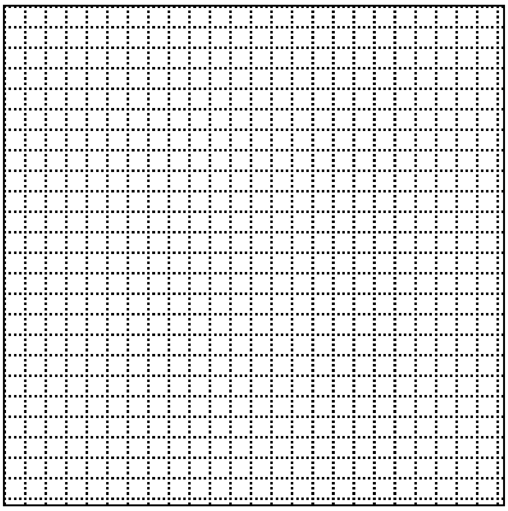
> *Let a be a positive real number, We already know what a^r means, if r is a rational number. What if r is an irrational number? ...* This is followed by the approximation using rational numbers, tabelaric representation, some properties and graphs, and at the end perhaps a few examples with the growth of bacteria, bank loans, ...

If we just follow the proposed explanation (even without the construction of approximations), we have to face numerous questions and a negative attitude in the form of "of what use could this ever be in our everyday life". Afterwards we are confronted with the fact that the majority of students does not understand the subject nor is able to use their knowledge of exponential function. It is also interesting that even if we present them with several cases where exponential growth occurs, they usually do not see that these examples have the same background.
Various interviews with students show that perhaps the least understood subject is the growth of exponential function. The usual approach with plotting an exponential function, showing its explosive growth, is often unsuccessful, especially when teachers present examples by themselves. Students do not realize how a process sky rockets when it grows exponentially.
To overcome this obstacle we developed a series of working materials in the hope that using them will help the students to grasp the feeling for exponential growth and exponential function. Working materials are intended to show several situations where we need to have a feeling of what exponential growth is. Among these examples are such well known stories as the ones about the reward for inventing chess, hamster population, etc. Very successful examples (in the view of students interest in the subject) are also money pyramids, which were very popular in Slovenia a few years ago, good luck chain letters, spreading of E-mail viruses like Melissa and Papa, ...
With the aid of DERIVE, TI-89 or 92 or similar tools we can let students try simulating all those events by themselves. On the basis of these explorations we then continue with investigating the role of parameters in functions of the form F #e^(At) + G. In the talk we will take a look at these materials.

WATER LILY

Janez has planted a water lily in his pond and also bought a supergrowth fertilizer for water lilies. A salesman warned him that he must follow instructions strictly as too much fertilizer can make the lily to grow so quickly that the are occupied by it would double every week. But Janez was lazy so he just threw some (of course too much) fertilizer into the pond. In the following weeks he could observe the explosive growth of his water lily.

Draw a scheme of how the events in the pond happened. Use different colors, shading or numbers to indicate growth in the first, second and subsequent weeks. Make a legend. Choose how many squares the lily covers at the beginning.



What is the area covered with water lilies if the lily initially covers an area of 1, 3, and 5.2 square decimeters?

week	area occupied	area occupied	area occupied
0	1	3	5.2
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

If the lily in the pond covers 8 dm^2 at the moment when Janez uses the fertilizer, how long will it take for the lily to cover the whole pond with the following area (some numbers are already inscribed). In the second half of the table write in the minimum number of weeks needed to cover the whole area.

pond area	weeks
1600 cm^2	
32 dm^2	
64 dm^2	3
256 dm^2	
10.24 m^2	7
20.48 m^2	
81.92 m^2	

pond area	weeks
32000 dm^2	
5000 m^2	
42000 m^2	
160000 m^2	
650000 m^2	23
2.5 km^2	
10 km^2	

Describe how you got the answer:

How many weeks does it take for the watter lily to cover one half of the area?

How long it will take the lily to cover the whole Slovenia (20256 km^2), if we start with a lily with the surface of 1 dm^2?

Suppose the lily has at the beginning a surface of 1 unit. Write down the expression for the area covered after the specified number of weeks.

week	area covered
0	1
1	
2	
3	
10	
15	
30	

Now try to generalize the expression. Write the expression for the area covered after n weeks

Now, let's denote the initial area covered by d. Write down the expression for the area covered after the specified number of weeks.

week	area covered
0	d
1	
2	
3	
10	
15	
30	

Generalize the expression. Write the expression for the area covered after n weeks, if the initial area is d.

What about if Janez uses an even stronger fertilizer, which causes the water lily to triple, quadruple, ... each week? Fill in the table:

week	area occupied		
	doubling	tripling	quadrupling
0	1	1	1
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

Write the expressions for the area covered after n weeks for each case.

doubling: ....

tripling: ....

quadrupling: ....

If we denote the factor of growth by g, write down the expression for the area covered after n weeks

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PAPER FOLDING

Take a sheet of paper. Fold it in half, do it again and so on. How many times can you do this? Assume that the thickness of paper is 0.09 mm. With each folding we double the thickness. Fill in the table that shows how thick the paper would be if the given number of foldings were possible. In DERIVE expression 0.1^{71} is entered as 0.1^71.

folding	thickness (mm)	thickness (km)
1		
2		
5		
10		
12		
20		
30		
40		
50		
100		

Find out how many folding would you need to get thickness (theoretically of course) equal to the given distance. If you know how to, you can calculate your answers or you can help yourself with experimenting.

from Kranj to Ljubljana (25 km) ----------------------------

from Ljubljana to Maribor (135 km) ----------------------------

form London to New York (5576 km) ----------------------------

from Earth to Moon (384,000 km) ----------------------------

from Earth to Sun (149,600,000 km) ----------------------------

from Earth to Sirius (85,147,200,000,000 km) ----------------------------

INTERNET MONEY GAMES

Janez got the following message. Read it and find out how the number of participants grows in this scheme.

```
All the best!

IT'S TIME TO LIVE YOUR DREAMS!
THE MOST HONEST ONE TIME PAY PROGRAM AVAILABLE!
WE SHOW YOU THE MONEY!

FINALLY a MLM Gifting Program that CAN'T FAIL, because we do all the work and
Guarantee Your Success! We print all the Flyers, we stuff all the Envelopes, and we do
all the mailing using our own Exclusive mailing list.

We guarantee each participant 10 people in their Downline, because we mail until each
participant has 10 people no matter how many flyers we have to mail. YOU CAN'T FAIL!!!
You may also do mailings on your own and increase the speed of your gifts. There has
never been a program like this! Making $$$ from your computer has never been so easy!
This is Not a GET RICH QUICK PLAN. This is a 100% HONEST and CONTROLLED program that
will insure you $10.000 in Gifts. Our participants usually start receiving their Gifts
in a few short months. There are Thousands of New Participants joining Weekly and with
Only Three Levels, Everything Moves INCREDIBLY FAST!

THIS PROGRAM IS LEGAL, ETHICAL, MORAL, HONEST, AND CONTROLLED
YOU CAN'T LOSE!
FORGET THE REST, JOIN THE BEST!

"THIS IS HOW IT WORKS"
Position 3. You join and we Postal mail and Email until you have "10"
Participants.....10
Position 2. We Postal Mail & Email until they have 10 Participants Each.....10 X 10 =
100
Position 1. We Postal Mail & Email until they have 10 Participants Each.....100 X 10 =
1,000
Those 1,000 Participants Gift You $10.00 EACH =........$10.000

YOU NOW HAVE $10,000!!! SIMPLY FOLLOW THE INSTRUCTIONS!

INSTRUCTIONS:
1. Dump Out & Print or type your personal information in the Box Below, or go to our
instant sign up order form, where we except Visa, MasterCard, and other personal
checks by email, fax, or phone instantly. There after, you will be able to get started
the same day. You will be emailed all membership information within a few hours, and
have access to our website. You will be able to direct anyone you wish to sign up with
your membership number instantly.

2. Enclose or make your payment for $20 & 60 First Class Postage Stamps.  ($10 for
person in #1 position & $10 for setup and monitoring fee). Or Enclose or make your
payment instantly via our instant order form for $41 and We'll buy the stamps.
"YOUR TOTAL INVESTMENT $41.00"

WE DO ALL POSTAL & E-MAILINGS USING OUR OWN EXCLUSIVE MAILING LIST TO GUARANTEE YOUR
SUCCESS!

*****YOU WILL RECEIVE $10,000 IN CASH GIFTS!!!!!*****
*****CHANGE YOUR LIFE! JOIN US TODAY!*****
```

Questions:

1. Fill in the following table, which shows how many new players are needed at each new stage. Remember – if you join the game for example at the 4^{th} stage – you will have to wait for the 7^{th} stage to get the money promised!

stage	new players
0	1
1	2
2	
3	
5	
6	
7	
8	
10	
15	
20	
25	
30	

2. What is the amount of money organizers will get at each stage? Suppose they are completely honest and really keep "just" 10$ for each player. How much they will earn at each stage?

stage	money paid	organizers earnings
0		
1		
2		
3		
5		
6		
7		
8		
10		
15		
20		
25		
30		

3. If we assume that for each person who participates in this scheme organizers need just 1 second for all the tasks needed to be accomplished, how long will it take to process new players at each stage. If needed, express time in days or larger time units.

stage	time needed
5	
6	
7	
8	
10	
15	
20	
25	
30	

4. The next table shows how many people who entered the scheme are "waiting" to get the money. Draw the data as a graph and also mark where the number of people waiting exceeds

4.a. the number of students in this class
4.b. the number of students in the school
4.c. the population of Slovenia
4.d. the world population

stage	people waiting
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
15	
20	
25	
30	

RICE ON THE CHESSBOARD

Indian emperor was so delighted about chess, that he promised the reward of his choice to the inventor. The "humble" monk wished for "only" as many rice grains as necessary to be able put a single grain of rice on the first chess field, two on the second, four on the third, eight on the fourth and so on. It was soon obvious that even for the extremely rich emperor it would be impossible to fulfill the monk's wish.

As we already know, we can express the number of grains on the k-th field with 2^{k-1}. So for the last field the emperor would need 2^{63} grains of rice. Let's find out how big this number is. First fill in the required data in the following table

approximate number of grain per kilo	
average crop	t/ha
population of Slovenia	
world population	
measurements of a box containing 1kg of rice	cm cm cm

Now let's calculate. Fill in the table, which shows how many kilograms of rice should be on each chess field.

field	number of grains	kg
1		
2		
5		
10		
20		
30		
40		
50		
60		
61		
62		
63		
64		

Can you calculate the mass of all rice on the chessboard?

Answer the following questions:

a) Suppose all citizens of Slovenia start eating rice for breakfast, lunch and supper. At each meal they eat 8 dag of rice. For how many days does all the rice on the 10^{th}, 20^{th}, 30^{th}, 40^{th}, 50^{th} or 60^{th} field suffice? Explain how you got the answer.

field	number of meals
10	
20	
30	
40	
50	
60	

b) For how many days would the rice on the last 10 fields be sufficient for all people on Earth?

c) Calculate the dimensions of the cube made of rice boxes on the 61^{st} field.

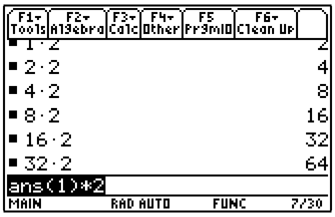
d) What area will bee needed if we want to harvest enough rice as is needed for

field	area
15	
25	
35	
45	
55	
60	

DOUBLING, TRIPLING, ...

Let's play with TI-89. (x) denotes pressing the key marked by x, ↓ the combination of keys (written after MEM and between / /) leading to command MEM.

- First reset it: ↓ `/(2nd)(6)/, (F1), (1), (Enter).`
- Type *1*: `(1) (Enter)`
- Type *ans(1) * 2*: `[ANS] /(2nd) ((-))/ (×) (2) (Enter)`
- Repeat pressing `(Enter)` until you get the following picture



Write down the number in a more compact form with the help of command *factor*:

factor(64): `(F2) (2) (6) (4)()) (Enter)`

We got 2^6. Now factor all numbers we previously got:

number	compact form

We repeat the same procedure but now we triple the numbers

- Type *1*: `(1) (Enter)`
- Type *ans(1) * 3*: `[ANS] /(2nd) ((-))/ (×) (3) (Enter)`
- Type *factor(ans(1))*: `(F2) (2) [ANS] /(2nd) ((-))/ ()) (Enter)`

Repeat the last two steps 10 times

number	compact form

number	compact form

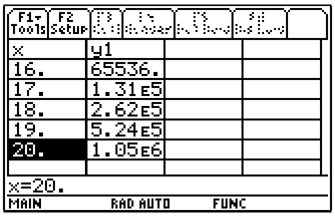
What will be the compact form of the number we get if we double the number and repeat the procedure 15 times?

What will be the compact form of the number we get if we triple the number and repeat the procedure 25 times?

Suppose we denote the number of steps of doubling with x. What will the expression for the number we get look like if we start with 1 and double the number x times.

TI-89 can produce tables for such formulas. First we use y = Editor, where we write down our formula:

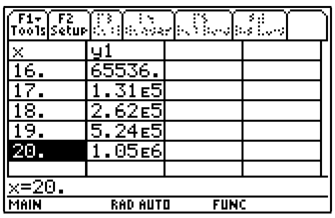
`[Y=] /(APPS) (2)/`
Type in the formula: `(2) (^) (x) (Enter)`



Now we can look at the values

`[TABLE] /(APPS) (5)/`

With the blue down arrow (∇) we can look at other values too. After several presses we get the following picture



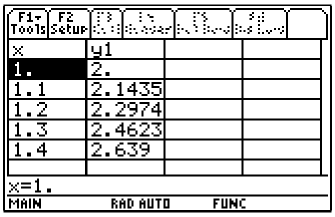
If we start with 1 and double the number 20 times we get more than a million. How many triplings are needed to reach 1 million?

`[Y=] / (APPS) (2) /`
Type in the formula: `(3) (^) (x) (Enter)`

Does our formula (2^x) work just for whole numbers? Let us try and set our table in a different way:

setup: `(F2)`
tblStart 1: `(1)`
Δtbl 0.1: `(∇) (0)(.)(1) (Enter)`

Followed by one more press of (Enter) we get



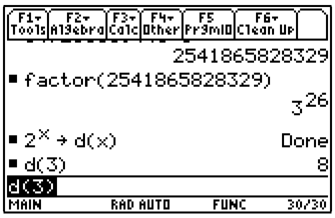
Try the same with different steps! It looks like we can use the formula 2^x for rational numbers too. But what about irrational arguments like √2, √3, π, and others. Let us define a function in our TI-89:

Return to main screen: (HOME)
Define function d(x):

$2^x \rightarrow d(x)$: `(2) (^) (x) (STO▷) [d] /(alpha) (,)/ (() (x) ()) (Enter)`

Now we can evaluate d(x) for various x.

d(3): `[d] /(alpha) (,)/ (() (3) ()) (Enter)`



Try with d(12), d(1.02), d(15.67), ...
How about d(√2)

d(√2*)*: `[d]  /(alpha) (,) / (() (3) ()) (Enter)`

With d(x) we have defined an exponential function with basis 2. Define also other exponential functions with bases 3, 5, 10, 1.1, 100 and fill in the table. Lower rows and last columns can be used for values and the basis of your choice.

x	3^x	5^x	10^x	1.1^x	100^x		
1							
√2							
2							
√5							
π							
5							
10.2							
20.23							
100.56							

VIRUS PROPAGATION

In the first half of 1999 there was a lot of media coverage about "disaster" various computer viruses made on computers causing even servers of such well known firms as Microsoft to crash. All those viruses have a common point – they all spread through E-mail. That allowed them to reach an enormous number of users. Among the most known were Melissa, Papa, Happy99, SKA virus and others.
But how can such viruses spread so quickly and how can they produce such enormous amounts of E-mail to bring E-mail servers to their knees? Let's have a look at perhaps the most ill named of them, Melissa.

The Melissa works in the following way. A user gets an e-mail message into his mailbox that has a special document attached. The mail consist of a message saying that it is very important for the user to open the attached document immediately. When the user opens the infected document, the virus will attempt to e-mail a copy of this document to up to 50 other people, using Microsoft Outlook. So per each opened infected document, up to 50 copies of e-mail will start their journey.

Questions:

2. Assume the worst. All infected e-mails are opened and causing 50 new e-mails to be sent. Fill in the following table, which shows how many new e-mails are produced at each step.

circle	new mails
0	1
1	50
2	
3	
5	
6	
7	
8	
10	
15	
20	
25	
30	

3. In which circle does the number of e-mail messages sent exceed the population of Slovenia?

4. In which circle does the same happens compared to world population?

5. Suppose an e-mail server is capable of processing 10000 E-mails in a second. How many such servers are needed to process all e-mails caused by Melissa in k-th circle in 1 minute, 1 hour, 1 day, 1 month or 1 year.

circle	year	month	day	hour	minute
5					
6					
7					
8					
10					
15					
20					

6. As a lot of e-mails remain unopened (and so incapable of spreading the virus), assume that out of 1000 e-mails received just 600, 200, 100, 50, 30, 20 or 10 are opened and cause a copy to be sent to 50 addresses.

circle	"harmful" e-mails out of 1000 received e-mails						
	600	200	100	50	30	20	10
	infected e-mails sent						
0							
1							
2							
3							
5							
6							
7							
8							
10							
15							
20							
25							
30							

Can you explain the numbers you got in the last two columns?

7. As well as that all users do not use Microsoft Outlook or have 50 addresses in it. So let us have a look at what happens if per 1000 e-mails received a copy is sent to 35000, 20000, 15000, 10000, 5000, 3500, 1500, 1100, 1050, 1010, 1000 addresses instead of 50 x 1000 = 50000. For easier calculation assume that at circle 0, we already have 1000 e-mails with the virus.

circle	"harmful" e-mails out of 1000 received e-mails				
	35000	20000	15000	10000	5000
	e-mails sent				
0	1000	1000	1000	1000	1000
1					
2					
3					
5					
6					
7					
8					
10					
15					
20					
25					
30					

circle	"harmful" e-mails out of 1000 received e-mails					
	3500	1500	1300	1100	1010	1000
	e-mails sent					
0	1000	1000	1000	1000	1000	1000
1						
2						
3						
5						
6						
7						
8						
10						
15						
20						
25						
30						

What happened in the last column?

As the number of e-mails sent in the last table is not extremely big, you should remeber that it is possible for the circles to occur quite rapidly – even more circles during the same day. So lower numbers just mean that the explosive growth occurs at a somewhat slower pace:

circle	"harmful" e-mails out of 1000 received e-mails					
	1600	1500	1300	1100	1010	1000
	e-mails sent					
20						
25						
30						
50						
100						
150						
200						
250						
500						
1000						
1500						
5000						
10000						

CHAIN LETTERS

Metka got the following letter:

.... Send a book of your own choice to the first address and send a copy of this letter to five friends of yours, dropping the first address and attaching your address at the end. This must be done in a week after you get this letter. In a month and a half you will get more than 3000 books ....

Promising, isn't it! You just spend money for 5 letters and one book and in a return you get more than 3000 books. Of course Metka could not resist. She immediately rushed into a bookstore, bought a book, envelopes, and stamps and promptly followed the orders. All excited she waited and waited but the books never showed up.

You have probably also received such a letter. These are the so called chain letters. In all of them you are expected to send the same letters in a few days to the specified number of addresses. If you do that, you will get various rewards or get extremely lucky. Some of the letters also warn you against not following the orders as not sending letters means you should expect various disasters. The rewards promised are different – from cash to books, postcards or winning the lottery.

Let us have a look at what happens. Suppose we got a letter with five addresses (listed in ceratin order), and the letter states we must send a similar letter to seven friends of ours. We will count how many people should send/receive the letter so that we will be on top of the list in these letters.

1. krog: We send a letter to seven friends. Our address is the fifth on the list.

2. krog: Each of the seven friends sends a letter to seven friends. In these 49 letters we are on the 4^{th} position.

3. krog: 49 people send seven letters with our address on the third place to $49 \times 7 = 343$ addresses. Till now the postmen have already delivered 1 (the letter we received) + 7 + 49 + 343 = 400 letters.

Using TI-92, TI-89 or DERIVE finish the table:

circle	our address at place	new people	letters together
1	5	7	8
2	4	49	57
3	3	343	400
4			
5			

Calculate how many people must have entered the game if those who entered in the 3^{rd} circle got to the first place in the list. Calculate also how much money is needed just for stamps so that all those who entered in the first five circles get to the first place. Assume that the cost of sending one letter is 21 tolars.

As these chain games are present in several forms, we will take a look st some numbers if we are expected to send the letters to n addresses.

$n = 10$

circle	new people	letters together
1	10	11
2		
3		
4		
5		
6		
7		
8		
9		
10		
15		
20		
30		
50		

$n = 3$

circle	new people	letters together
1	3	4
2		
3		
4		
5		
6		
7		
8		
9		
10		
15		
20		
30		
50		

$n = 6$

circle	new people	letters together
1	6	7
2		
3		
4		
5		
6		
7		
8		
9		
10		
15		
20		
30		
50		

$n = 2$

circle	new people	letters together
1	2	3
2		
3		
4		
5		
6		
7		
8		
9		
10		
15		
20		
30		
50		

Now add another three columns to the table where we sent letters to 6 addresses. Data should state how many people received a book (got to the first place) if we started in the k^{th} place.

circle	letters together	books / k = 5	books / k = 3	books / k = 7
1	7	1	1	1
2	43	7		
3				
4				
5				
6				
7				
8				
9				
10				
15				
20				
30				
50				

Is there any change in the columns? Explain!

How many books does each one receive if your address starts at 5th, 4th or 7th place and you send letters to *n* addresses

place	books / n = 5	books / n = 3	books / n = 7
2			
3			
4			
5			
6			
7			
8			
9			
10			

What is the difference if the number of addresses in the letter is 4, 5, or 7?

Can you explain why such schemes do not work?

CHAIN LETTERS – PART II

In preceding work we took a look at what happens to chain letters and why the best solution for such letters is throwing them into the basket. If you are still not completely convinced, let's have a look at some more examples.
Remember we have already found out that the number of letters that should be sent in the k^{th} circle is expressed as an exponential function with basis n and argument k, where n is the number of addresses at which each participant is required to send a copy of a letter.

$$\text{number of letters at stage } k := n^k$$

Exercises:

1. Suppose just Slovenians send letters to each other. Try to find out in which circle each citizen of Slovenia could get a letter if each person who receives the letter sends it to 2, 3, 5, 10 addresses.

circle	address			
	2	3	5	10
1				
2				
5				

2. A letter states that it started to circulate before 5, 7, 10, 15, 20 years. Assume every receiver sent a copy of a letter to 2, 3, 5, 10 addresses and that it happened 10 times per year. Calculate how many letters will be sent in just the last stage (when you received the letter).

start before years	letters			
	2	3	5	10
5				
7				
10				
15				
20				

3. As a lot of received letter finish in the wastebasket, we will calculate what happens if per 100 received letters 136, 115, 110, 101 letters are sent out. Will it happen in a certain circle that the number of letters sent exceeds the population of Slovenia? With experimenting try to find out the first circle when this happens.

circle	of 100 received letters the number of letters sent out			
	136	115	110	101

Findings:

first circle where number of letters exceeds 2 million	of 100 received letters the number of letters sent out			
	136	115	110	101

4. Suppose we want that in the 20th, 15th, 10th, 5th circle at least one million letters are sent. Try to find the minimum average number of letters out of 100 that must be sent in the preceding circle.

letters out of 100	circle			
	20th	15th	10th	5th

Findings:

number of letters out of 100 received letters	circle			
	20th	15th	10th	5th

Tables in preceding exercises were filled in by trying. Now put down corresponding equations that solve our problems.

1. Each letter is send to 7 addresses. Write down the formula that states the number of letters sent out in the 50th circle.

____

____

____

2. Each letter is sent to 5 addresses. What is the equation, which helps us to calculate the circle where the number of letters sent reaches one million?

3. Per 100 letters received 117 letters are sent. Write the formula when the number of letters sent in one circle reaches one million.

4. Write the formula, which states how many copies of a letter should be sent so that in the 15^{th} circle 1 million letters are sent out.

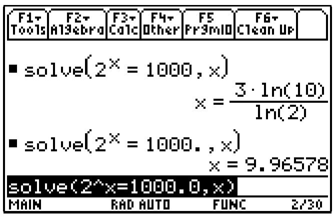
We can solve the equations with TI-89 in the following way:

Suppose the equation is $2^x = 1000$. Procedure for solving:

solve(2^x = 1000, x): `[solve() /(F2) (1)/`
`(2) (^) (x) (=)`
`(1)(0)(0)(0) (,) (x) ()) (ENTER)`

If we are interested in decimal form we type 1000.0 instead of 1000:

solve(2^x = 1000.0, x)



With the help of the above procedure solve preceding exercises.

1. Number of letters in the 50th circle where each letter is sent to 7 addresses:

2. Circle when the number of letters sent out exceeds 1 million (5 copies of each letter):

3. Circle when the number of letters sent out exceeds 1 million (117 copies sent per 100 received letters):

4. Number of letters to be sent so that in the 15th circle 1 million letters are send:

CHAIN LETTERS – PART III

In preceding work we took a look at what happens to chain letters. But we assumed that in each circle the number of letters sent out is still bigger than the number of letters received. What about if the opposite happens?

Exercises:

5. Can you explain under which circumstances the chain will be broken? Suppose that at a certain moment there are 100000 letters circulating. In each circle only 90%, 80%, 50%, 30%, 15%, 10% of letters are sent out. How many letters are circulating in each circle?

	90%	80%	50%	30%	15	10%
0	100000	100000	100000	100000	100000	100000
1						
2						
3						
4						
5						
6						
7						

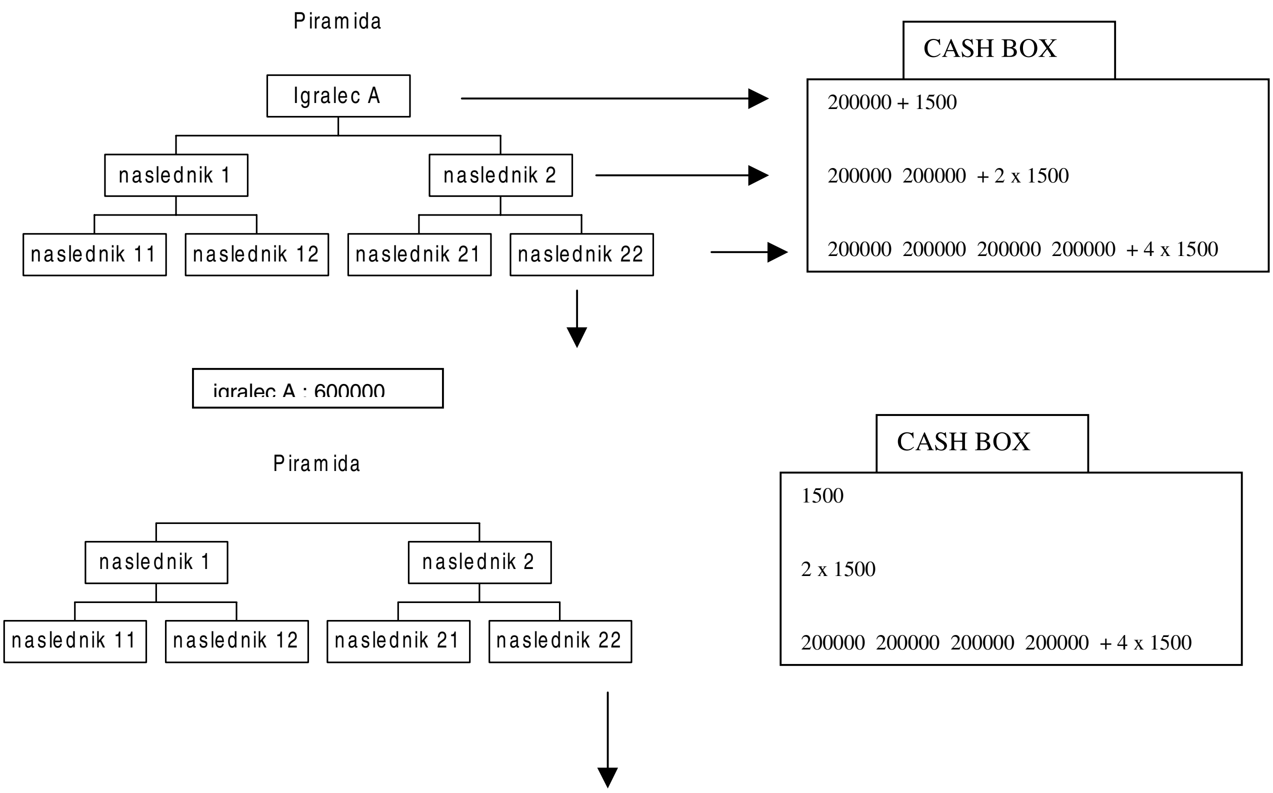
6. How long does it take that for the chain to be completely broken (calculated number of letters is less then 1) if the initial number of letters is 1 million, 100000, 1000, 100.

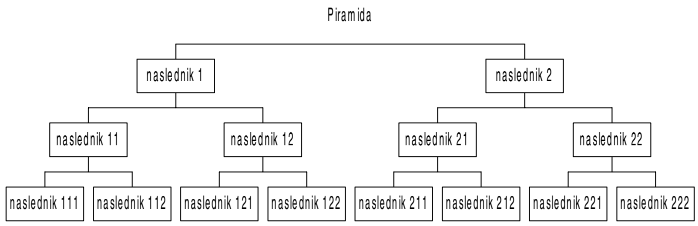
	1000000	100000	1000	100
90%				
80%				
50%				
30%				
15%				
10%				
1%				

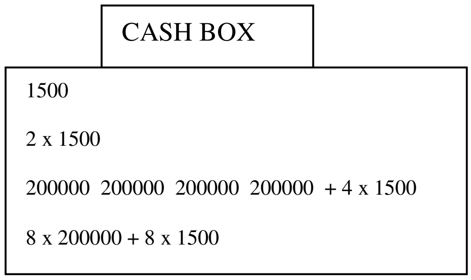
CATCH THE CASH

A few years ago a money game called the Pyramid was very popular in Slovenia. A lot of Slovenians participated and many of them lost a lot of money. As a matter of a fact the headquarters were in Austria (in the village of Klobasnica) due to more appropriate (for the organizers of course) laws, but that does not matter. The official name of the game was Fair Play. They even had the law and consulting service for the players. In the subsequent years quite a lot of similar games called Catch the cash, Network twenty-one and others were organised. All stated that they were completely honest and fruitful for all players.
With the aid of a computer we will take a look at these games and find out why it is impossible for the participants to be all richer at the end of the game. As a matter of a fact a very simple reasoning shows why. If all participants get more money than they invest – where does this additional money come from? If we put some people in the room and they give each other money – is it possible that in the end each one will have more?
But as it seems that such an explanation does not convince everybody we will take a detailed look. We will exploit the power of CAS, which can calculate with almost unlimited numbers. First take a look at a fragment of the lecture so called propagators of the game presented

Pyramid is a scheme, which help you to get reach without any risk. Player A must convince two other players (followers) to enter the scheme. Each one has to put 200000 SIT into the cash box as well as 1500 SIT for expenses. The task of each of the followers is to find two more followers (who also have to deposit the same amount of money). At this moment player A gets 600000 SIT and exits the game (he can of course enter again) and his followers become players A. So for 201500 SIT invested everyone gets 600000 SIT.





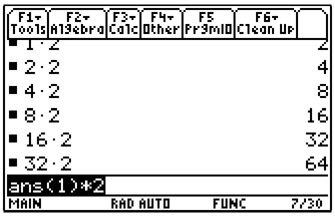


As you can see from the pictures this is a completely honest scheme which can not fail.

But the promoters fail to show how quickly the number of players needed grows. As you can see it doubles at each step. In the first few steps the numbers are not too big but, as your table will show, the numbers are soon extremely big.

For calculating numbers you can use TI-89. (x) denotes pressing the key marked by x, ↓ the combination of keys (written after MEM between / /) leading to command MEM.

- First reset it: ↓ `/(2nd)(6)/, (F1), (1), (Enter).`
- Type *1*: `(1) (Enter)`
- Type *ans(1) * 2*: `[ANS] /(2nd) ((-))/ (×) (2) (Enter)`
- Repeat pressing `(Enter)` until you get the following picture



step	new players
0	1
1	2
2	4
3	
4	
5	
6	
7	
8	
9	
10	
15	
20	
25	
30	

Calculate how many people must enter the game if all those who entered at the 3rd step are to get their promised 600000 SIT.

Calculate also how much money the organizers get (for expenses) if all those who entered at the first five steps get their money.

What is the amount of money that is put into the cach box at the 10th step? Express that in cars or new houses. Assume you pay 2 millions SIT per car and 25 million SIT per new house?

What is the amount of money that is paid for expenses at the 10th step? Express that in cars or new houses. Assume you pay 2 million SIT per car and 25 million SIT per new house?

Fill in the number of players who had got their money and those who are still waiting for their money

step	new players	players paid	players waiting to be paid
0	1	0	1
1	2	0	3
2	4	1	6
3			
4			
5			
6			
7			
8			

9			
10			
15			
20			
25			
30			

At which step does the number of players entering the game exceed the population of Slovenia (2 000 000)?

At which step does the number of players entering the game exceed the world population (6 mrd.)?

At which step the money of players entering the game exceed the gross income of Slovenia (in 1994 7140 USD per capita)?

Graphically show how the number of players grows.

EXPONENTIAL FUNCTION

We will examine the function of the form $f(x) = a^x$, which transforms real numbers into real numbers. Such a function is called an exponential function. Number a is called the basis and x the exponent. Our task is to find out

1. Which bases (a) are valid?
2. What does the graph for various bases a look like?
3. Common properties of graphs.

Commands from *DERIVE*, used in the exercises:

- `Author` entering expressions
- `Simplify` transforming expressions
- `approX` approximate value of the expression/ transformation into decimal form
- `Plot` ploting
- `^` exponent

First take a look at function $f(x) = 2^x$. If x is any rational number, we already know what the notation 2^x means. So fill in the table with exact values in the first row (`Simplify`) and with approximate ones in the second row (calculate them using `approX`):

	0	1	2	10	1/2	5/3
exact						
approximate						
	0	-1	-2	-10	-1/2	-5/3
exact						
approximate						

What if x is an irrational number? What does it mean for example $2^{\sqrt{2}}$ or 2^{π}? As we know that $\sqrt{2}$ is approximately 1.414213562 (which is a rational number), calculate $2^{1.414121356}$ and $2^{1.414213565}$. Their difference is ________________________.

Calculate also $2^{1.4141213562}$________________________.

Try to calculate $2^{\sqrt{2}}$ (use `approX`). Answer: ____________________

Similarly, calculate also $2^{\approx\pi}$:

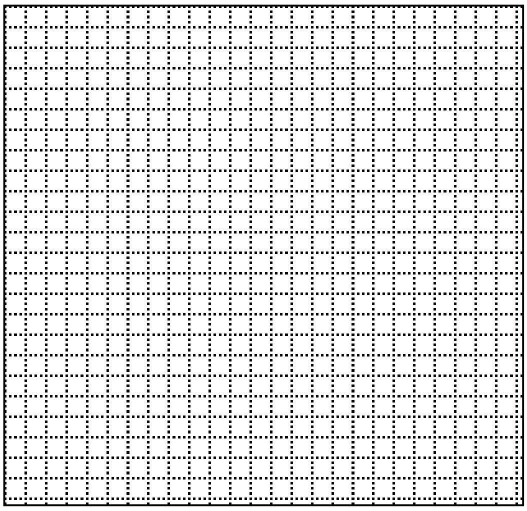
2^	3.14159		3.1415926		3.1415926535	
	3.1416		3.1415927		3.1415926536	
	3.141592		3.14159265			
	3.141593		3.14159266		π	

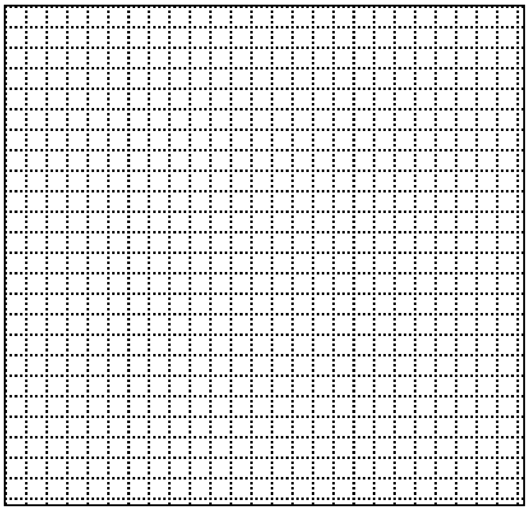
Calculate also $2^{\approx\sqrt{3}}$:

$2^\wedge$	1.73		1.73205		1.7320508075	
	1.74		1.73206		1.7320508076	
	1.732		1.7320508			
	1.733		1.7320509		$\sqrt{3}$	

What do you think about the existence of $2^{\sqrt{2}}$ and all others where the exponent is an irrational number?

On the left side draw a graph of $f(x) = 2^x$ and on the right the one you arrive at using *Derive* (`Plot`). If the pictures differ, try to calculate the value of a function in the points where you spot the differences.





Try the same with $f(x) = 3^x$, $g(x) = 5^x$ and $h(x) = 10^x$. First fill in the table. Afterwards draw a graph by yourself on the left side and on the right side with the aid of *Derive*.

	0	1/4	1/2	1	$\sqrt{2}$	2	10
f(x)							
g(x)							
h(x)							
	0	-1/4	-1/2	-1	$-\sqrt{2}$	-2	-10
f(x)							
g(x)							
h(x)							

What do you think about the graph of the following function $f(x) = \left(\frac{1}{2}\right)^x$? Recall that $\frac{1}{2} = 2^{-1}$.

Compare your picture with the one you got using *Derive*.

What about the graph with basis 1?

Let us now examine the negative basis. Fill in the table for bases -1, -2, -3. Be careful! As we defined our exponential functions to map from real numbers into real numbers we discard all complex answers we get (just make a dash in the table).

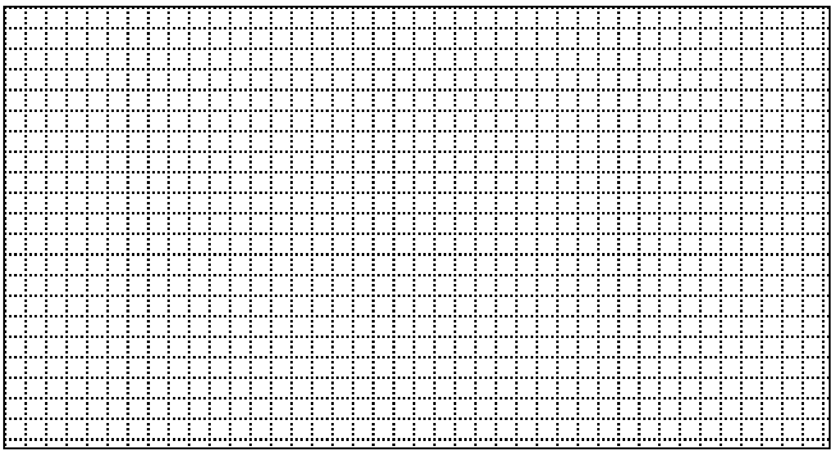
	0	1/4	1/3	1/2	1	2	3
$(-1)^x$							
$(-2)^x$							
$(-3)^x$							

	-1/4	-1/3	-1/2	-1	-2	-3	-4
$(-1)^x$							
$(-2)^x$							
$(-3)^x$							

Fill also the following table:

	0.4	0.6	0.2	π	0.416	$\sqrt{3}$
$(-1)^x$						
$(-2)^x$						
$(-3)^x$						

What do you think about the graph of an exponential function with a negative basis? Try to plot such an exponential function with *Derive*.



Write down your findings:

1. Exponential function $f(x) = a^x$ is defined, when the basis ___________________________.

2. What do the graphs with bases a > 1 have in common?

____

____

____

____

____

3. What do the graphs with bases 0 < a < 1 have in common?

____

____

____

____

____