Using Computer Algebra to Extract Meaning from Parameters

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Introduction

Understanding the meaning of a mathematical equation, model, or result comes from a variety of sources. It may come from the ability to estimate, a graphical intuition, or an understanding of the phenomena underlying the result. Ultimately, however, the most valuable insights are gained by understanding the parameters that are used to define the mathematical objects being considered.

For example, the behavior of a cubic, $ax^3 + bx^2 + cx + d$, depends on the size, sign, and relationships that exist among the coefficients, a, b, c, and d. The fact that the polynomial is a cubic determines the general shape of the curve, but it is the parameters that determine the center of symmetry, the location of the maximum and minimum points, if any, and the distance between these points. In effect, the degree of a polynomial determines the general shape, and the parameters define the quality of the polynomial's behavior.

While the analysis of parameters associated with a family of curves can give significant insights into the behavior of the family, there are other uses for this type of analysis. For example, parameters can be used within the context of a mathematical model to determine a policy for the implementation of the model in a real world setting. Examples of this use will be taken from a calculus based analysis of a mathematical model, and the discussion of the behavior of a model described by a differential equation. In each case we will show that the ability to keep the parameter in its symbolic form enables an analysis that goes beyond the standard analysis that may be done in the standard situation using numerical coefficients.

Constructing Cubic Curves

The general definition of a function determines the general shape of a process, but analysis of the parameters associated with the function determine the quality of the process. For example, using our knowledge of trigonometry we can describe the process associated with the function, $f(x)=ax + b\sin(\alpha x + p)$ as an oscillating curve that lies along a ray passing through the origin. The parameter, a, determines the angle the curve makes with the x-axis, while b, α, and p determine the height, frequency and phase shift of the oscillation. The latter characteristics are important if one is describing a radio wave being broadcast to a destination. In a similar way the parameters associated with a polynomial function determine the behavior of the polynomial. We will first consider a cubic polynomial.

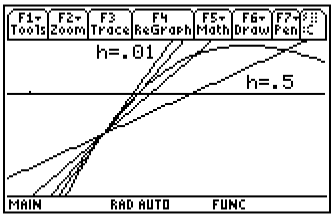
Begin the analysis by considering a very straight forward cubic function of the form

$$f(x) = \alpha x^3 - \beta x$$

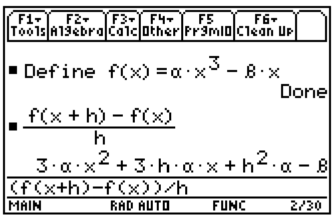
We can learn a lot about this curve by doing some elementary reasoning and exploration

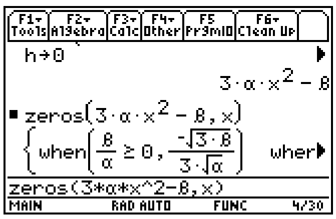
1. The roots are located at $x = 0,\ \sqrt{\dfrac{\beta}{\alpha}}\ ,\ -\sqrt{\dfrac{\beta}{\alpha}}$
2. The graph is symmetric about the origin, i.e. $f(-x) = -f(x)$
3. The graph has a “peak” and “valley” at $x = \pm\sqrt{\dfrac{\beta}{3\alpha}}\ \textit{or}\ \mp\sqrt{\dfrac{\beta}{3\alpha}}$ if and only if α and β have the same sign.

The last fact is easily determined using calculus techniques if the students have had an introduction to differential calculus. However, if they have not some elementary graphical analysis and algebraic manipulations using a CAS have the potential to make the exploration even more exciting. For example, one approach is to begin with a graphic justification of the tangent line as a limiting position for secant lines through a fixed point and a sequence of points becoming ever closer to the given point as is illustrated in the following screen taken from a TI-89 session.



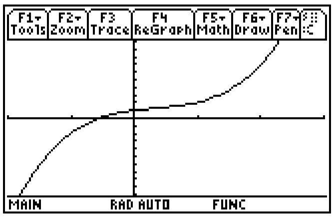
Each student can be given different points on the graph and a different sequence of points (values of h) each of which terminate at a point ‘close’ to the fixed point. The graphical evidence is convincing but not a proof. The CAS allows for a more convincing argument after the graphical case has been considered. It also leads us to the conclusion given in (3) above.

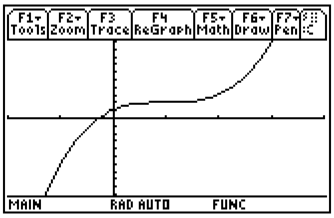


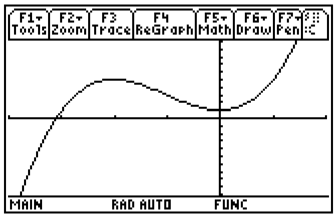


The screen on the left allows the student to easily manipulate the expression for the difference quotient. The obvious question is what happens as h becomes smaller and smaller in magnitude. This leads to the idea of a limit. The student now knows how to find the slope, and as a result the equation, of the tangent line to the graph of $f(x)$ at any point on its graph. Now, the next question is what makes peaks and valleys interesting? Of course, these are points where the tangent line is horizontal! So the student solves for the points where the slope of the tangent is zero, and the students have shown statement number 3. In the process they see that the location of the peak and valley are related to the location of the roots of the cubic. In fact if the location of a root on the x-axis is multiplied by $\frac{1}{\sqrt{3}}$ the x-coordinate of either the peak or valley is found.

Now let's turn our attention to a cubic curve, $f(x) = ax^3 + bx^2 + cx + d$. Using either elementary calculus or a process similar to the above algebraic and limit arguments with the aid of a CAS, it is relatively easy to show that the graph of $f(x)$ will have a maximum and minimum if and only if $b^2 - 3ac > 0$. Thus, the parameters also exert control over the





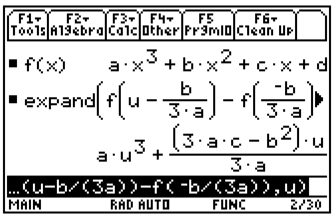


general shape of the curve.

The three graphs shown above illustrate graphs of cubic curves where $b^2 - 3ac < 0$, $= 0$, and > 0, respectively. Note that the case $b^2 - 3ac = 0$ causes the graph to have a saddle point. We will spend the rest of this section examining the effect of the parameters in the case where the graph has a peak and a valley.

It is an easy CAS activity to show that four points on the graph of a cubic curve uniquely determine the curve. It is simply a matter of substituting in the x and y values at each point and solving the resulting equations. In this section we will show how an understanding of the role played by the parameters associated with the curve coupled with the use of a CAS make it possible to easily determine the curve in question with information on only two points lying on the curve. Of course, they must be the correct two points, i.e. the local maximum and minimum points. As mentioned in the previous paragraph, we are dealing with the case, $b^2 - 3ac > 0$.

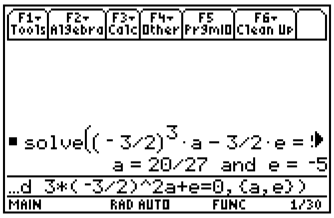
The first fact that we use is that a cubic curve is symmetric about its (lone) inflection point. The location of the inflection point $(-b/a, f(-b/a))$ is an easy calculus exercise and certainly does not require the use of a CAS. To prove that the graph is symmetric about this point requires understanding of the properties of graphs and a manipulative skill that is best passed off to the CAS.

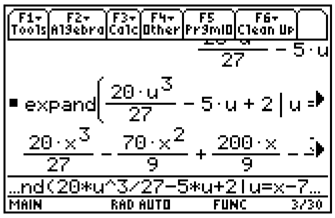


The important intellectual work involved in the process shown above is the transformation of the coordinate system from the (x, y)-system to the (u, v)-system that is centered at the inflection point. The next important step is to realize that the resulting function is an odd function. Thus, the graph is symmetric about the (u, v)-origin, i.e. the inflection point of the original function. It should also be noted that the translated maximum and minimum point are sufficient to describe the curve. We merely define a new parameter, e, to be the coefficient of the term of order 1. Solve for a, and e and then translate the curve back to $(-b/a.\ f(-b/a))$.

As an example of this process, we will find the equation of a cubic curve that has a maximum point at (2, 7) and a minimum point at (5, –3). The inflection point, based on our observations is at (3.5, 2). If we use the point (u, v)-coordinate system centered at the inflection point, we see that in this system our function is $f(x) = ax^3 + ex$, the maximum point is at (–1.5, 5), and the minimum point is at (1.5, -5). We find a and g by using the fact that (–1.5, 5) is a maximum point and solving the equations.

$$(-1.5)^3 a - 1.5e = 5$$
$$3(-1.5)^2 a + \quad e = 0$$





For the above we used the fraction 3/2 instead of 1.5 in order to obtain an exact instead of an approximate result. The resulting cubic polynomial in the variable x is:

$$\frac{20}{27}x^3 - \frac{70}{9}x^2 + \frac{200}{9}x - \frac{331}{27}$$

The role of the parameters associated with the cubic is illustrated by the above example. We first note that the center of symmetry for the cubic is determined by the ratio of the parameters, a and b. It is $-b/(3a)$. If the coordinate system is translated so that the origin is at the center of symmetry, the location of the local maximum and minimum is at $\pm\sqrt{(-e/(3a))}$, where e is the coefficient of the order one term in the translated system. Thus, we see that a is a scaling and stretching factor for the equation. If a is fixed, varying b will change the location of the center of symmetry, and varying b and c will change the distance of the local maximum and minimum from the center of symmetry. The parameter, d, associated with the constant term, moves the curve vertically.

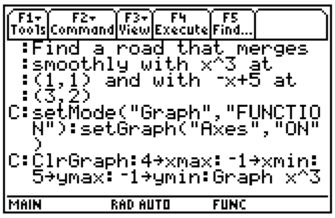
This discussion of the role played by the parameters associated with a cubic curve is quite elementary and obvious to a person who is sophisticated in their understanding of mathematics, but it is not obvious to a student just learning about the graphs of polynomials. It may be totally lost to the student if the discussion centers on the roots of the polynomials and methods for finding those roots. While root finding is an important activity and needs to be learned. Much of it is easily accomplished using a CAS. The important fact that is often overlooked is that a function represents a process where important things may happen at places other than the roots. The parameters associated with that process control the behavior and quality of the process.

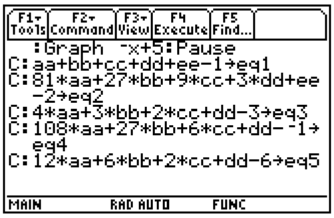
Extending The Techniques to More General Polynomials and an Application

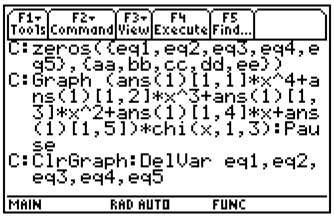
The following example extends the ideas of the previous section to construct a new function that joins two other functions at specified points on the graphs of the functions. We want the junctures to be as smooth as is possible given the properties of the original functions. We pose the problem as one of finding a slip ramp from one road represented by the graph of the function $f(x) = x^3$ and another road represented by the graph of the function $g(x) = 5 - x$. We specify the two points on the road that are to be joined. If the ramp is to leave and enter the roads smoothly, more information is needed. Two points only determine a straight line.

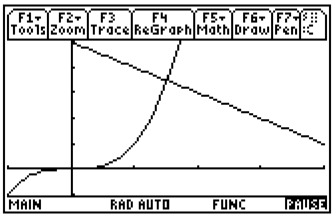
Given what we have learned in the previous section, we can match the tangents to the graph of the slip road to the tangents of the joining road at the given points. This is enough to give a segment of the graph of cubic function for the slip road. We can increase the degree of function by one if we use the second derivative of x^3 at the point of intersection. Doing this opens the way for a discussion about why matching the second derivatives further smoothes the juncture. What information is contained in the second derivative of a function?

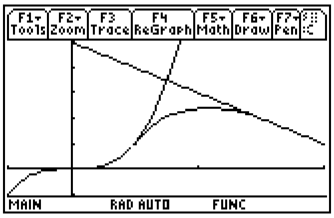
The following three screens show a script that will solve the problem for the given situation. The center screen and the first command of the screen on the left find the coefficients of the fourth degree polynomial whose graph defines the slip road.











The screens above show the two roads prior to constructing the slip road and after the function for the slip road has been deteremined.

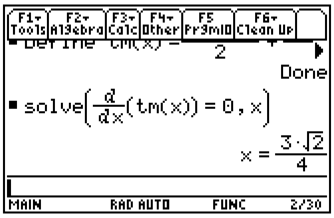
There are many variations that can be played on this theme. One of them is to determine the flight path for smoothly landing an airplane that is at a present position on a runway at an airport.

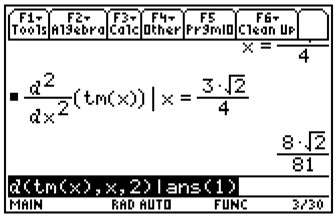
Parameters for Decision Makers = Policy for Rule Followers

In this section we will look at how using a CAS can use a well known optimization problem found in a calculus course move it from a templated problem whose solution technique is memorized by the student to a means of obtaining insight into the process behind the problem. We begin with the statement of the problem as it may be found in a calculus textbook.

> *A courier is to be dispatched to a ship that is anchored three kilometers off shore to a rendezvous point that is located 10 kilometers down the shore to the east of the ships position. If the courier can row at a rate of 2 kilometers/hour and run along the shore at a rate of 6 kilometers/hour, at which point on the shore should the courier land in order to make the journey in the shortest possible time?*

The solutions to this and other calculus optimization problems can be templated. Once the basic model for the problem is established (a very worthwhile activity), the procedure for finding the optimum value can be programmed or scripted on any CAS. Using the TI-89, the figures shown below essentially solve the problem.

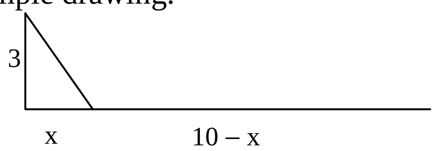




To explain the construction of the model that resulted in the function called tm(x) in the above figure, we recall some basic facts from elementary physics and the situation described in the problem.

Total time = time rowing + time running
time rowing = distance rowing/rowing rate
time running = distance running/running rate

Now we need to make a simple drawing.



and identify the terms in the objective function.

Tm = total time for the trip
x = point west of ships position where the courier lands
distance rowing = $\sqrt{9+x^2}$
distance running = $10 - x$

Finally, we construct the objective function.

$$Tm(x) = \frac{\sqrt{9+x^2}}{2} + \frac{10-x}{6}$$

The attentive student may have learned a great deal from the solution to this problem. Unfortunately, most students have memorized a technique to solve a class of problems. They extend the model to any situation where they are traversing a specified distance where the agent covering the distance travels at two different rates (be they speeds or costs, etc.). Unfortunately, they miss some important observations that can help them solve the problem in a general and efficient way while operating in a more general domain.

Let's generalize the solution of the problem and ignore some of the particulars that are given. In short, we replace all of the specifics by parameters.

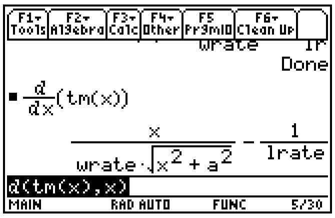
a = distance of mother ship from shore
b = distance of rendezvous point down shore
wrate = rowing rate (rate on water)
lrate = running rate (rate on land)

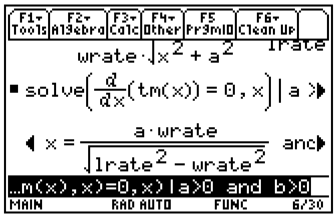
The objective function becomes

$$Tm(x) = \frac{\sqrt{a^2 + x^2}}{wrate} + \frac{b - x}{lrate}$$

The solution to the problem in this form can be found without the use of technology. However, the algebraic computations become cumbersome and the students miss the point of the exercise.

We employ the solution techniques that we used to solve the specific problem. The CAS does not mind the presence of parameters instead of numbers. It treats both as symbols. The figures below are comparable to the ones shown above when solving when we were solving the specific problem.



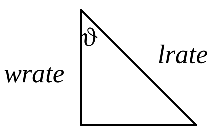


The entire solution to the equation is not shown above. The CAS is more careful than are our students as well as many of us.

The first thing that can be observed from the above is that the distance down shore of the rendezvous point, except for the case when it is so close that an endpoint solution becomes optimal, plays no role in the choice of the landing point. More intriguing is the ratio in the solution

$$\frac{wrate}{\sqrt{lrate^2 - wrate^2}}$$

It virtually smacks of Pythagoras! If we draw the following triangle,



we see that the solution is cos(ϑ) scaled by the off shore distance of the ship.

This makes the following policy decision possible:

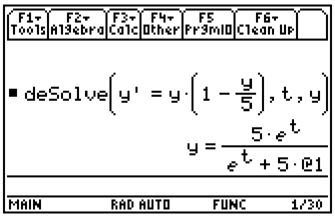
> *We can determine at what angle to send the courier without knowing the ship's distance from shore or the distance down shore of the rendezvous point (except in the case of endpoint solutions). All we need to know are the rowing rates and running rates of the courier. Simply make a right triangle similar to the one shown above, place it on the starting point for the courier on the ship, orient it correctly, and dispatch the courier in the direction indicated by the hypotenuse. The exception to this rule is when the ray in this direction does not go past the destination. In this case send the courier directly to the destination.*

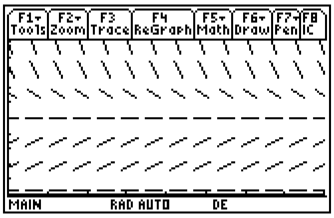
The fact represented by this statement is hidden by the templating approach of the solution to the particular problem. It may become obvious after a great deal of experimenting with particular values of the distances. It is, however, sitting there waiting to be discovered in the algebraic solution of the problem with parameters. Many other optimization problems lend themselves to policy statements made possible by this more general approach. The combination of using the CAS to employ the techniques learned in calculus with a careful analysis of the solution is valuable for all of our students, both those who will be mathematics majors and those who will major in other fields. It empowers them early in their academic career.

Parameters For Understanding an Ecological "Fact of Life"

Consider a standard logistic equation for a population of animals developing in an environment having limited resources. The first order differential equation shown below is one of many that describes this process.

$$y' = y(1-by)$$



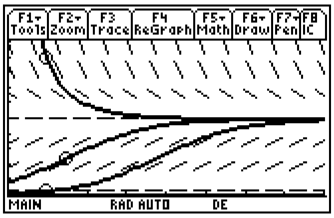


The parameter ,*b*, gives the value of an equilibrium population, 1/*b*, enforced by this model. We used a value of 1/5 to show an analytic solution and to also show the direction field generated by this equation. For this figure, $0 \le t \le 8$ and $0 \le y \le 10$. The horizontal line across the figure on the left is located at the value $y = 5$ or 1/(1/5). All other slope lines lead towards this equilibrium position. This equilibrium denotes the carrying capacity for the population within the environment described by the model.

We can convince ourselves that the equilibrium value represents a stable equilibrium by graphically illustrating that for a selection of initial conditions we have that

$$\lim_{t\to\infty} y(t) = \frac{1}{b}$$

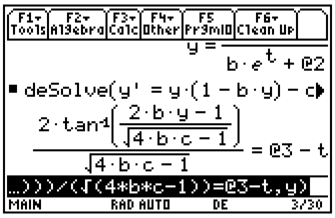
This result is easily deduced from the analytic solution shown above. It is sometimes to our advantage to first show the graphical result and then apply Computer Algebra to the analytic result.

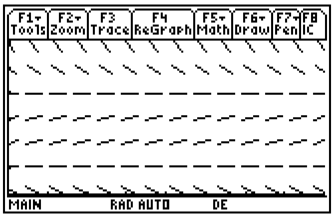
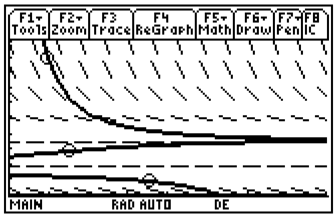


The model becomes more interesting if we introduce a term to indicate the effects of harvesting upon our model. For example, suppose that we are considering a fish population along the fishing banks off the coast of the US and then introduce at parameter into the differential equation model to represent the effects of a fishing fleet that takes a yearly annual harvest.

$$y' = y(1-by)-c$$

An analytic solution to this equation can be found using a CAS, but it really does not lend much understanding to the overall process described by the equation.



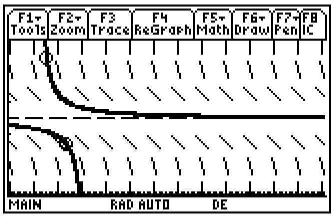
The figure on the left shows the analytic solution to the model when it is modified to include harvesting (hunting, fishing, etc.) of the population. The middle panel shows the slope field again with the range of y modified to [0, 6]. (For the purposes of drawing the graph we used $b = 1/5$ and $c = 1$.)   This figure drives home the point that the introduction of the harvesting parameter introduces two equilibrium values for the solution, $y(t)$. Notice in the figure on the right that in the upper two regions the model behaves as before with a stable equilibrium, not at $y = 5$, but at $y = 3.6$. It is in the region below the lower of the two equilibrium values that we notice a disturbing behavior. This equilibrium value is unstable.   If the population is greater than this value, it will eventually stabilize at the upper (stable) value. However, if the population falls below the unstable value it dies out rather quickly. Thus, an unforeseen perturbation in the population size could be the difference between a stable population and an extinct one.

To find these two equilibrium values, it is a matter of simply solving the quadratic equation

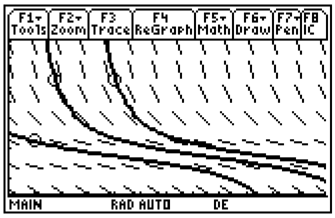
$$y(1 - by) - c = 0$$

for y. The discriminant of the resulting quadratic is $1 - 4bc$. This suggests the interesting question of what happens when the harvesting parameter $c \geq 1/(4b)$?

For our example the critical value for *c* is c = 1.25. For this value of *c*, the lone equilibrium value attracts trajectories that start above it and repels those that start below. In this case the equilibrium value is *y* = 2.5. In the figure on the left we extended the *t*-axis to 100 to illustrate this phenomenon.



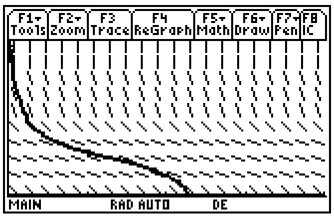
$0 \leq t \leq 100$



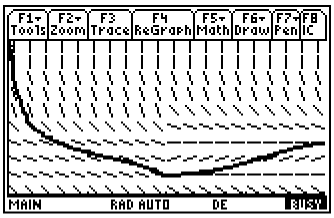
$0 \leq t \leq 12$

To illustrate what happens when the roots of the quadratic become imaginary, choose a value for c = 1.5. The behavior in this case is different than that of any of the previous cases. No matter what initial conditions we choose the population becomes extinct. Thus, if care is not taken in setting the harvesting rate for the population, extinction is inevitable. Or is it, here is where the ability to use our knowledge of parameters comes in!

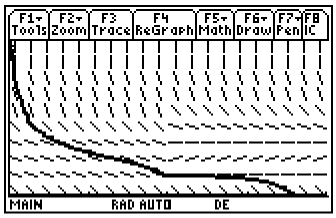
Suppose the situation shown on the right above has been going on for a period of 10 years. Now, we see in the figure on the left that we are disastrously close to extinction. If we change the value of the parameter c at this point to c = .92. This is a large reduction from the former value of 1.5, but preferable to completely eliminating harvesting. At this level a new equilibrium is established and as long as the harvesting is controlled at this new, lower, level harvesting can continue. The figure on the right shows what happens if the harvesting is not tightly controlled.



c = 1.5



c=1.5 c=.92



c=1.5 c=.98

In this discussion the convincing evidence based on the parameters connected with the differential equation came as a result of the numerical/graphical displays. However, the decisions on choosing the values to generate these displays came as a result of algebraic investigation. For this simple equation, the algebraic manipulations for our insights were not difficult. For more complicated examples, use of the CAS may be absolutely essential for gaining insight.

Conclusions

Those of us who have been long time advocates of the use of the CAS have often stated that the intelligent use Computer Algebra means that we must change not only the way we teach, but also what we teach and emphasize in our courses. We need to think again about where our students are going to be using their mathematics. Most of them will be consumers of mathematics in a more general marketplace than one that values the solving of equations and doing efficient manipulation of symbols. They need to be able to look at processes as functions and interpret the behavior and the quality of the process from the parameters associated with the function.

It is this ability that will mean more in the long run than the manipulative skills that are often emphasized. Originally the manipulative skills were developed in order to find answers to facilitate an understanding of the general behavior of processes. Unfortunately, not all students are talented in developing manipulative skills. Because the manipulative ability requires so much effort, they and their teachers see the ability to master these skills as a desirable. The result is that mathematics becomes a set of pointless exercises instead of a means of understanding the nature of processes that affect their daily lives. In short, it does nothing to enrich their lives and understand the world in which they live.

The title of the one section in this report summarizes my philosophy: Parameters Determine Policy. That is an understanding of how the parameters affect the general process can help us to adjust them and control the process or, at the very least, understand the outcome of the process and prepare for the results. Without this ability we are merely rule followers who are at the mercy of the result. We have lost the ability to control that comes from a deeper understanding of what is happening.

A failure to understand the behavior of a process and its quality can lead to disastrous decisions. For example, consider the differential equations model of harvesting a population. A failure to understand the power of mathematical modeling and the relationship between the parameters in this model and the future of the cod population in the fishing banks off the coast of Massachusetts in the USA is having disastrous results. People are trying to extrapolate from past experience and are influenced by the immediate effect of a fishing ban on their income. They do not listen to the ecological arguments based on a model similar to, but more sophisticated than the one presented in this paper. They can not see that continued fishing at the current rate will lead to a permanent end to their livelihood. Mathematics has failed in its mission. Instead of empowering individuals with the ability to use it to control their destiny, it has created a distaste that causes them to dismiss any result that does not support their position as meaningless and not worthy of consideration.

We need to establish an atmosphere in our classrooms that mathematical modeling provides a vehicle for learning about processes and thus, provides us with a vehicle to understand our world. Creating mathematical models can be started early in students' mathematical careers. Using the CAS and computational support as a black box, we can teach elementary modeling techniques that are based on the relationship of the important variables within the process. The students spend their time identifying the variables, the relationships between them, and the strength of these relationships. The process within the black box will generate numerical and graphical data as feed back to the qualitative statements of policy. This provides concrete evidence of the importance of the strength of the relationships within the model. As the black box becomes illuminated, students acquire a deeper sense of the role of the parameters within their model. This is the point where empowerment begins.

In addition to teaching mathematics in a way that may be more meaningful in the lives of our students, mathematical modeling is an activity that excites students and provides numerical and visual feed back for "what if" hypotheses. Students are interested in answering questions that are related to their lives. We can take advantage of this curiosity by providing them with a tool that helps them to talk intelligently about things that matter to them. In return we are teaching them to use mathematics and develop a mathematical intuition. In effect, we can lure them into a desire to learn more about mathematics and mathematical objects. Topics in advanced mathematics, both of an applied and abstract nature, can be previewed and developed once we have their interest.

While this paper has concentrated on the use of the CAS and computational technology, it must also be realized that manipulative, or paper and pencil, technical skills are important and must not be ignored. Students need to understand the nature of these skills as a start to understanding how models work. It is for this reason that the first example in the paper dealt with an abstract mathematical object, a family of cubic curves. If one is going to describe a process and its role within a model, it is necessary to be able to draw upon an experience with such mathematical objects. Students need to develop a library of functions for their use. They need to be able to describe the properties of these functions. In short, they need to be able to do some of the basic skills such as factoring polynomials, describing the trigonometric functions, and the basic properties of exponentials and logarithms. The development of such skills and knowledge of technical algorithms help to hone their skills in understanding the roles of the parameters within a model. They need to be familiar with them, but they do not need to be "experts" in their use. The art of teaching comes in being able to blend these topics with others in a way that is interesting and such that it excites the students' curiosity.

Once we have piqued the students' interest, it is possible to proceed with the help of the CAS. There is no need to repeat the same operations over and over again until they become mechanical. The CAS has mechanized the operations. If the CAS is used merely as an answer book to check students' work, we are missing its full value. Possessing a tool that can quickly do manipulations and create visual displays that might otherwise take a great deal of time allows our students to concentrate on developing skills that are truly important to their understanding and appreciation of mathematics.