Exam questions when using CAS for school mathematics teaching

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Using computer algebra systems for teaching 12-18 year old students is going to change the teaching methods, the contents of what we teach, and, last but not least, the exams. We look at traditional exam questions and categorize them according to their usefulness in a CAS environment.

1. CAS-induced changes in teaching and learning of mathematics

Computer algebra systems (CAS), being one of the technologies offered by information sciences, inevitably influence the teaching and learning of mathematics. Teaching and learning mathematics using CAS requires us to change the teaching methods. CAS opens up new dimensions of teaching and will cause teachers to change topics and/or to shift the focus within existing topics. Teaching and learning must no longer be "algorithmically centered" as it mostly has been in traditional teaching and learning. The goals of mathematics lessons are going to be changed. Now, understanding the mathematics in the content can become the central goal.

Most traditional mathematics lessons are very much centered around the craftsmanship of learning and executing algorithms. The main de facto goal in school mathematics was (and is) the learning and practicing of the ability to *perform (a large variety of) mathematical operations*. Since CAS perform most of these mathematical operations much better (much faster and much more reliably) than even the best human mathematician can do it, we ought to shift our educational goals from *performing* mathematical operations to *using mathematical operations*. This goal is closely related to *understanding the meaning of mathematical concepts* within and outside mathematics. Both these goals were pushed into the background, because the teaching and learning of procedures gained more and more importance. In traditional mathematics lessons we use most of the time for practicing calculation procedures, little time for exploring meanings of concepts, and very little for using mathematics (Kokol-Voljc, 1998).

One could say: *"In climbing up a tree we forgot why we climbed it up."* We are like a sculptor who wants to make a statue, but then spends all our time and energy to manufacture the tools for making the sculpture – while forgetting to make the sculpture itself. We want our students to learn to use mathematics, but we spent most of our time and energy teaching and practicing the tools (operations) we need to use mathematics.

CAS provides us with ready-made mathematical tools with which we now can pursue our initial goals, namely teaching understanding of mathematical concepts and the applying of mathematical concepts. Therefore, not only the methods, but also the focus of teaching becomes different. (In my point of view, this is the most important consequence of using CAS in teaching and learning of school mathematics.)

Two important goals of mathematics teaching are: the development of the theoretical meanings of mathematical concepts and their applications. CAS can provide major support for the development of theoretical meanings, see for example (Kutzler 1995), (Schneider 1996) and (Böhm&Pröpper 1998). CAS also can be decisive in enabling or improving the teaching of the application of mathematical concepts.

In mathematics teaching, the application of mathematical concepts is done through solving mathematical problems. Mathematical problems play an important role as a tool of generalization within the process of developing mathematical concepts in the form of exercises and homework. They are also a key for both the teacher and the student for getting feedback about their efficiency during the learning process. Here we call them *exam questions*. Compared to exercises and homework, exam questions are different in intention/function and contents.

Today more and more teachers are in a state of transition from traditional to “modern” (in the sense of “technology-supported” or “CAS-supported”) teaching of mathematics. For many teachers a first step is the *changing of teaching styles* (e.g. from teacher centered to student centered group work, and from deductive to inductive learning). The inevitable next step is to *refocus on the teaching goals* and, hence, *to change the mathematical problems* treated, i.e. the exercises, the home work, and the exam questions.

When choosing exam questions one has to consider some important facts (Meyer&Winkelmann 1991, S.126):

- Exam questions test *general goals* using specific contents.
- *General goals* are not *skills* (computer can do them better) but *abilities.*
- The abilities aimed at have to be made a subject of discussion and be taught in lessons preceding the exam.

Students use CAS for activities, which require many of the "traditional skills" such as factoring, graphing, and computing a derivative. This means that now the exam questions can be newly focused on the concepts behind these skills (Laughbaum 1998).

Each particular exam question has to be analyzed with respect to the following:

- To what extent does it test basic abilities (in general and belonging to particular concepts)
    - modeling real world situations
    - reflecting about mathematical content
- How large is the share of algorithmic and calculations skills required.

In this paper, the aim is not to show newly invented exam questions to be used in mathematics lessons with CAS, but to analyze some of the exam questions in use today, looking at their value in a CAS-supported learning environment. The goal is giving teachers new perspectives with which to analyze existing exam questions.

2. Exam questions in a mathematics lesson without CAS

In traditional teaching, i.e. one that is centered mostly around algorithmic methods, exam questions predominantly test the ability to perform calculations:

Following are examples of exam questions in traditional mathematics teaching:

1) Simplify:

$$\left(\frac{x}{x+3}-\frac{x^2}{x^2+x-6}\right):\left(\frac{x}{x+3}-\frac{x^2}{x^2-9}\right)=$$

2) Calculate:

$$\left| (1+2i)^2 + \frac{25}{3+4i} + \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) \cdot \left(\frac{1}{2} - \frac{1\sqrt{3}}{2}\right) - i^{207} \right| =$$

3) Calculate:

$$\int \frac{\cos x}{(1 - \sin x)^2} dx =$$

4) Given two functions: $f(x) = x^3 - 3x + 2$ and $g(x) = - x^2 + 3x + 2$.

a.) Determine zeros, local extrema, and inflection points, then draw the graphs of the functions.
b.) Determine the intersection points.
c.) Show, that one of the zeros of the function $f(x)$ and both intersection points are collinear. Give the equation of the corresponding line.
d.) Calculate the area between the two graphs.

5) We throw three dies simultaneously. What is the probability of:
a.) obtaining three different numbers?
b.) obtaining a sum of 17?

The majority of these questions focus on calculation procedures, i.e. on the use of calculation rules and techniques, and very little on updating knowledge about conceptual matters or on using concepts outside mathematics, e.g. modeling of (half-)open situations.

3. Exam questions in mathematics lessons with CAS

What will be changed when using CAS?

Usually (exam) questions are classified into two groups:
- theory oriented questions and
- application oriented questions.

From the point of view of CAS, with respect to their significance of testing abilities and skills, traditional exam questions such as the ones from section 2 can be classified as follows:

3.1. CAS-insensitive questions

Some exam questions remain as they are, for instance question 5 from section 2. Further such examples are:

1) The prime factorization of 10! is $7 \cdot 5^2 \cdot 3^4 \cdot 2^8$. Explain, why there is one 7 and why there are two 5s, four 3s, and eight 2s.
2) A function $f(x)$ has a constant first derivative $f'(x)$. What does this mean for the graph of $f(x)$?

For questions such as these the use of CAS – or any other calculation tool – is only of very limited help. The performing of calculations plays only a minor role in finding the answer, the focus is on understanding the theoretical issues which underlie mathematical concepts. We call such questions *CAS-insensitive questions*.

3.2. Questions changing with technology

For other traditional exam questions the main focus will change drastically, as is the case for instance for Question 4 from Section 2, or questions such as:

1) Prove that the following function is continuous at x = 0:

$$f(x) = \frac{ln(\cosh(x))}{x^2}$$

2) Draw the Graph of the following function and compute the area between the function and the *x*-axes in the interval [0,1]:

$$y = \frac{1}{x^3 - x^2 - 14x + 24}$$

3) Which cylindrical metal can with given volume *V* requires the least amount of sheet metal? How much sheet metal do we need for a can with a capacity of 0.2l?

Without CAS, the solving of such problems was (is) strenuous because they involve a lot of mechanical calculations, or because they require "complicated" multi-step solving strategies. For many students, such craftsmanship activities outshine the actual point (goal) of the problem, hence they often forget to answer the original question after successfully completing the necessary calculations.

In a traditional paper and pencil environment tasks such as solving equations, finding derivatives, and computing integrals require most of the time used for answering these questions. Therefore for both teachers and students the feedback centers on testing the performing of operations. When using CAS, the time needed for such questions is reduced drastically, hence the significance of these questions for providing feedback for teachers and students changes as well.

The above are the questions/problems for which the use of CAS means a shift of focus from technical/mechanical/routine work to mathematical/semantical/conceptual/applicational work.

Before using these questions in a CAS-supported exam, they need to be analyzed and possibly questioned with respect to their value as a feedback instrument for the teacher. Some of them provide only very limited information about what a student knows (e.g. question 3) – they rather test students' technical skills than mathematical abilities. Nevertheless, some teachers may be tempted to consider these the most appropriate questions when allowing CAS with the argument that now they can pose these questions with more complicated or more realistic parameters. However, only changing the parameters in such questions (e.g. replacing a simple function with a complicated function) does not turn the question into a "good" or "appropriate" one from the point of view of testing with CAS. Despite such a change these question will test skills – and not abilities. When changing traditional questions, one should strive to change the emphasis of what is tested towards the understanding or the application of mathematical concepts.

3.3. Questions devalued with CAS

For another group of traditional exam questions the use of CAS means a strong devaluation for testing mathematical knowledge, as is the case for instance for questions 2 and 3 from section 2, or questions such as the following:

1) Compute:

$$\lim_{x \to 2} \frac{\sqrt[3]{6+x} - \sqrt[3]{12-x^2}}{x-2} =$$

2) Compute the first derivation $f'(x)$ of $f(x) = x^2 \sin x + \frac{1-x}{\cos x^2}$ at the point $x = x_0$.

This group of questions is characterized by the fact that in a paper-and-pencil environment the students need a very specialized (or highly developed) knowledge about calculation procedures, which in some cases is irrelevant to the underlying mathematical concept. Typically, such examples require the application of exotic transformations or rare tricks (e.g. question 3 from section 2). Such questions exclusively test skills, i.e. the capability to perform specific operations/calculations. In general, questions such as these even may divert students from the essence of the related mathematical concept.

When using CAS, all that remains with such questions is the testing of the technical *ability to use the CAS*. They become worthless for obtaining any feedback on student's mathematical abilities.

Questions like these lose their purpose in a CAS environment, because CAS reveals the insignificance of their traditional purpose. Such questions served only one very narrow purpose, namely the development of a craftsmanship (without even touching its usefulness or applicability), which is a purely mechanical goal. In fact, such questions never were good exam questions, not even in a traditional paper and pencil environment. (Mathematics education researchers consider them the "worst" type of exam questions.) CAS help us to make this obvious.

3.4. Questions testing basic abilities and skills

Finally there are questions such as question 1 from section 2. In a way, these questions become trivial when using CAS, as such a system would produce the answer right away – even without requiring any specialized CAS know how. On the other hand, the goal connected with such questions is a very important one: Beside the skill of transforming an expression using given (mathematical) rules the students also need to have a knowledge about the syntactical structure of the expression.

When using CAS, what remains with such questions is the testing of the knowledge about the syntactical structure of the expression. The student requires this when entering the expression into the CAS. (One might test other abilities by adding questions such as "*simplify the denominator*".) Questions such as no. 1 from section 2 provide information about students' basic knowledge (in this case: knowledge about the syntactical structure of an expression) and, therefore, should be used for testing with or without CAS.

3.5. "Rediscovered" questions

Classified and analyzed above are the so called "traditional exam questions" by their value as a feedback instrument for the teacher. The use of CAS guided the analysis and served as an instrument for finding the actual mathematical benefit of a question. Looking at the traditional exam questions is the first step when searching for exam questions for lessons using CAS. The next step is, of course, to find additional "new" questions – or perhaps "forgotten ones". There are a lot of rarely used or, maybe, even forgotten questions, which support creativity, fluency, and flexibility. Their rare use in traditional lessons without CAS may be explained by the large variety of possible answers or because of the difficulties in evaluating and grading them.
Traditionally, teachers were interested in testing if a student can execute a specific algorithm. Typically there was only one way to do this and teachers wanted their students to provide this one solution only (Laughbaum 1998).

Examples of such questions are:

1) Given two functions $y = 3t^2 + 5t - 1$ and $y = -2t + 3$. They describe a time/distance dependency. Use a maximum number of different methods to find the points in time when both functions are at the same position. Determine the meeting points.

2) Give at least three different (but equivalent) word problems leading to the mathematical problem "Solve the equation $7x^2 - 4xt - 2 = 0$".

3) $x_1 = -1$, $x_2 = ...$, $x_3 = 2$ are the only real solutions of an equation. Find at least two appropriate equations of 3rd, 4th, and 5th degree.

The above classification is most relevant and useful, in case we compare the usability of more or less traditional exam questions when testing with and without CAS. In case we focus just on mathematics lessons with the use of CAS, then the following classification scheme seems very useful.

4. Classifying exam questions according to the use of CAS

Above are classified exam questions with respect to their *significance of testing abilities and skills*. Another way of looking at exam questions is from the point of view *of using a CAS for their solution*. Kutzler suggested a respective classification scheme (Kutzler, 1998).

Kutzler proposes to classify exam questions according to the role that CAS plays in answering them. First he looks at how significant the use of the CAS is (*primary* versus *secondary use*), then he looks at how well the student needs to know the CAS (*routine* versus *advanced use*).

- *Primary CAS-Use* means that using a CAS is the major activity in solving the problem, hence the problem could not (or hardly) be solved in lessons without the CAS.
- *Secondary CAS-Use* means that using a CAS is a minor activity in solving the problem, hence the major contribution to its solution requires abilities for which CAS offers no support.
- *Routine CAS-Use* means that a superficial knowledge of the CAS suffices for solving the problem.
- *Advanced CAS-Use* means that in-depth knowledge of the CAS is required for solving the problem.

Finally, there are exam questions which require none of the skills offered by a CAS.

- *No CAS-use* means that a CAS is of no help in answering the question.

Like with any classification scheme, there is no clear-cut dividing line between the various categories, because the reality is continuous – not discrete. Hence, for some exam questions it may appear arbitrary to put them into one or the other category.

Kutzler presents his exam questions classification scheme in form of the following matrix:

	Routine CAS-Use	***Advanced CAS-Use***
Primary CAS-Use	see examples in section 4.1	see examples in section 4.3
Secondary CAS-Use	see examples in section 4.2	see examples in section 4.4
No CAS-Use	see examples in section 4.5	

4.1. Examples of *Primary Routine CAS-Use* questions

1) Draw the family of graphs defined by the functions $f(x) = \ln(x-p)$ for $p\in[-3,3]$, $p\in Z$. Which transformation is determined by the parameter p?

2) Proof that the function $f(x)$ is continuos at x=2 and explain the result using the graph.

$$f(x) = \frac{\sqrt[3]{6+x} - \sqrt[3]{12 - x^2}}{x-2} =$$

3) What is the slope of $f(x)$ at $x=x_0$?

$$f(x) = x^2 \sin x + \frac{1-x}{\cos x^2}$$

3) Question 1 from section 3.2.

4) Question 2 from section 3.2.

5) All questions testing basic abilities and skills (section 3.4.)

4.2. Examples of *Primary Advanced CAS-Use* questions

1) Compute the length of that chord of the ellipse $(x-1)^2/54 + (y+2)^2/27 = 1$, which goes trough the center of the ellipse and bisects the angle between the axes of the ellipse. Give the equation of the line through the chord.
2) An Audi TT shows the following dependence between revolutions per minute and fuel consumption:

rpm	1000	2000	3000	4000	5000	6000
fuel cons. (l)	7	7,2	7,9	9,6	11,5	13

a.) Represent the dependence graphically.
b.) Find the fuel consumption at 3500 rpm
c.) For which number of revolutions per minute is the fuel consumption 7,6 l (11 l)?

3) Question 4 from section 2.

4.3. Examples of *Secondary Routine CAS-Use* questions

1) Find functions with the following behavior. Give their symbolic forms.
   a) has a hole at x=2
   b) has a vertical asymptote at x=-1
   c) passes through (2,1)

2) Given the function $f(x) = |x|$. (and explain why) this function has no tangent at the point T(0, 0) . Why does this function have no tangent at the point P(1, 2)?

3) Question 3 from section 3.2.

4.4. Examples of *Secondary Advanced CAS-Use* questions

1) A comet's orbit has an eccentricity of 0.98. It's closest distant to the sun is 20×10^6 km. What is the longest distant from the sun along the orbit?

2) The braking distance for a truck with the speed measured in km/h is $v^2/100$ meters, the "reaction distance" (distance driven during the reaction time) is about v/4. For save driving, the distance between one truck driving behind another truck should be at least the sum of the braking distance and the reaction distance. At which speed will a convoy of trucks have the highest number of trucks passing a point along the road (within a time unit)?

4.5. Examples of *No CAS-Use* questions

1) The function A assigns to a radius r the area $A(r)$ of the corresponding circle. Describe the functional dependence with an equation (equation of function).

2) All CAS-insensitive questions (section 3.1.)

5. Comparision of the above two classification schemes

The classification scheme in section 3, which is on the basis of the significance for testing abilities and skills, is *educational* insofar as we look at the educational value of mathematical problems used as exam questions. The classification scheme in section 4, which is on the basis of the usefulness of CAS, is *technical* insofar as we look at the role of technology for answering the question. Any exam question used in a mathematics lesson can be classified according to either one of the two schemata. In this section we look at possible correspondences between the two. We look at the technical categories from the educational point of view and vice versa.

An overview of this correspondences is described by the following matrix, where a smiling face shows a correspondence between the two respective groups.

	Primary Routine CAS-Use (4.1)	*Primary Advanced CAS-Use (4.2)*	*Secondary Routine CAS-Use (4.3)*	*Secondary Advanced CAS-Use (4.4)*	*No CAS-Use (4.5)*
CAS-insensitive questions (3.1)					☺
Questions changing with technology (3.2)	☺	☺	☺	☺	
Questions devalued with CAS (3.3)	☺	☺			
Questions testing basic abilities and skills (3.4)	☺				
“Rediscovered” questions (3.5)			☺	☺	☺

From the educational point of view it is important that all *Primary CAS-Use Questions* aim at testing mathematical abilities, as otherwise they would be worthless. These questions have to center around the mathematical meaning of concepts or their applications, while using the CAS allows the students to focus their attention on concepts without being interrupted by mechanical work.

Primary Routine CAS-Use Questions (4.1) are problems, for which using a CAS is the major activity when solving them, though superficial knowledge of the tool suffices. Most of the questions from 3.2 (*Questions changing with technology*), most of the questions from 3.3 (*Questions devaluated with technology*), and all the questions from 3.4 (*Questions testing basic abilities and skills*) belong to this technical category.

In *Primary Routine CAS-Use Questions* the interruption by mechanical work typically happens only once or twice and a straightforward application of the CAS prevents this interruption. For question 4.1.3 these two interruptions are the determination of the first derivative and the evaluation of the derivative at the given point. Both calculations are very easily done with a CAS. For these questions it is vitally important that their formulations ensure the testing of mathematical abilities, as otherwise they would be worthless. Examples 3.3.2 and 4.1.3 help us clarify this. While from a technological point of view they appear identical (one has to compute the first derivative of a function, then evaluate it at a point). The formulation “What is the slope of ...” used in 4.1.3 requires the student to know about the application of the derivative, hence this question remains appropriate. The formulation “Compute the first derivative ...” used in 3.3.2 makes this a straightforward computing exercise which becomes worthless when using CAS. *Primary Advanced CAS-Use Questions* (4.2) are problems, for which using a CAS is the major activity when solving them, while advanced knowledge of the tool is required. Some of the questions from 3.2 (*Questions changing with technology*) and some of the questions from 3.3 (*Questions devaluated with technology*) belong to this technical category.

In *Primary Advanced CAS-Use Questions* the interruption by mechanical work typically happens more than twice as is either enforced by the formulation of the question itself or by the set of basic operations offered by a specific CAS. (The later demonstrates how much the tool influences the choice between categories *Routine* and *Advanced*. With a CAS not offering a function for solving a system of linear equations, such a question would fall into the category *Primary Advanced*, while with a CAS offering such a function it would be *Primary Routine*.)

Primary Advanced CAS-Use Questions require a multi-step solving strategies (either by the nature of the question itself or by the basic operations offered by the tool). This alone preserves their value as exam questions for testing mathematical abilities.

Other as with *Primary Advanced CAS-Use Questions*, *Primary Routine CAS-Use Questions* are more likely to be devaluated through technology. Most of the time an appropriate reformulation will ward off this danger.

In *Secondary CAS-Use Questions* the aspect of mathematical content or application already is in the foreground.

Secondary Routine CAS-Use Questions (4.3) are problems, for which using a CAS is a minor activity when solving them, while superficial knowledge of the tool suffices. The more sophisticated questions from 3.2 (*Questions changing with technology*) and the "easiest" questions from 3.5 (*"Rediscovered" questions*) belong to this technical category.

Secondary Advanced CAS-Use Questions (4.4) are problems, for which using a CAS is a minor activity when solving them, while advanced knowledge of the tool is required. Most of the questions from 3.5 (*"Rediscovered" questions*) and eventually some of the application oriented questions from 3.2 (*Questions changing with technology*) belong to this technical category.

Most of the questions in this group are application oriented, hence the students have to start by finding a mathematical model.

No CAS-Use Questions (4.5) are problems, for which the CAS is of no help in answering the question. All questions from 3.1 (*CAS-insensitive questions*) and some of the questions from 3.5 (*"Rediscovered" questions*) belong to this technical category.

We feel that, as a tendency, the five categories of the CAS-use-induced classification scheme from section 4 show the following ranking in terms of their value for testing mathematical abilities:
1) *Secondary Routine CAS-Use* (4.3) & *No CAS-Use* (4.5)
2) *Secondary Advanced CAS-Use* (4.4)
3) *Primary Advanced CAS-Use* (4.1)
4) *Primary Routine CAS-Use* (4.2)

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The above comparison leads to some important observations:

- Using CAS facilitates the pursuing of specific mathematical teaching goals.
- Using CAS reveals the educational value of specific exam questions.
- Using CAS forces the teacher to more consciously use exam questions. (See the above table, where the *Questions changing with technology* correspond with all four CAS-use-induced categories.)
- Using CAS means a rebirth of the *Rediscovered questions*.

6. Conclusion

The *act of understanding* as well as the *act of overcoming an obstacle* are both acts of the learning process, requiring intellectual concentration and leading to emotional tension. In mathematics lessons intellectual concentration and emotional tension culminate within the exam situation, which finally becomes a learning situation as such.

Exam questions play a very important role as didactical tool in teaching mathematics, not only as a key for getting feedback during the learning process, but also as an important learning situation. As such they inevitably have to follow two major educational goals:

- understanding the theoretical meaning of mathematical concepts (reflection about mathematics subject matter)
- using mathematical concepts in modeling real situations (modeling open real world situations)

These are also the goals for which using CAS in teaching and learning mathematics are most helpful. Therefore, using CAS for teaching and learning as well as for testing student's achievements appears to be most natural.

However the teacher has to decide, for any particular question, if it is appropriate to be used as an exam question: Does it test general goals (e.g. abilities) on specific contents or does it test mainly the skills which a student has acquired (collected)? When choosing exam questions, the teacher need to know what needs to be tested. This becomes even more important in a CAS teaching and learning environment.

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