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The necessary fundamental algebraic competence in the age of Computer Algebra Systems

In the same manner as we discussed the necessary numeric fundamental competence when we introduced the numeric calculator in the Seventies, it now becomes vital to explore the algebraic fundamental competence . This lecture will deal with these questions partly only by formulating new questions but I will also try to give first answers based on the experience of the Austrian CAS projects.

1. The situation at the beginning of the CAS-age

In the middle age there existed a honorable guild, the guild of "calculating masters". They died out when people became able to calculate themselves. Will the math teachers also die out in the age of CAS?

The progress of the humanity is documented by her tools . Tools, on the one hand, are results of recognition and on the other hand new recognition is not possible without tools. [Claus, 1990, p 43]

In the 17th century Leibnitz tried to invent a calculating machine because *"for human beings it is unworthy like slaves to lose hours by doing stupid and monotonous calculations"*.

But looking at math lessons or tests, calculating skills are still dominant.

In the didactic literature one can find dubious arguments, like [Köhler, 1995]:

Calculating
- supplies math education with necessary materials for practicing
- makes a feeling of success possible also for the weaker students
- relieves teachers and students.

Or another argument which I found in an didactic article:
Technological progress causes the disappearence of routine exercises which brought a feeling of success for weaker students by practicing hard.

My opinion is: It is senseless to hold on to senseless tasks or goals because they give weaker students a feeling of success. This is a danger for the subject of mathematics because people would look for more sense in other subjects as you can see in articles in Austrian newspapers titled "Environment instead of mathematics".

My first thesis:

Thesis 1: In the CAS-supported math education it is also possible or even more easily possible to find routine exercises suitable for more meaningful goals and to automate skills which give weaker students a feeling of success.

He who holds on to the dominating of calculating skills should look at the history of mathematics. A main goal of mathematics always was to develop schemes and algorithms which make lengthy calculations obvious, that means to automate unnecessary mathematical activities or in other words, to trivialize. Progress in mathematics means creating free capacities by automatism for more important activities on a higher level. Tools always play an important role in this process..

As the results of our CAS projects and the developed didactic concepts show, mathematics in the age of CAS does not necessarily mean uncomprehending work with black boxes. Especially the White Box/Black Box principle demands at first a white box phase in which the algorithm is developed also trying to understand the process. and in which the learners have to acquire certain fundamental manual calculating skills. Only afterwards operating is done by the CAS as a black box. The CAS plays an important role in both phases: In the white box phase as a didactic tool to help the learners to make the box "white" and also as a controlling tool , in the black box phase as a calculating tool and as a tool for testing and interpreting.

But the theme of my lecture is not only manual calculating competence. Fundamental algebraic competence is more than calculating, but the main change is the change of the importance of calculation competence.

The tool computer forces us to think about problems which we had to think over a long time ago [Schubert, 1994].

2. Important elements of fundamental algebraic competence

To begin with, I must confess that I have no answer to the often posed question: „Which minimal algebraic competence is indispensable?" It has been shown in many discussions that at least the average of all absolutely necessary calculating competences is empty. We first have to ask ourselves what the educational value and the goals of the subject mathematics are and then consider the question of partial algebraic competence. As this discussion would go beyond the scope of this conference, I have based my thoughts concerning the grander picture of mathematics on the following definition by Bruno Buchberger

> ***Mathematics is the technique, refined throughout the centuries, of problem solving by reasoning***

Let us now turn to those competences which, in my opinion, are important:

2.1 The competence of finding terms or formulas

Of the three phases of the problem solving process, modelling—operating—interpreting, the operating phase has always dominated. The tools of CAS now make it possible to more evenly distribute the importance of the three phases. This means that developing formulas gains more importance in comparison to calculating with terms.

The activity often is a brain activity by remembering an earlier learned formula or knowing where the needed formula could be found.
This competence also demands abilities of other areas of mathematic activity like competence in geometry for deriving geometric formulas or the important competence of translating a text into an abstract formula.

The translation process from the student´s language into the language of mathematics mostly takes place in three steps:
Step 1: A sentence in the colloquial language
Step 2: A "word formula"
Step 3: The symbolic object of the mathematical language

Didactic concepts which support the acquisition of this competence could be taken from language subjects, e.g. creating a dictionary for translating from German into mathematics.
Example: The sentence "*vermehre um p Prozent*" is translated into "$.(1+p/100)$"

The influence of CAS:

- The CAS allows the students to transform the word formula directly into a symbolic object of the mathematical language by defining variables, terms or functions or writing programs.
- The CAS allows a greater variety of prototypes of a formula and also offers some which were not available before. While in traditional mathematics education often only one prototype is available and used, now the CAS offers several prototypes parallely in several windows. This fact causes a new quality of mathematical thinking. We name this mathematical activity the "Window Shuttle Method"
- The CAS offers and allows a greater variety of testing strategies, in this case testing if the formula is suitable for the problem and mathematically correct.

Example 2.1: A financial problem

One takes out a loan of K=$ 100.000,- and pays in yearly instalments of R=$ 15.000,-.
The rate of interest is p=9%. After how many years has he paid off his debts?

In traditional mathematics education such problems could be solved for the first time in 10th grade, because the students need geometric series and calculating skills with logarithms. The computer offers a new model, the recursive model. From that pupils now work such problems in 7th grade

The first step is finding a word formula, which describes what happens every year:

Interest is charged on the principal K, the instalment R is deducted.

Translated into the language of mathematics:

$K_{new} = K_{old}*(1+p/100) - R$

After finding a formula in the phase of modelling operating is the next activity. Using a CAS calculating takes on a new meaning. The TI-92 offers a special way to come to a better understanding of a recursive (or better an iterative) process: The activities of storing and recalling make the pupils conscious of the two important steps of a recursive process: Processing the function and feedback (K_{new} Y K_{old}) (figure 2.1). Looking at the list of values the quality of an exponential growth becomes much clearer than by calculating with logarithms. The typical problem of paying in instalments can be recognized: During the first phase the loan is nearly equal because the greatest part of the instalment is used for the interest (figure 2.2). The experimental solution is obtained by repeated usage of the enter-key until the first negative value appears (figure 2.3). The variable n shows the number of the years.

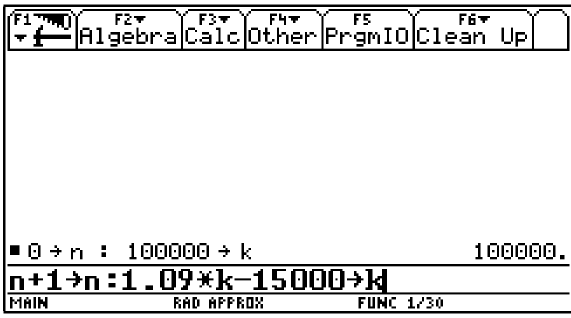


figure 2.1

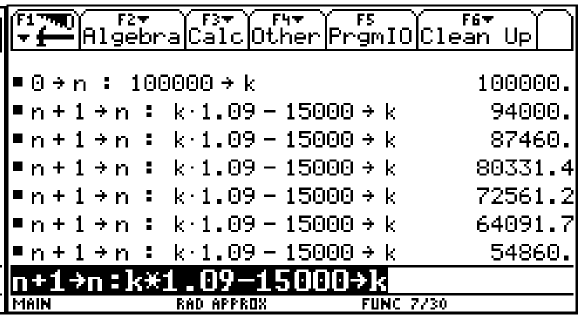


Figure 2.2

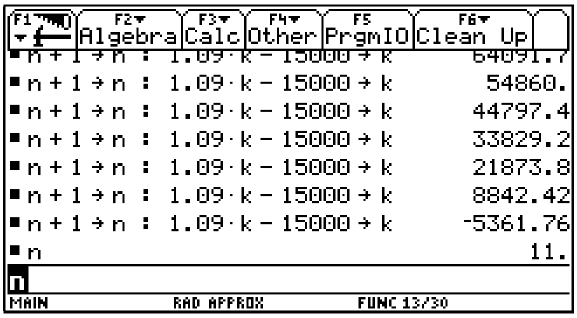


figure 2.3

In this learning phase the pupils should explore the fundamental idea of a recursive process by experimenting and working step by step. We call such a phase White Box Phase, a phase of cognitive learning.

By using the Sequence Mode in the next learning phase - the Black Box Phase – a new prototype of the formula is available (figure 2.4). The learners can easily experiment with several rates of interest and installments. Simulating is done by the CAS. The students have to find a suitable model and to interpret the results, either the table or the graph (figures 2.5 and 2.6).

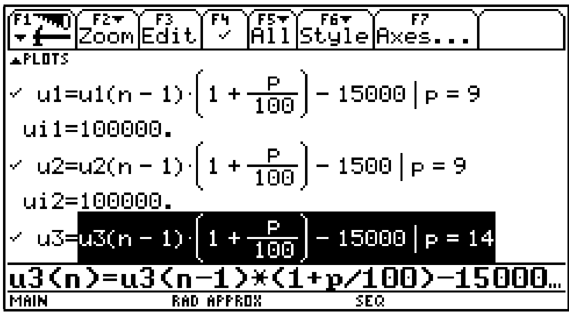


figure 2.4

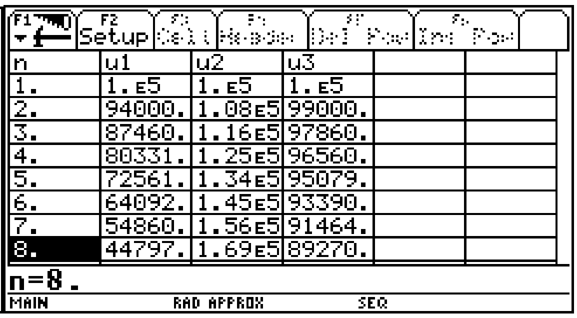


figure 2.25

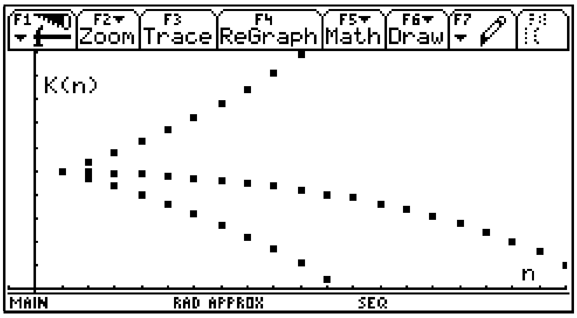


figure 2.6

Another way of finding a formula for this financial problem is the traditional way of using one´s knowledge about geometric series. This way also needs a higher competence of calculating , because a more difficult equation has to be solved and knowledge about logarithms is necessary.

Example 2.2: First experience with the function concept in the 7th grade: Direct - indirect proportion

This example is part of an investigation, called „observation window“ in the Austrian CAS project [Klinger, 1997]. The goal was to observe the pupils´ behavior: The learners should choose a prototype of a function suitable for a given problem and they should discover and use strategies for the proof of a definite functional relation.

The initial problem for indirect proportions was rather simple:

> The distance between Vienna and Innsbruck is 500 km. Calculate the driving time for several mean velocities.

After calculating the time for velocities in the given table the pupils had to find a formula. The TI-92 offers several possibilities:

- Using the y-Editor. This prototype offers the approach to other prototypes such as the Table or the Graph (figure 2.7 to 2.9).
- Defining the Formula in the Home Screen. This way opens several possibilities of calculating (figure 2.10 and 2.11)
- The Data/Matrix Editor offers the strategies used in Spreadsheets.

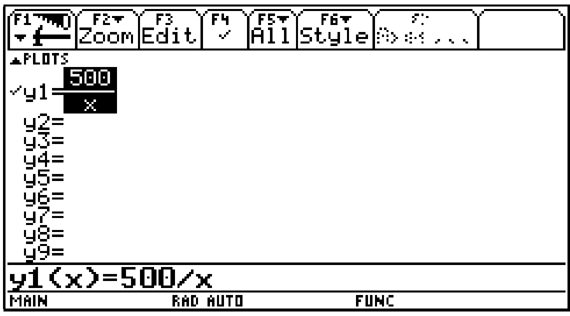


figure 2.7

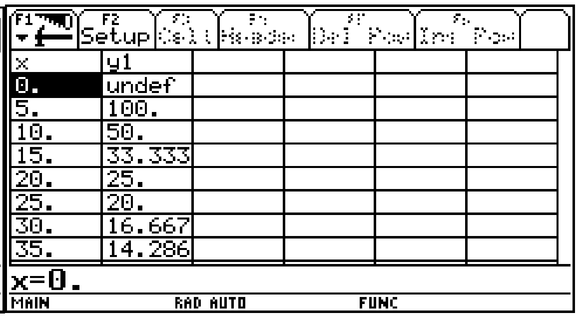


figure 2.8

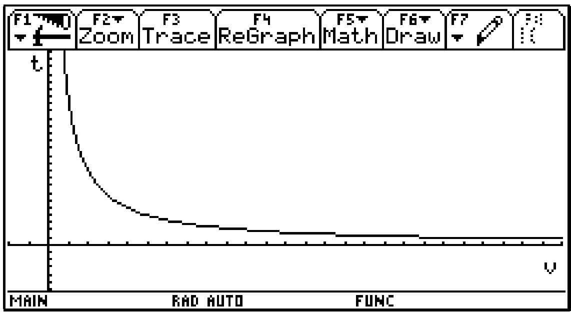


figure 2.9

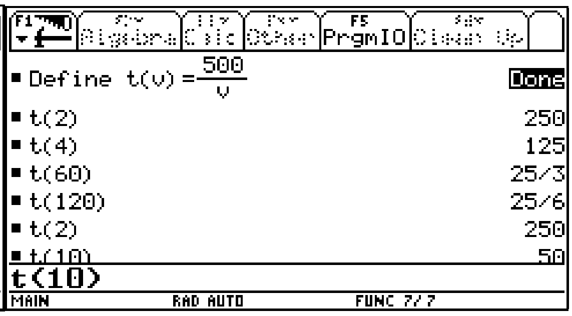


figure 2.10

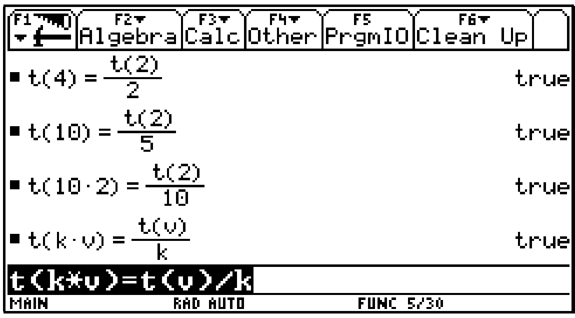


figure 2.11

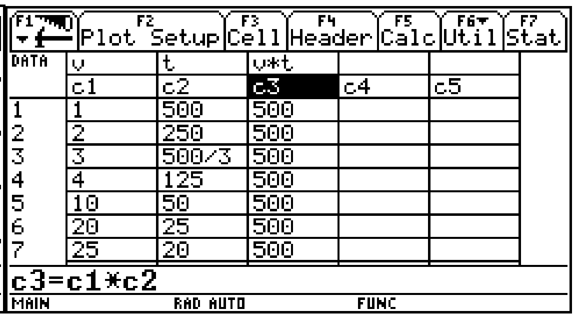


figure 2.12

Some results of pupils´ behavior:

Pupils use the possibility of having several prototypes of the function parallely at their disposal. Shuttling between several prototypes becomes a common practice and allows them to use the advantages of certain prototypes.

Several pupils develop preferences to several prototypes. In traditional math education, the table often is the only prototype which is at their disposal. I did not expect that pupils of the 7th grade to also use the graph and the defined function (see figure 2.10), the last one is prefered by more gifted children.

It is not only easier now to get tables, the opportunity to calculate with whole rows is the main importance of function prototypes in the Data/Matrix Editor.

The testing strategies strengthen the decision competence according to the type of the function.

2.2 The competence of recognizing structures and recognizing equivalence of terms

This competence is necessary when developing a term, when deciding upon or entering a certain operation and also when interpreting or testing. This competence has always been of great importance as research, such as that of Günter Malle, has shown us that the most commonly made mistakes during algebraic operating are those of recognizing structures.

Recognizing equivalence of terms which the learners have developed or recognizing results of calculations done by the CAS is a part of the competence of recognizing structures.

A prerequisite for this competence is the knowledge of basic algebraic laws. Such decisions cannot be done successfully by using the CAS as a black box without this mathematical knowledge.

The influence of CAS:

- When using a CAS the first step, the input of an expression, needs a structure recognition activity.
- Using the CAS as a black box for calculating a recognition of the structure of the expression is necessary before entering the suitable command. Blind usage of commands like *factor* or *expand* is mostly not successful.
- The learner must interpret results and recognize their structure which he himself did not produce.
- The individual results of various students doing experimental learning must often be checked for their equivalence.
- CAS sometimes produce unexpected results and students do not know whether they are equivalent to their expected results or whether they differ.

Example 2.3: New goals for working with traditional complex expressions [Böhm, 1999]

In traditional mathematic school books one can find such complex terms for practicing manual calculations. Using a CAS like the TI-92 or Derive the new goal is structure recognition when entering the expression (figure 2.13 And 2.14). Especially the linear entry line demands this competence.
The calculation is done by the CAS as a black box (figure 2.14 And 2.15). We think that the competence of manual calculating such a complex expression necessary in the age of CAS.

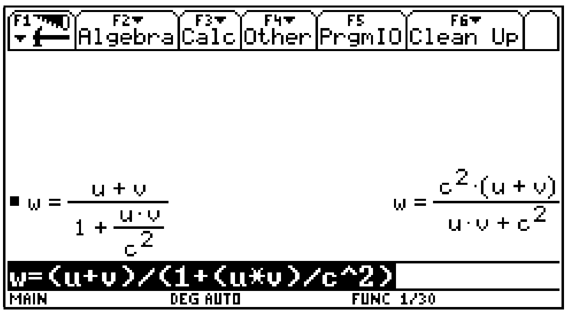


figure 2.13

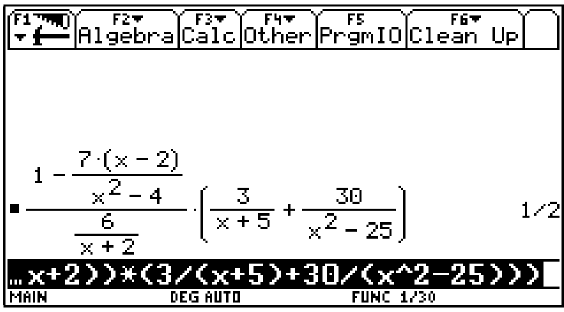


figure 2.14

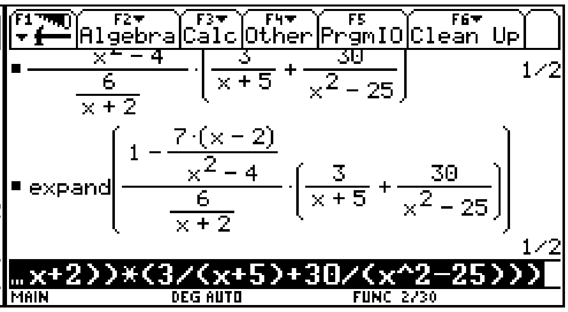


figure 2.15

Example 2.4: Better understanding of formulas by experimenting with CAS

Formulas like $(a+b)^2=a^2+2.a.b+b^2$ are derived by calculating areas of squares and rectangles. The next step is practicing, i.e. recognizing the structure of a term and applying the suitable formula.
The role of CAS is primarily a testing tool and not a calculating tool. At first pupils have to work with pencil and paper.

The following tables were given to the pupils. The goal was not only to find the binom but also to recognize the structure. They also had to discover wrong examples like in line 4 and at last they had to invent their own example (figure 2.16):

Given term	square of a binom	Formula type	$x =$ $x^2 =$	$y =$ $y^2 =$	$2.x.y =$
$4g^2 + 12g + 9$		(1) $(x+y)^2$ (2) $(x-y)^2$			
$25 - 20b + 4b^2$					
$D^2 – 8d.r + 16r^2$					
$4a^2 + 4a.b +4b^2$					
$4p^4 + 4.p^2.q^3 + q^6$					

figure 2.16

After having completed this worksheet pupils could use the TI-89 or TI-92 for testing (figure 2.17)

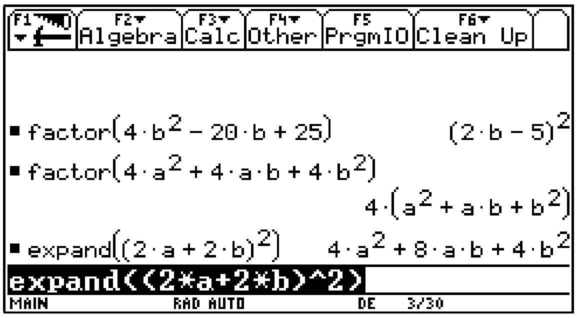


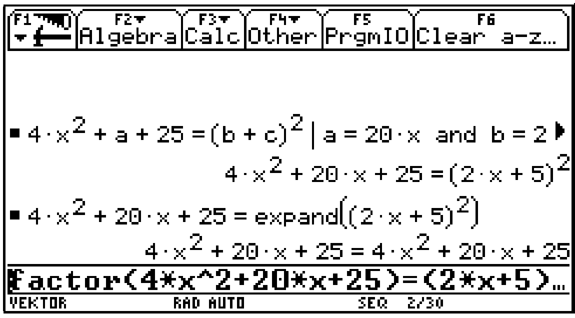
figure 2.17

The next step was completing formulas. Given were tables like the following one:

Given	a =	b =	c =	the completed formula
$4.x^2 + a + 25 = (b + c)^2$				

figure 2.18

Pupils did not use the calculator. After having solved the problem by pencil and paper, they experimented with the calculator. They substituted the supposed expressions using the ***with***-operator. The reaction of the TI-92 (figure 2.19) and the TI-89 (figure 4.20) is quite different: The TI-89 immediately gives the answer „true“, the TI-92 shows the changed equation.



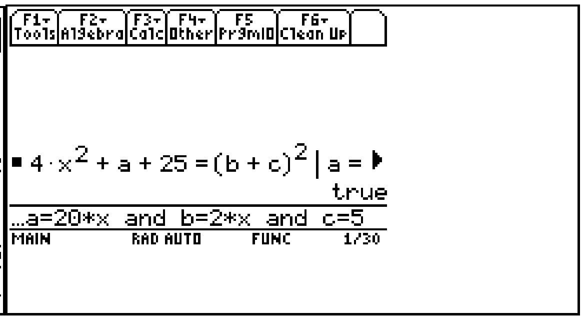


figure 2.19 figure 2.20

Independent of the calculator type pupils had to explore and to use testing strategies after having a supposition for the wanted variables:

Use the ***with***-operator and

- expand one side of the equation
- factorize one side of the equation
- calculate the difference of the both sides

Example 2.5: Comparison of transformation strategies for solving equations

Pupils can compare the effectiveness of different solution strategies and find their individual strategy. The decision concerning a certain strategy needs the recognition of the structure of the terms.

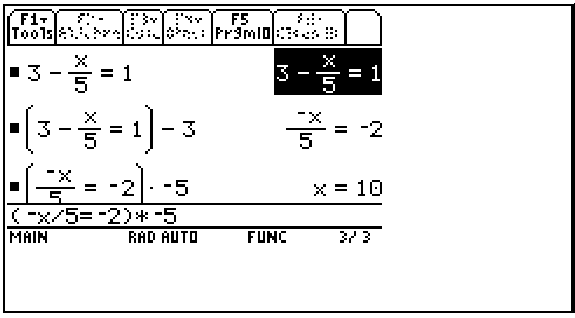


figure 2.21

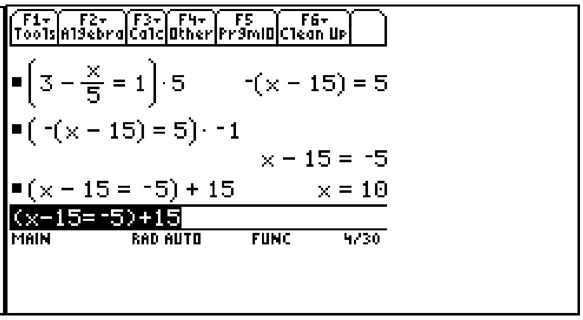


figure 2.22

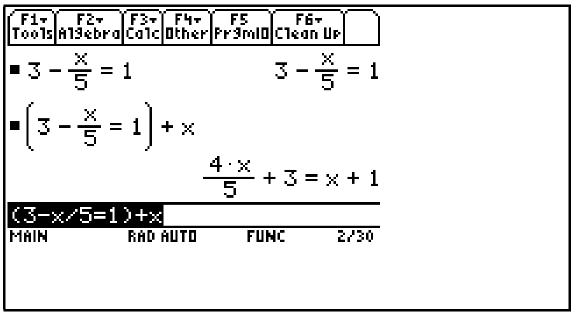


figure 2.23

As seen in figure 2.23, the student is able to recognize the ineffectiveness of incorrect strategies, which he probably would not have recognized when working on paper. On paper the equation 3.x=12 might be improperly solved by using the strategy of subtracting 3 on both sides, resulting in the incorrect assumption that x= 9; working with CAS the student would discover that 3.x-3 =9, a step which is not a real contribution to the situation.

Example 2.6: Riemann sums

Calculate the definite integral $\int x^2dx$ taken from a to b by using the definition of the definite integral. Compare three ways: „midsums", „lower sums" and „upper sums". Make at first a sketch and test the result with the formula of the definite integral for a= -2 and b=4. Use also the TI-92 as a black box for testing (12th grade)

Way 1: Midsums

The expected result is $b^3/3 - a^3/3$ but the CAS offers a quite different result as you can see in figure 2.26. The best way to come to the expected result is a structure recognition which leads to the decision to use the *expand*-command

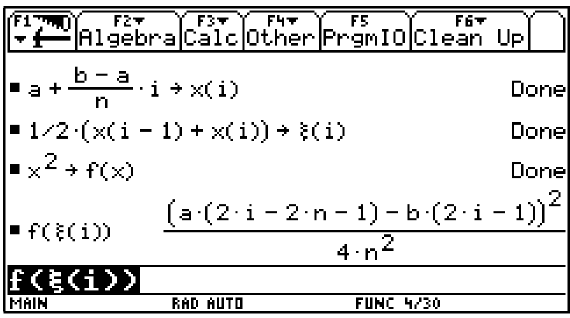


figure 2.24

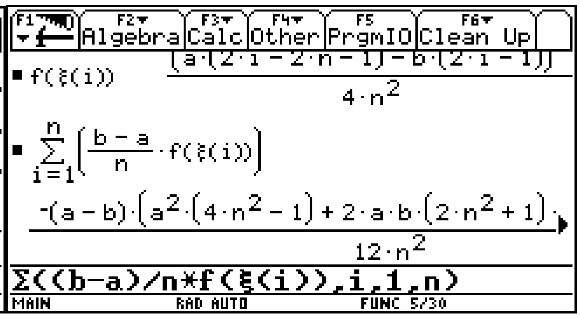


figure 2.25

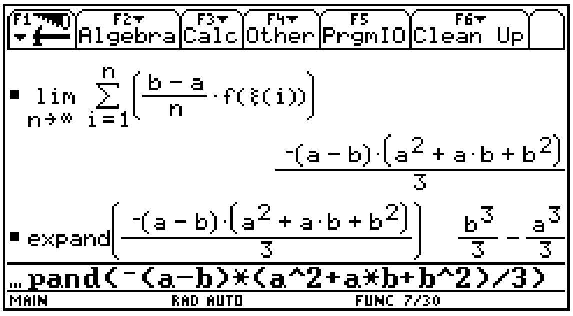


figure 2.26

2.3 The competence of testing

Ever since math has been used as a problem solving technique, it has been necessary to corroborate the correctness of the solutions and to interpret them. The teacher has to offer the learners testing strategies or make them able to find some themselves.

In traditional mathematics education testing means activities like substituting numbers, trying another way of solution, checking the usefulness of the mathematical solution for the applied problem, remembering the definition of a concept a.s.o.

A central result of our CAS projects is a more experimental and independant learning process, whereby the expert is not so much the teacher as the CAS. This means that testing becomes even more in important. The stronger emphasis on modelling and interpreting also demands a higher competence in testing.

The influence of CAS:

- The CAS enables the learner to carry out tests both more effectively and quickly.
- Completely new possibilities are available as far as algebraic and graphic testing are concerned
- Using CAS causes a new problem: The learner has to examine and to interpret results which he himself did not produce. The expectation of the sort of the solution or the form of the algebraic term sometimes differs between the learner and the machine.
- The variety of paths leading to solutions and therefore the number of different results increase dramatically. One will not often find the "algorithmic obedience" of the classical math classroom, in which the majority of the students simply imitate the strategies presented by the teacher. Therefore the equivalence of the numerous results has to be tested.
- The more applied mathematics which we see in the CAS-classrooms demands more testing of the correctness of the model, testing of the usefulness of the mathematical solution according to the given problem and testing of the influence of parameters.

Due to the growing importance of testing , new strategies are necessary which show the learner how to make use of the potential possibilities of the CAS. All those who fear that the use of the computer will lead the learner to experimenting with black boxes without the faintest comprehension of what he is doing should realize that in order to carry out an activity on the computer, the learner must, very definitely, have a grasp for algebra and the underlying algorithm. In fact he needs a wider comprehension than if he were to do the problem by hand. A coincidental trial and error method would not be successful.

Example 2.7: Interest is paid on the capital k at the percentage rate p (7^{th} grade)

Determine a formula for the new capital after one year

Pupils found several formulas, some of the results were wrong (figure 2.27 and figure 2.28)
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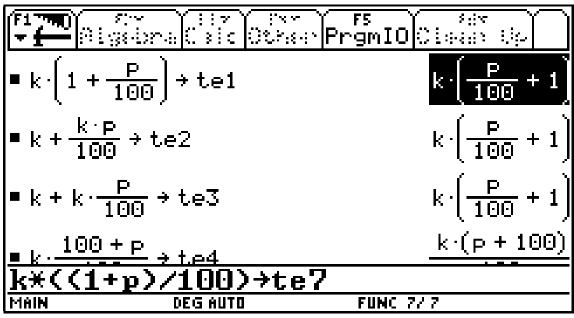


figure 2.27

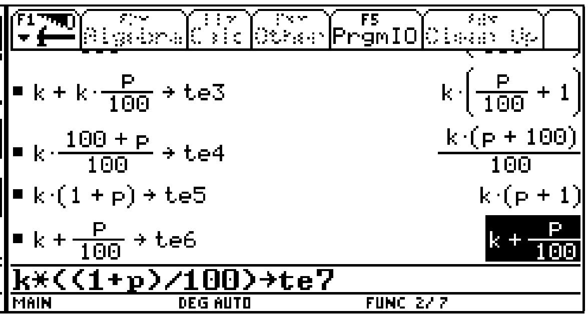


figure 2.28

Now it was necessary to find strategies to prove the correctnes and the equivalence of the terms.

Strategy 1: Using the algebraic competence of the TI-92

After entering an expression in the Entry Line, the TI-92 produces entry/answer pairs. The answer is the simplified version of the entry term, normally simplified by factorizing. When the pupils compare the factorized answer terms they find out, that the terms te1, te2 and te3 are equivalent. They were not sure whether te4 is also equivalent, but it is rather sure that te5 and te6 are not equivalent to te1, te2, te3 (figure 2.28).

Strategy 2: Using the difference of the terms

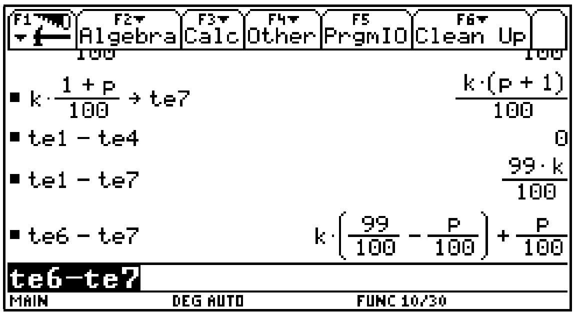


figure 2.29

If the result is 0, pupils recognize that the terms are equivalent. Building the difference of te1 and te7 one can see, the terms are not equivalent.

Strategy 3: Using equations

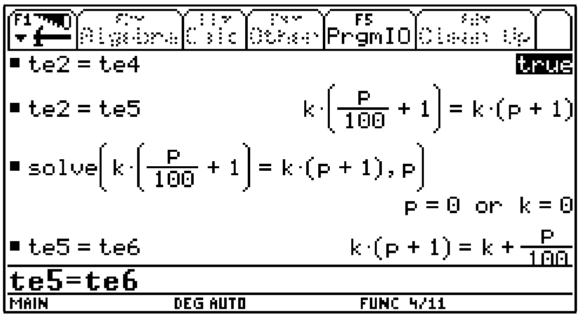


figure 2.30

If the answer is „true“, the equation is soluble for all numbers of the domain. So the pupils know the terms are equivalent. In all other cases the equation is only soluble for certain values and therefore the terms are not equivalent.

Strategy 4: Looking for a factor

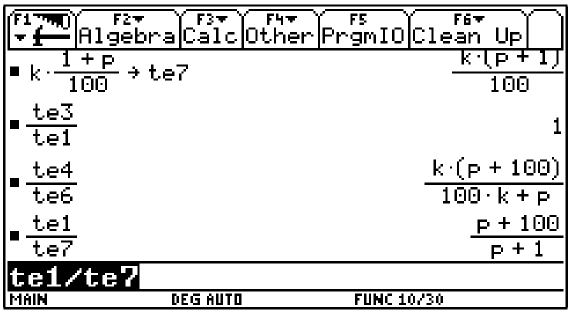


figure 2.31

If the quotient of two terms is 1 they are equivalent.

Strategy 5: Substituting numbers

This traditional strategy can now be used more easily because calculation is done by the CAS

The following example comes from 11th grade. In addition to the algebraic problems new subjects e.g. the derivation of complicated functions bring new problems. Using CAS for more complicated operations allows the learner to solve interesting problems of applied mathematics and also to find new strategies for testing.

Example 2.8: Planck´s radiation formula - surprises by using the strategy of substitution [Dorninger, 1988]

As you can see in figure 2.32 the emission e of a black corpus is a function of the wavelength 8. Determine the maximum value of the function.

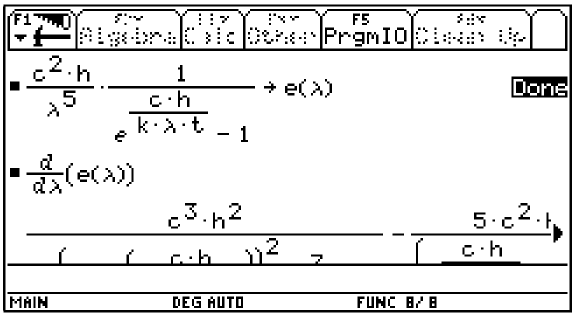


figure 2.32

Pupils decided to use several ways of finding a solution:

Way 1: The derivative of e with respect to 8 is a very complicated expression (figure 2.33). Not the whole expression can be seen on the screen of the TI-92. A commonly used strategy is the substitution of a partial term. In this case the exponent is substituted by the variable x. Thus the first derivative is a function with respect to the variable x (figure 2.34)

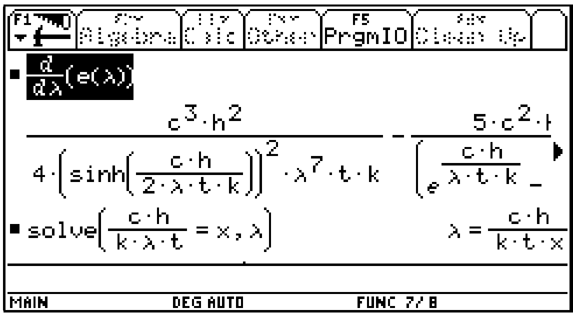


figure 2.33

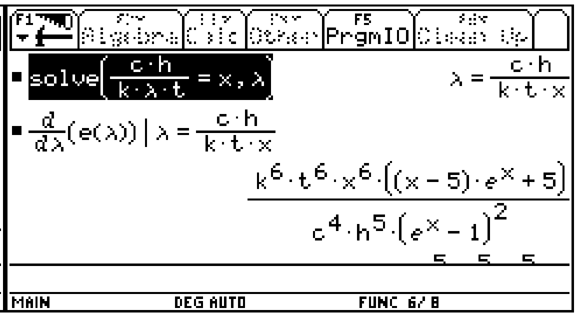


figure 2.34

Way 2: Some pupils decided to substitute before differentiation. They had to look for the 1st derivative of a function e(x) (figure 2.35)

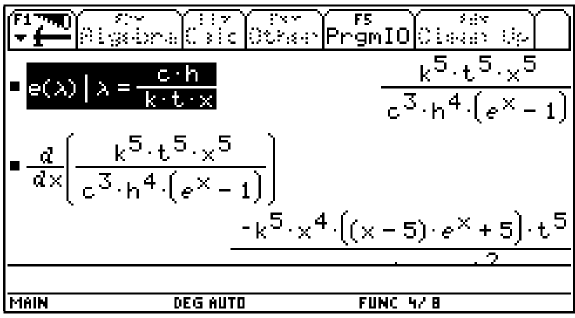


figure 2.35

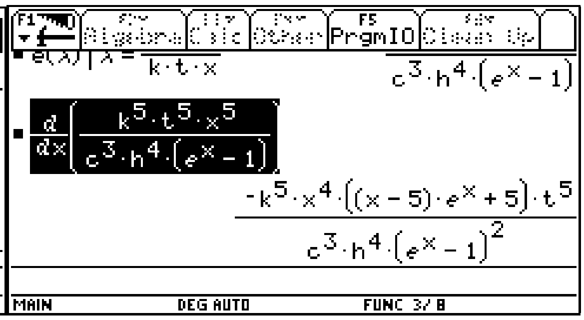


figure 2.36

When several groups compared their results they realized that they were not equal. Looking for the reasons they decided to determine the factor so that the solution of way 1 multiplied with this factor is equal to the solution of way 2. (figures 2.37 and 2.38). Now a discussion started about the meaning of this factor. By remembering the theory and by experimenting with the CAS they discovered: The factor is the „inner“ derivative of 8 with respect to x.

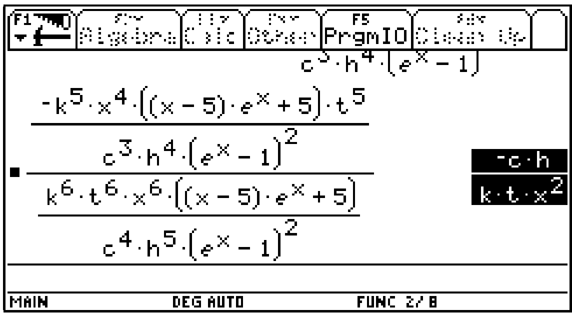


figure 2.37

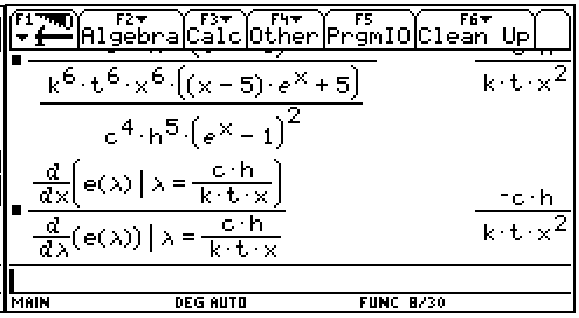


figure 2.38

2.4 The competence of calculating

Before discussing what sort and what extent of competence is necessary for the students to have we might first define what calculation competence is:

Definition: Calculation competence is the ability of a human being to apply a given calculus in a concrete situation purposefully.

Seriously we should distinguish between calculus and algorithm. But this short lecture does not allow a deeper discussion of these concepts.

Briefly we can say:
Calculus] a system of rules
Algorithm] a certain succession of operations (application of rules)

In 1995 there was an interesting conference in Wolfenbüttel in Germany, entitled
“Calculating Skills and Generation of Concepts”
This title is therefore so well chosen. I agree with H. Hischer´s thesis in the keynote lecture that these two components a re not separable:

> *A mathematical concept arises only in a communicative situation by establishing relations on the one hand between objects on a so called level of experience and on the other hand between symbols on a level of calculation [Hischer, 1995].*

One consequence is the following

Thesis 2: **For mathematics to develop within a learner a certain calculation competence is needed.**

We cannot completely leave the calculations to the computer as a black box.
In the age of CAS we have to distinguish between calculation competence and manual calculating skills, because calculation competence could also mean being able to decide on the suitable algorithm and to delegate the execution to the computer.

Another point of view:

Richard Skemp [Skemp, 1976] distinguishes between **relational understanding** and **instrumental understanding** (or shortly between understanding and skills):

Instrumental understanding: Mathematical usage of rules when solving problems without necessarily knowing why the rule is valid.

Relational understanding: The ability of deriving rules, interpreting and possibly proving, to see them as rules in a net of concepts ("knowing both, how to do and why).

This point of view leads to some questions:

Question 1: Is instrumental understanding a prerequisite or a support for a higher level of relational understanding?

Question 2: Does relational understanding support the necessary skills of instrumental understanding?

The influence of CAS:

- A shift in emphasis from calculating skills to more conceptual understanding, to more modelling and interpreting.
- A shift from doing to planning.
- A reduction of the complexity of manual calculated expressions.
- A shift from calculation competence to other algebraic competences, like structure recognition competence or testing competence.
- A better connection between the formal aspect of mathematics and the aspect of contents.

By that I did not give the answer to the central question:

What is the fundamental calculation competence?

0r formulated with the striking words of W. Herget [Herget, 1995]:

"***How many term transformations does a human being need?***"

(1) I am convinced that there is no unequivocal answer to this question.
The answer depends on the context in which this calculation is needed. Calculation competence should not be an end in itself it depends on the mathematical contents.
An example:
We discussed the necessity of the "p-q-formula" for solving quadratic equations. My opinion was it would not be necessary. The competence of factorizing or visualizing the term is more important for the goals of this chapter and in the Black Box phase of learning factorizing could also be done by the CAS. But another member of the discussion group replies: I need this formula because I need the discriminant in geometry when I will derive the condition of tangency for conic sections.

(2) In the age of CAS the answer is a connection between two aspects
- manual calculating skills which are necessary for calculating simple expressions and for the other algebraic competences like the competence of finding formulas, testing or recognizing structures and
- skills of using the actual algebraic calculator efficiently.

I will give some more answers to the questions of this topic in chapter three where I will present some investigations of our CAS-Projects, especially the Project II, the TI-92 project

2.5 The competence of visualizing

A special quality of mathematics is the possibility of graphic representation of abstract facts. Visualizing was also important in traditional mathematics education but it was not easy to get the graphic prototype of a concept or a function. Apart from free hand drawings, it is difficult to develop graphs without using a computer. Finding the most important points and characteristics of functions in order to be able to draw the graph is the main goal of discussion of curves in analysis.

The influence of CAS:

- CAS allows the learner to get the graph faster and more directly.
- Other prototypes of a concept or specially a function are also available much more easily, like a table or lists of values or matrices in a Data/Matrix editor.
- The CAS allows the learner to use several prototypes parallely, while in traditional math education only one prototype was given e.g. the term and it was hard work to get other prototypes like the graph or a table. Now the given term allows the pupil to draw the graph directly by entering the command *graph* and deciding on a suitable window area.
- The learning process consists of shuttling between several prototypes that means shuttling between several windows. Therefore we call this didactical concept the Window Shuttle Method.
- These facts also allows to solve algebraic problems graphically.

Example 2.9: Graphic solution of an unequation. Solve the equation

x - 2 - 1 < x/3 + 1

The *solve*-command does not help the learner to find a solution (figure 2.39). Graphic solution means to comprehend the left and the right side of the unequation as functions (figure 2.40), to draw them and to look for the intersections in the graphic window (figure 2.41). These can be found by calculating when using the CAS as a black box or by experimenting or walking along the graph by using the Trace-mode.

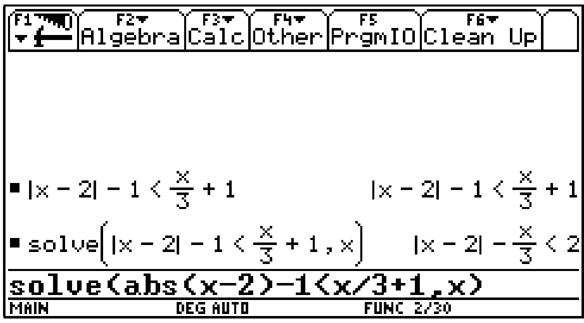


figure 2.39

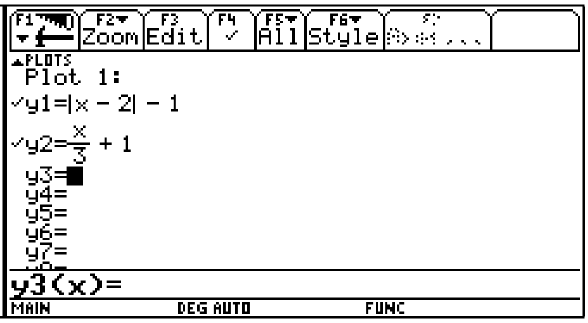


figure 2.40

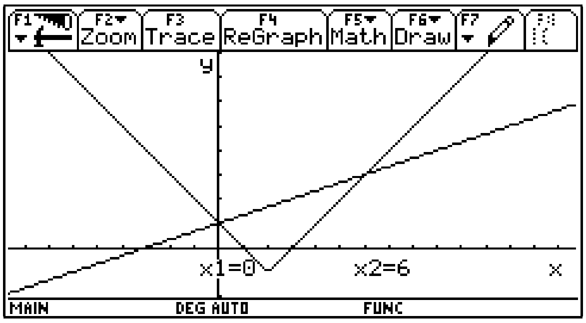


figure 2.41

2.6 The competence of working with modules

Using modules is not new for the learners. Every formula used by the pupils can be seen as a module e.g. Hero's formula for the area of a triangle or the use of the cosine rule in trigonometry. Knowing such a module means that the student has a model for his problem but when using this module he has to do the calculation himself.

The influence of CAS:

- The computer, and especially CAS, opens a new dimension of modular thinking and working. By defining or storing parts of a complex expression as a variable, students can simplify the structure of the expression making it more comprehensible and they can calculate with such modules.
- One weak point of the TI-89 and TI-92 promotes and strengthens the use of modules as new language elements: the small screen. More complex expressions cannot be totally seen on the screen and therefore operations with such expressions become confusing. So the structure of such operations is more comprehensible and clearer if students use the name of the expressions instead of the expressions themselves.

Example 2.10: Solving systems of equations. Discovering algorithms like substitution method, Gauss algorithm a.s.o.

Watching our students from seventh to tenth grade in the White Box phase of learning we observed three steps of abstraction:

Step 1: Working "***into the equations***" as in traditional math education.
Step 2: Working "***with the equations***".
Step 3: Working "***with the names of the equations***", that mean working with modules.

Step 1: In traditional math education students have to work ***„into the equations“***. They have to do the calculating themselves:

(I) 3.x - 2.y = 12*+2.y
(II) 7.x + 2.y = 8

(I) 3.x = 12 + 2.y*:3
(II) 7.x + 2.y = 8

(I) x = (12 + 2.y)/3
(II) 7.(12 + 2.y)/3 + 2.y = 8*.3

(II) 84 + 14.y + 6.y = 24*-84

(II) 20.y = -60*:20

(II) y = -3

a.s.o.

Step 2: Using CAS students can work ***„with the equations“***. Calculating, substituting and using algorithms to solve the single linear equations, which they learned in former White Box phases, is now done by the computer as a Black Box. Students have to decide on the operations, the CAS have to do them (figure2.42).

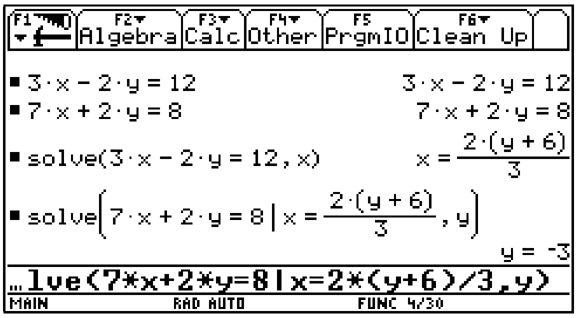


figure 2.42

Linear systems of equations in ninth grade

Step 3: In our project where we observe students who are familiar with the CAS we found out that the operations are not carried out with the equations but ***„with the names of the equations“*** which the students had defined as new elements of the mathematical language.

Using the idea of the Gauss algorithm to find out the Cramer rule.

After storing the equations as named variables students can use the names to form suitable expressions. The next step is to speak about strategies in their colloquial language and to translate it into the mathematical language. E.g.: *„We have to multiply the first equation with a_{22} and the second equation with $-a_{12}$ and have to add the two equations. After that we have to solve the new equation for the variable x_1 and so on.“*

At first for a better understanding of the several calculating steps it would be better to separate the particular steps:

Word formula	**Abstract expression**
Multiply the first equation with a_{22} and the second equation with $-a_{12}$ and add the two equations.	***a_{22}.equ1 + ($-a_{12}$).equ2***
Store the new equation with respect to the variable x_1 in the variable equ2n	***a_{22}.equ1 + ($-a_{12}$).equ2 Yequ2n***
solve the new equation for the variable x_1	***solve(equ2n, x_1)***

figure 2.43

Students having more experience with using CAS and especially the more talented students more and more prefer to translate the word formula of phase 1 into one abstract expression:

Multiply the first equation with a_{22} and the second equation with $-a_{12}$ and add the two equations. Now solve the new equation for the variable x_1	***solve(a_{22}.equ1 - a_{12}.equ2, x_1)***

figure 2.44

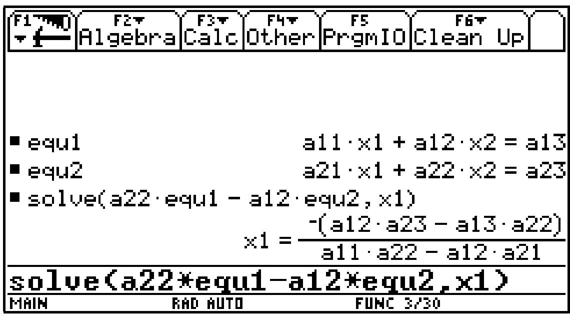


figure 2.5

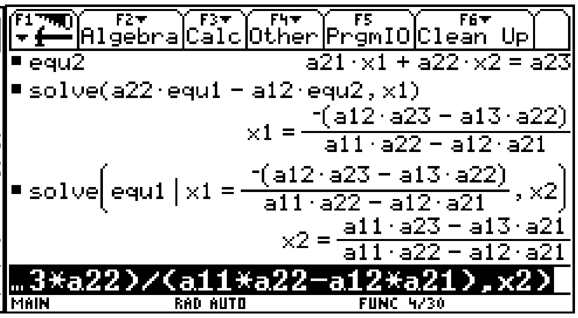


figure 2.46

This result shows a new quality of mathematical thinking caused by CAS (figure 2.44). The tool of CAS does not only support cognition, it becomes part of cognition.

2.7 The competence of using the chosen CAS

> *Besides the actual mathematic contents I now have to additionally concentrate on skills which are necessary when using the calculator.*

This statement of one of our students shows the necessity of the “computer-using-competence”.

The influence of CAS:

- The use of CAS causes additional demands and problems for the students. The operation of the electronic tool needs additional skills which also have to be practiced as calculation skills.
- The evaluation of our last project shows that the measured growing joy and interest in mathematics is significantly higher by those pupils who have no problems with the operation of the computer.
- Another significant result is the gap between boys and girls. Both groups show a growing joy and interest in mathematics but boys significantly more than girls. Girls more often observe to have problems with the operation of the calculator.
- The necessary commands, operations and modes have to be offered to the students in small portions. Practicing and repeating in regular intervals are necessary.
- The use of CAS as a Black Box for problem solving demands an agreed documentation of the way of solution, especially in the exam situation.

Some handling skills which are necessary for the fundamental algebraic competence when using a TI-92:

- Input Y recognizing the structure of the expression.
- Storing and recalling variable values.
- Most important commands in the algebra menu are *factor*, *expand* and *solve*.
- Substituting numbers, variables and expressions.
- Setting modes which are necessary for algebra, like the decision exact/approx.
- Defining functions for graphing, displaying Window Variables in the Window Editor. Using Zoom and Trace to explore the graph.
- Generating and exploring a Table, Setting Up the Table Parameters.

3. Some results and answers to the topic coming from the Austrian CAS-Projects.

3.1 Investigation of the impact of the TI92 on manual calculating skills and on the competence of reasoning and interpreting.

This investigation was part of our 2nd CAS project, the TI-92 project, and was carried out by Walter Klinger and Christian Hochfelsner [Klinger,1998].

Framework

Participating classes of the 7th grade with a total of 653 pupils:

	Project classes 1)	Control classes 2)
Gymnasium 3)	1	9
Realgymnasium 3)	8	8 (1 Derive class)

1) Every pupil is equipped with a TI-92 which is used in every learning situation, during the lessons, at home and in the exam situation.
2) Classes with traditional mathematics education. Every pupil has a numeric calculator like a TI-30. Exception: The Derive class, a CAS class where every pupil has Derive available.
3) Gymnasium and Realgymnasium are college preparatory high schools, beginning with 5th grade and lasting 8 years. The Gymnasium is more language-oriented with 3 math lessons a week, the Realgymnasium is more science-oriented with 4 math lessons a week

Description of the test

The core of the test comes from Prof. G. Malle from the Klagenfurt University [Malle, 1986], example 6 was substituted by 2 new examples . Especially new is the row for written reasoning.

Worksheet: Test your algebraic competence

Name: .... Class: ....

Calculate the given examples and describe verbally the way of solution. Use mathematical concepts and arguments.

	Calculate	**Describe your way of solution**
Example 1:	**6a.3b – (5ab + b.2a)=**	
Example 2:	**4.(3a+5)–(4a-7).3=**	
Example 3:	$x^2y^2(xy)^2=$	
Example 4:	$(a^2\text{-}3b).(\text{-}3a+5b^2)=$	
Example 5:	**2a – a/3**	
Example 6: Complete	$(.. - 7x)^2 = .. - 56xy + ..$	
Example 7: Correct	$(-2b^2 + 3b)^2 = 4b^4 + 12b^4 + 9b^2$	

figure 3.1

Procedure
This test should be given in the first lesson following the summer holidays.
No aids may be used during the test.
The test should take no longer than 25 minutes.

Most of the classes followed the proceedings, but nevertheless it is impossible to prove if all the pupils actually took the test in the first lesson following the holidays without any review sessions beforehand.

Method of evaluation:
The test was corrected by the math teachers according to the following criteria:

Calculating skills:	correct – false - answer not given
Interpretation competence:	excellent/good - poor/false - answer not given.

Due to the individual correction of the teachers there is a certain amount of uncertainty concerning the evaluation of the interpretation competence. The authors of the investigation, however, share the opinion that this has no major effect on the total results of the test as these irregularities are balanced out in the end.

Goals of the investigation:

- Investigation of the instrumental understanding as well as the relational understanding. Looking for the answers to the two questions which I formulated in chapter 2.4 (calculation competence):
  - Question 1: Is instrumental understanding a prerequisite or a support for a higher level of relational understanding?
  - Question 2: Supports relational understanding the necessary skills of instrumental understanding?
- Survey of the impact of the TI-92 used as a teaching and didactic tool on the manual calculating skills of students in the third form of the Austrian High School (13-14 year old pupils)
- Survey of the impact of the usage of an algebraic pocket calculator on the competence of reasoning and interpreting.
- Discovery of which areas and under which circumstances the usuage of the TI-92 as a didactical tool makes sense

Before the tests were carried out the following **hypothesis** were assumed:

Hypothesis 1:
The classes using TI-92 will show fewer skills in manual calculating and will have a much better competence in reasoning and interpreting.

Hypothesis 2:
Using the TI-92 as a didactical tool especially when applying formulas or finding mistakes will have positive effects.

3.1.1 Interpretation of the manual skills

The overall test results show a rather disillusioning picture of the basic calculating skills of pupils , in the third form 7^{th} grade. Fewer than half of the examples were solved correctly by the tested students whereby the difficulty of the first five problems is certainly not at a very high level, one could say these 5 examples are tests of the fundamental calculation competence which is also necessary in the age of CAS.
The sixth problem was a standard example taught in 7^{th} grade and only the seventh problem demanded higher algebraic skills . It is safe to assume that these skills might not have been trained in this manner in all the test classes. Only 15% of the tested students were able to solve this problem correctly.
It must be pointed out that the test was carried out in the first lesson of the academic year, the students just having returned from summer holidays and in most cases not having had the opportunity to review the skills before sitting the test.

The pupils of the Realgymnasium who were not using an algebraic calculator showed the highest percentage rate of correctly solved problems (50.8%), followed by the project classes (44.8%) and at last the pupils of the Gymnasium with 41.7%.. The pupils of the one DERIVE class were able to solve 50.2% of the examples correctly.

The difference between the results of the pupils in the control classes of the Realgymnasium classes and those of the project classes is indisputable, the former being able to solve 6% more of the examples correctly. Although it is not possible to determine how often the TI 92 was used as a didactic aid in the White Box phase it must be assumed that the calculating skills were left to be done by the TI 92 as a Black Box, especially in the Black Box phase.

On the other hand it is remarkable that the project classes have a better result than the classes of the Gymnasium

First Conclusion:

> **The first part of hypothesis 1 seems to be confirmed! Manual calculating skills are reduced when using an algebraic calculator !**

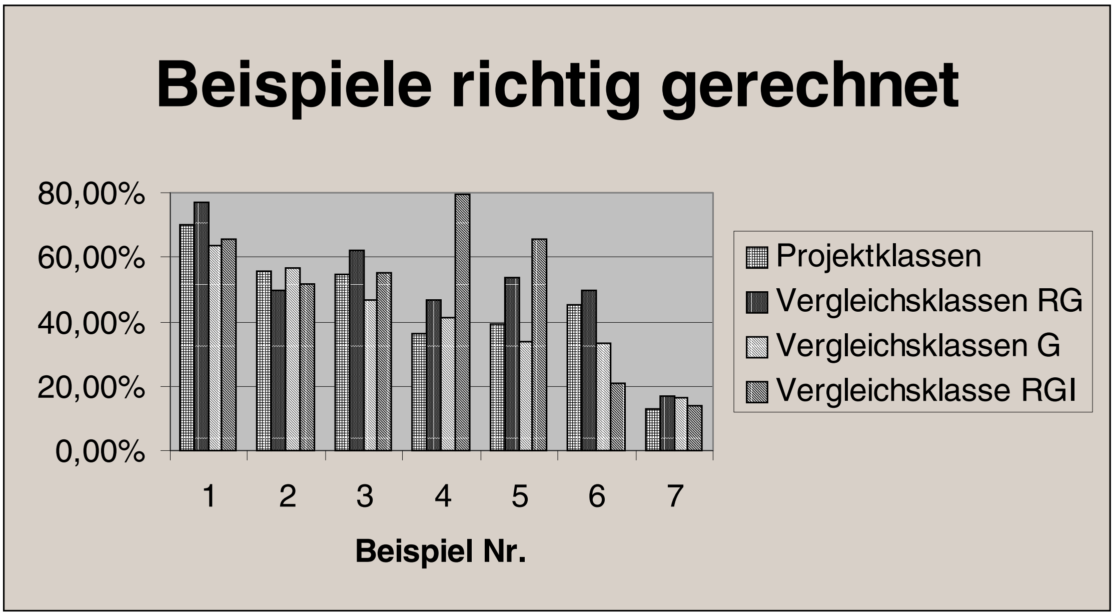
However, the following questions still remain to investigated and answered:

- Have the pupils of the project classes gained other basic skills through the usage of this didactic tool, such as techniques to compare terms, techniques of substituting a.s.o, in other words, skills which are necessary for the other fundamental algebraic competences.
- Can any advantages be seen in modelling and interpreting among the pupils of the project classes?
- Have the pupils gained a different understanding of term structures?

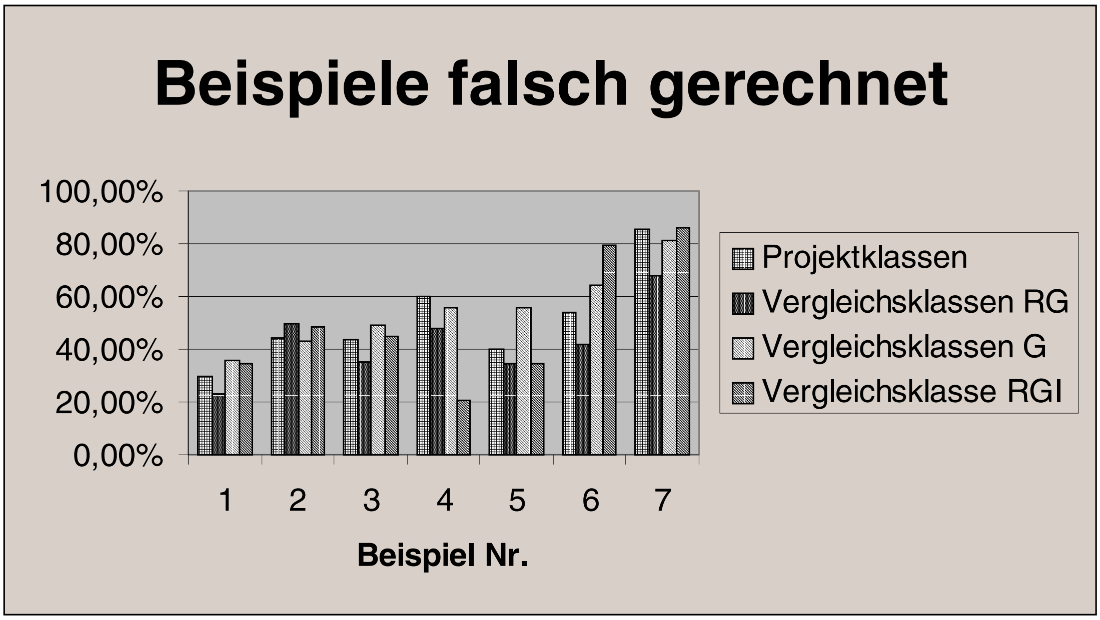
Please see the graphs for a more detailed interpretation of the results of each individual example.

Explanation:
Vergleichsklassen RG] Realgymnasium classes with traditional math education
Vergleichsklassen G] Gymnasium classes with traditional math education
Vergleichsklasse RGI] Derive class

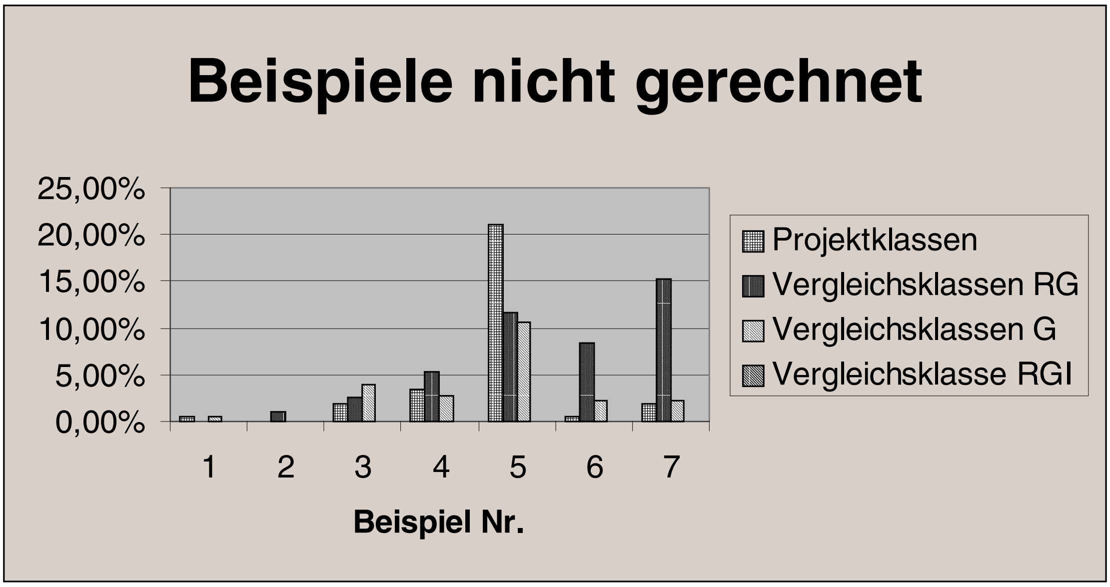
Correctly solved problems (Example 1 to 7)



Not correctly solved problems



Missing problems



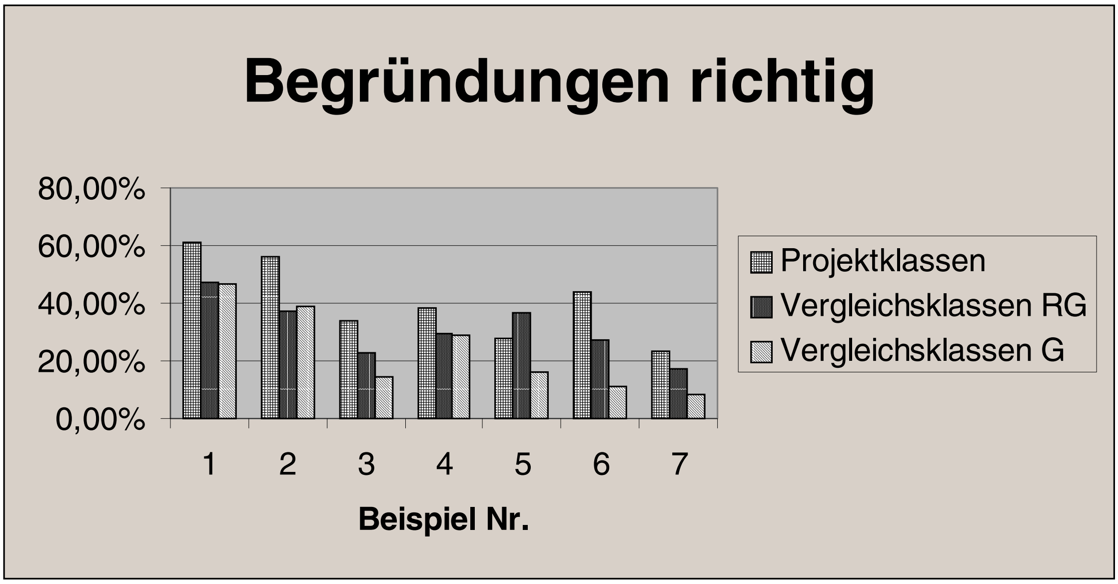
More than 20% of the project classes did not solve the fifth problem and problems 5, 6 and 7 were often left unsolved by the pupils of the Realgymnasium. Exactly why these problems were left out, remains unclear.

3.1.2 Interpretation of the competence of reasoning and interpreting

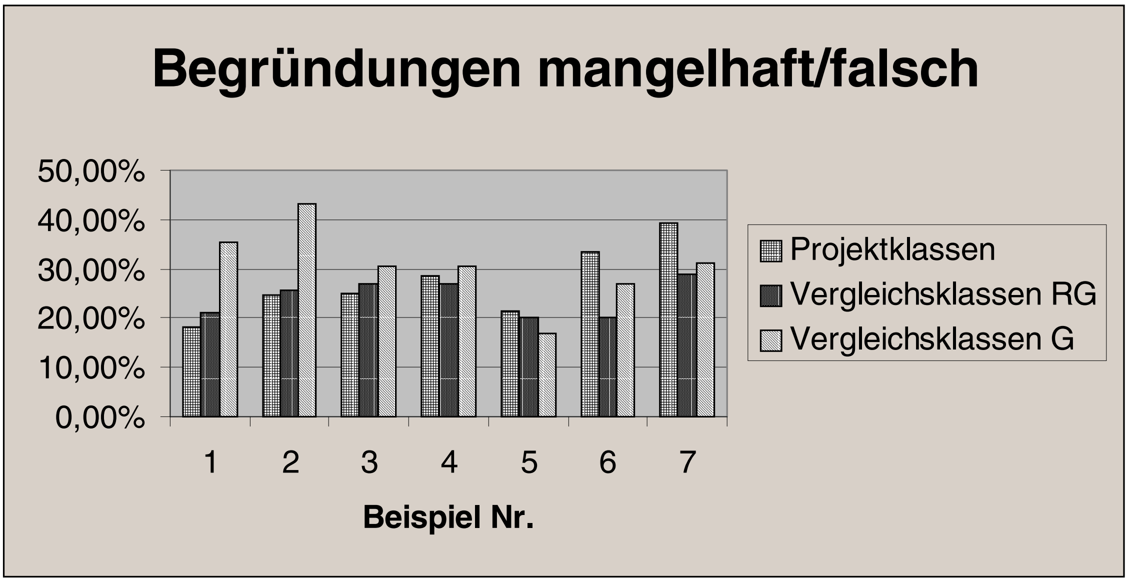
This competence requires a much deeper understanding of fundamental algebraic activities. Now it is not simply a matter of carrying through a certain calculation, but rather reflecting about the application of fundamental activities.
We did not expect the number of correct reasons to exceed the number of correctly calculated examples and it was seldom the case that the reason was correct, the math result incorrect. In some cases we found that in spite of well founded reasons calculation mistakes were made. Generally it can be said that correctly solved problems were often correctly interpreted whereas wrongly solved examples showed poor, incorrect or no verbal interpretations.

The project classes (8 out of 9 classes were evaluated) show the highest percentage of correct reasons and interpretations (40.7%), followed by the 7 RG classes and the 5 NG classes with 30.9% and 23.6% correct proofs respectively. The DERIVE class was not evaluated.

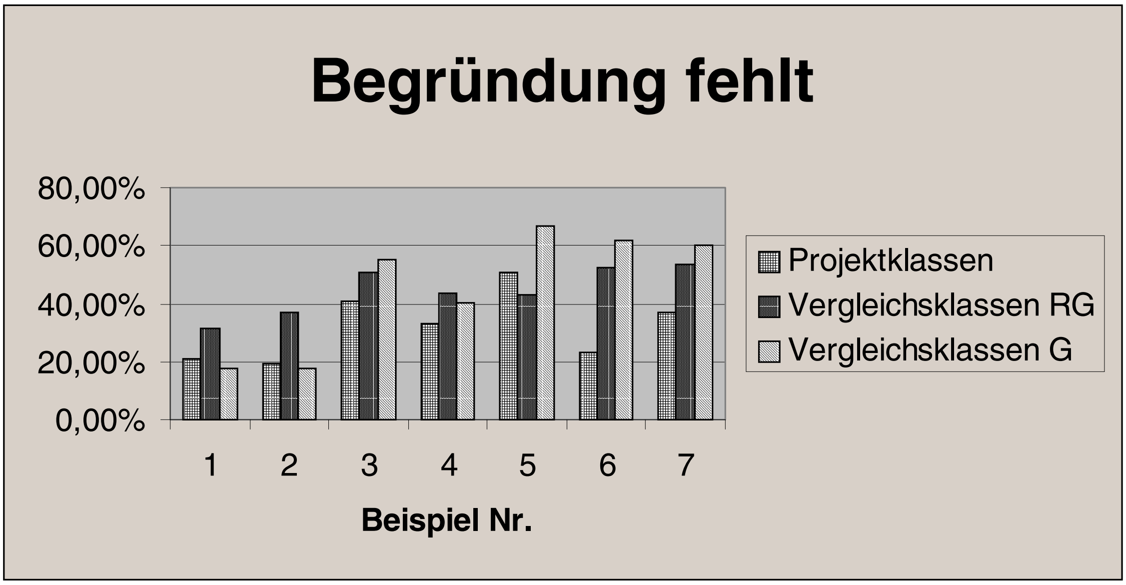
Correct reasons



Reasons not correct or wrong



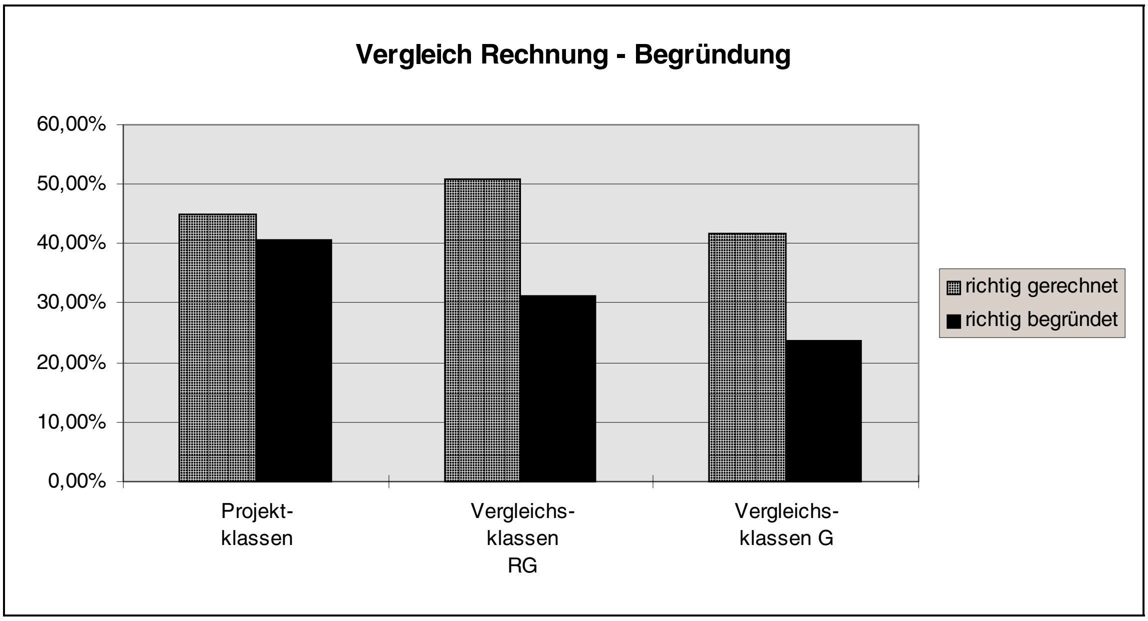
Missing reasons



3.1 3 Comparison of calculating skills to reasoning competence

This result is even more striking when we put the percentage of reasoning in relation to the number of correctly solved math problems. This relative value only shows the percentage of correctly solved and correctly interpreted examples but does not tell us anything about the relation between incorrect calculation and correct interpretations or vice versa.

This percentage is quite high in the project classes: Approx. 90% of the correctly solved examples were also correctly interpreted In the Realgymnasium classes the relative value lies at 60.8 % and in the Gymnasium classes at approx. 56.6%.



Explanations: First value (first rectangle)] correctly calculated
Second value (second rectangle)] correct reasons

These results open up various interpretations and questions:

- Was more value placed on discussing math in the TI-92 classes?
- Does working with the electronic medium encourage pupils´ willingness to discuss and argue through math problems?
- Is there a relation between better competence in reasoning and interpreting and deeper understanding of mathematical correlations?
- How does the competence in reasoning influence other areas such as argumentation, testing , modelling, interpreting and approach to open problems?

Second conclusion:

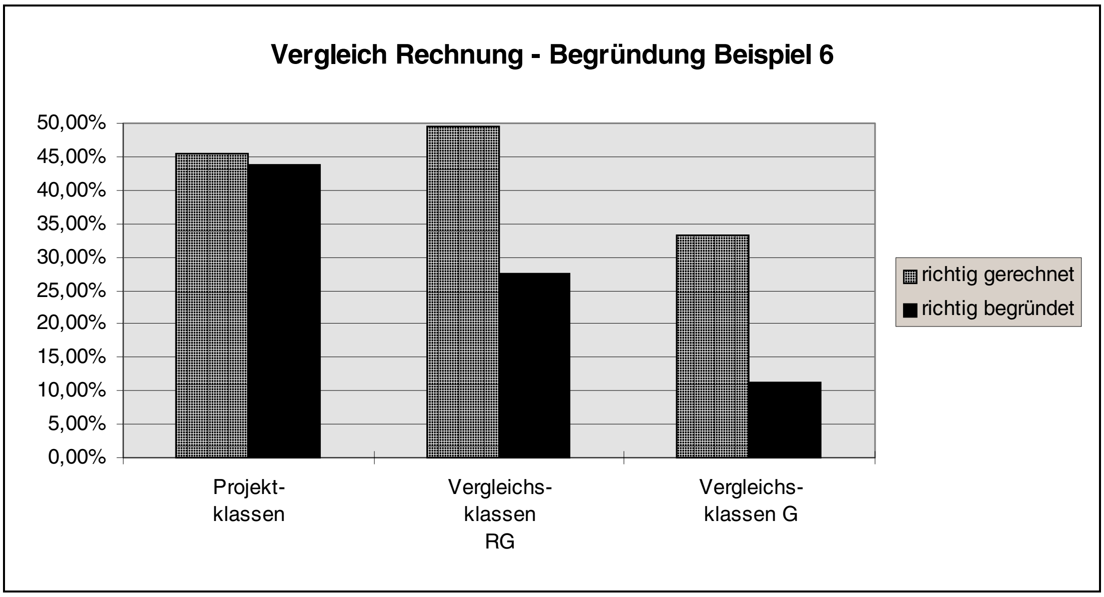
> **The second part of hypothesis 1 seems to be confirmed! The changes in teaching and learning mathematics caused by the use of a CAS support the possibility of reasoning and describing mathematical activities.**

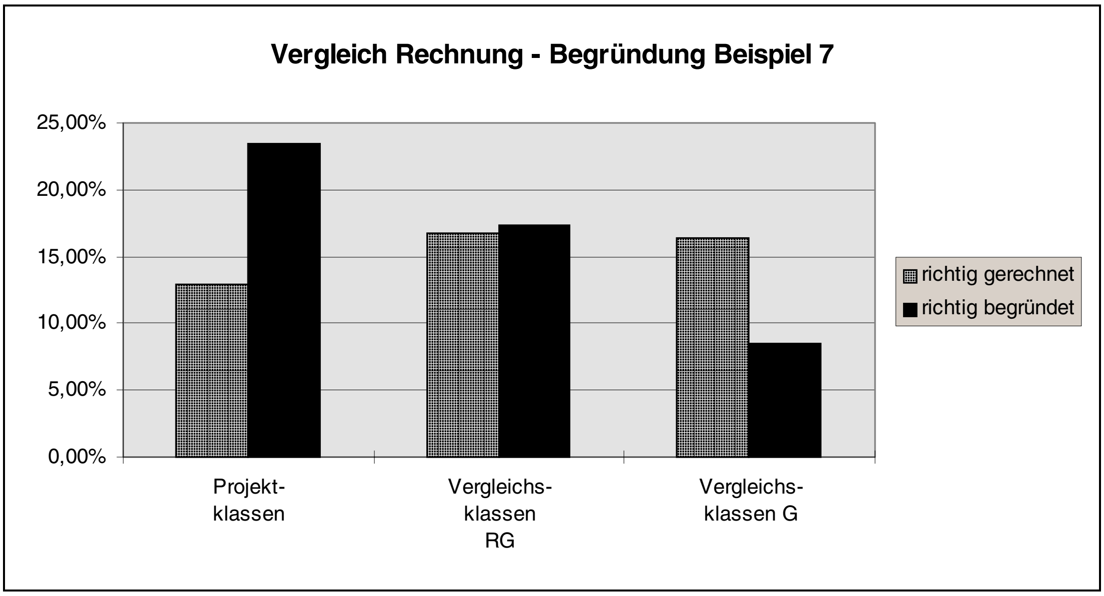
This conclusion, however, can only be drawn if the lesson time which is not used for the training of manual calculating skills is invested in working for these competences. It must be pointed out that in the lower grades a calculator should be used as a didactic tool rather than strictly as a calculating tool. The best CAS command remains worthless if the user can no longer decide which command he must choose in order to reach a specific mathematical goal.

Effects on the application of formulas and on the analysis of mistakes

To begin with, it should be pointed out that the project teachers tried to intensify the usage of the TI-92 in lessons of elementary algebra—in the areas of formulas, drills and mistake analysis. The materials of these „observation windows“ (goals, lesson plans and work sheets) can be found in the report and on the home page of ACDCA under the „observation windows of the 3rd form (7th grade)“.

Comparison of calculating and reasoning competence in examples 6 and 7





Explanations: First value (first rectangle)] correctly calculated
Second value (second rectangle)] correct reasons

As can be seen in both, the sixth and the seventh test examples**,**
the calculating skills of the project classes are the weakest of all the participating groups:
Project classes: percentage of correct answers to example 6 is 45.5% and to example 7 it is 12.8%. In the Realgymnasium 49.5% and 16.8% correct answers respectively and in the Gymnasium classes 33.3% and 16.4% correct solutions to the two named problems.

One possible explanation is that in TI-92-classes such examples are solved by systematical trial and error on the TI 92 and not “by hand”.

At the same time, the test results show that the project classes produce a relatively high percentage of correct reasoning:
Project classes: 43.7% and 23.5%; Realgymnasium classes 27.4% and 17.4% and the Gymnasium classes 11.3% and 8.5%.

In these two examples the entire percentage of proper reasons in the project classes lies even higher than the percentage of correctly solved problems.

A first answer to the question 1 of chapter 2.4:

> **Instrumental understanding is not an absolutely necessary prerequisite for a higher relational understanding. The ability to give reasons does not automatically go hand in hand with the ability to calculate**.

The seventh example shows a competence in reasoning which is almost twice that of calculating. On the one hand this is surprising, on the other hand we need to question how forcefully we should demand this reasoning competence of the pupils in the area of elementary algebra. A process which often can be explained in a matter of seconds is turned into a monsterous example if the interpretation is to be carried out in detail. Furthermore we must remember that not all items of knowledge, especially applied „tool thinking“ really need to be explained. Often it suffices simply to work by „feeling“ - a different realm of knowledge. A explanation is much more sensible when modelling, argumenting, interpreting or approaching open problems.

Third conclusion – answer to Hypothesis 2 and Questio 2 of chapter 2.4:

The second hypothesis does not seem to be confirmed! Even systematic, didactically planned usage of the TI –92 is no guarantee for the improvement of competences, like completing of formulas or analysing mistakes, if the calculator is not placed at the pupil´s disposal. An available reasoning competence does not automatically mean that the proper process will be applied.

Relational understanding does not necessary support the skills of instrumental understanding. Or in other words: To have relational understanding is not enough for having instrumental skills.

3.1.4 Correction work using the TI-92

Hypothesis 3:

Using the CAS as a calculating and testing tool the calculation results will be much better and the competence of reasoning is supported

When grading the test papers in the project classes the answers were simply marked as either right or wrong, regardless of what mistake and how bad a mistake was. In the following math lesson the papers were handed back and the pupils were asked to re-do those problems which were marked as wrong (using a different color pen). The pupils were told to use their TI-92 and to find and mark the mistakes that they had made.

The result:

Many of the pupils were able to find and correct their mistakes with the use of the TI- 92 and a lot of them were also able to understand their mistakes.

Result of the Question: How many wrong examples could be corrected by a repeated calculating when the TI-92 was also available (without a practicing phase between the test and the correction phase)?

Example	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Corrected mistakes	100%	100%	81.8%	70.6%	96%	83%	50%

figure 3.2

Result of the Question: How many wrong examples could be corrected by a repeated calculating when the TI-92 was also available and the sort of mistake was recognized (without a practicing phase between the test and the correction phase)?

Example	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Corrected And recognized mistakes	100%	62.5%	45.5%	41%	52%	42%	22.7%

figure 3.3

Fourth conclusion:

The third hypothesis seems to be confirmed

3.2 Changes in the exam situation

Procedure

Model 1:

The TI-92 can always be used. Calculations carried out by the TI-92 have to be documented. Some agreements about the style of documentation were taken. The teacher makes clear which examples have to be calculated by hand and which can be calculated by the CAS. The pupils can always use the TI-92 as a testing tool. For some examples the use of the calculator is demanded.

Model 2:

The written exam consists of two parts:
Part 1: No calculator is allowed.
Part 2: The CAS is permitted, documentations are needed.

Changes of the style and the contents of the examples

There are **fewer changes of the contents** noticed **but changes of the goals and the style of the questions**. Some observed changes:

(1) "check – compare – test"
Example: Several formulas of a trapezoid were given (expanded, factorized and also wrong examples). Pupils had to find out which of them are equivalent and which are wrong.

(2) Open questions
Example: Given are 2 pairs of numbers (1,40) and (40,1):
a) Find a suitable formula for an indirect proportion
b) Find a formula, suitable for these pairs which is not an indirect proportion.

(3) Reasoning and documenting
Questions like "Explain your assertion" or ""Describe the sequence of the commands which are necessary for a certain algorithm" can be found.

(4) Structure recognition:
Example: Which keys were activated?

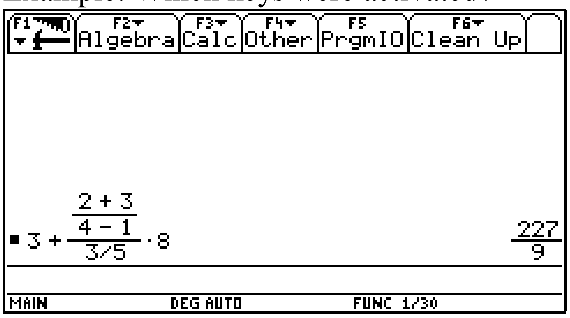


figure 3.4

(5) More often than in traditional math education text problems can be found

(6) The length of the examples was statistically growing.

(7) A trend to experimental working
can be noticed, on the one hand because the question demands it and , on the other hand, the time which is gained because the calculating is done by the CAS can be used for testing and experimenting.

(8) Changes in the description of the examples
It is necessary to set when the CAS can be used and when not. Sometimes it is necessary to demand the prototype which has to be used, like "use a table" a.s.o.

(9) The CAS as an expert for testing
The students always have their own expert available. The testing competence becomes very important, practice in testing is necessary.

Final Conclusion

Hypothesis and Questions	Answers and Conclusions
Question 1: **Is instrumental understanding a prerequisite or a support for a higher level of relational understanding?**	**Answer to question 1:** **Instrumental understanding is not an absolutely necessary prerequisite for a higher relational understanding. The ability to give reasons does not automatically go hand in hand with the ability to calculate.**
Question 2: **Does relational understanding support the necessary skills of instrumental understanding?**	**Answer to question 2:** **Relational understanding does not necessary support the skills of instrumental understanding. Or in other words: To have relational understanding is not enough for having instrumental skills.**
Hypothesis 1: **The classes using TI-92 will show fewer skills in manual calculating and will have a much better competence in reasoning and interpreting.**	**First conclusion:** **The first part of hypothesis 1 seems to be confirmed! Manual calculating skills are reduced when using an algebraic calculator** **Second conclusion:** **The second part of hypothesis 1 seems to be confirmed! The changes in teaching and learning mathematics caused by the use of a CAS support the possibility of reasoning and describing mathematical activities.**
Hypothesis 2: **Using the TI-92 as a didactical tool especially when applying formulas or finding mistakes will have positive effects.**	**Third conclusion:** **The second hypothesis does not seem to be confirmed! Even systematic, didactically planned usage of the TI–92 is no guarantee for the improvement of competences, like completing of formulas or analysing mistakes, if the calculator is not placed at the pupil´s disposal. An available reasoning competence does not automatically mean that the proper process will be applied.**
Hypothesis 3: **Using the CAS as a calculating and testing tool the calculation results will be much better and the competence of reasoning is supported**	**Fourth conclusion:** **The third hypothesis seems to be confirmed**

Based on the results of our CAS Projects

- I tried to find out a list of needed fundamental algebraic competences. Instead of a preponderance of the manual calculation competence in the traditional math education we observe a more equally importance of the listed competences.
- I tried to give answers to the questions about instrumental and relational understanding and to the hypothesis of our investigation.
- But I have no exact answer to the question about certain necessary manual calculation skills in the age of CAS.

Samour Papert had a dream: Just as you learn a foreign language best in a country where it is spoken you would learn the language of mathematics best in a “mathematic land” and he was sure that his Logo-learning environment would be such a “Mathematic Land”. Years later we can say Logo is not the “holy land”

We did not find such a “holy land” when working with CAS, not for the pupils and not for the teachers, but an interesting learning environment for a

- more meaningful,
- more interesting,
- and more future oriented

Mathematics Education

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