Karl-Heinz Keunecke

What caused the Collapse of the Tacoma-Narrows-Bridge?
or
Solving Motion Problems numerically applying
Newton's Fundamental Law of Motion

Fig. 1: The carriage way of the bridge is excited to torsional vibration[1]

1. Introduction

It is one task of physics teaching to impart the importance of Newton's second axiom for calculating movement processes. This axiom, also called **Fundamental Law of Motion** says:

> If a force is acting on an object then this object is accelerated in direction of the force.
> The acceleration is proportional to the force F and inversely proportional to the mass m of the object.

This relation is often described by $F = m \cdot a$, where a is the acceleration. However, from this expression one cannot recognize the importance of this law. For students $F = m \cdot a$ is one of many other formulae which they know from physics teaching. Not before acceleration is replaced by the second derivative of the position with respect to time it becomes clear that this equation is a differential equation,

$$F(x(t)) = m \cdot \ddot{x}(t), \tag{1}$$

which allows to predict the motion of a mechanical system, if the law of force and the initial values of the motion are known.

In class the general form (1) of the fundamental law can be used only in a few special cases, because when treating mechanics in secondary II the necessary knowledge of calculus to work with differential equations is generally missed.

In order to demonstrate nevertheless the universal validity of equation (1), solutions can be calculated numerically and then graphically represented. This needs only basic knowledge of difference quotients, limits and derivatives.

[1] https://sites.lsa.umich.edu/ksmoore/research/tacoma-narrows-bridge/

The following will describe a teaching course "Numerically solving differential equations". The calculations were performed using a TI-Nspire CX CAS, which was used in class, too.

It is wanted to provoke students' interest for the numerical solution of differential equations. So, a movie free available in the internet as motivation: "*The Collapse of the Tacoma Narrows Bridge*" is chosen.

https://www.youtube.com/watch?v=j-zczJXSxnw

https://www.youtube.com/watch?v=mXTSnZgrfxM

Then the question: "*What's the reason for this catastrophe*?" can be posed.

The movie shows that the bridge vibrates heavily, and this leads finally to its collapse. Obviously, a strong wind was responsible for this disaster.

First of all, the students should investigate en forced oscillations of a spring-mass-system. By this simple model they collected experimental experiences, how and under which conditions large oscillations can result from small exciting forces. Then they could set up the respective equations of motion and finally solve them numerically.

At the end of this project the students were able – based on their results – to interpret the collapse of the Tacoma-Bridge as a resonance catastrophe.

2. Methods for numerically solving of Differential Equations

This paragraph describes how to solve differential equations of order 2 numerically.

Most of the dynamic processes like motion, growth and decay, any change of state etc. can be described as a mathematical model using a differential equation. As it is not possible in all cases to find an analytical solution, one has to apply a numerical approximation method. In doing so, motion is only calculated stepwise in defined time intervals. The result can be given in a table and/or as a graph.

An undamped oscillation of a spring is chosen as introductory example. The students should know this from earlier lessons. They expect as result a sinusoidal oscillation with constant amplitude. So, they can judge the quality of the numerical approximation.

Starting with initial values t_0 and x_0 the position coordinate x_1 after time step Δt shall be determined.

Velocity v can be approximated by $v \sim \frac{\Delta x}{\Delta t} v$. The first values are $\Delta x = x_1 - x_0$ and $v = v_0$ which leads immediately to

$$x_1 = x_0 + v_0 \cdot \Delta t \qquad (2)$$

In order to calculate the next coordinate $x_2 = x_1 + v_1 \cdot \Delta t$ from the previous one iteratively, it needs the value of v_1. This can be achieved applying the law of force for the undamped oscillation.

There is a linear relationship between the force acting on a body of mass m and the excursion x of the oscillator during the oscillation

$$F(x) = -k \cdot x, \qquad (3)$$

where k is the spring hardness.

Because of Newton's 2nd axiom $F(x)$ can be replaced by $m \cdot a$. Introducing the approximation $a \sim \frac{\Delta v}{\Delta t}$

gives $\Delta v = -\frac{k}{m} \cdot x \cdot \Delta t$.

Using the initial values and equation (3) and replacing Δv by $v_1 - v_0$ gives for v_1:

$$v_1 = v_0 - \frac{k}{m} \cdot x_0 \cdot \Delta t. \tag{4}$$

For calculating x_2 and v_2 it only needs to increase the counting variables in equations (2) and (3) by one:

$$x_2 = x_1 + v_1 \cdot \Delta t \text{ and } v_2 = v_1 - \frac{k}{m} \cdot x_1 \cdot \Delta t. \tag{5}$$

Further calculations can be done in the appropriate way.

This iteration is called *Euler Method*.

2.1 Using a Spreadsheet

The method described in the paragraph above shall be performed using the spreadsheet application of the TI-Nspire CX CAS. However, the parameters must be defined. Thus, $\Delta t = 0.1$, $x_0 = 1$, $v_0 = 0$ and $\frac{k}{m} = 1$ are set. After inserting the *Lists & Spreadsheet* application three columns as time, position and veloc are defined. Then the initial values are entered in the first row. Cell *a2* becomes cell content of *a1* increased by 0.1.
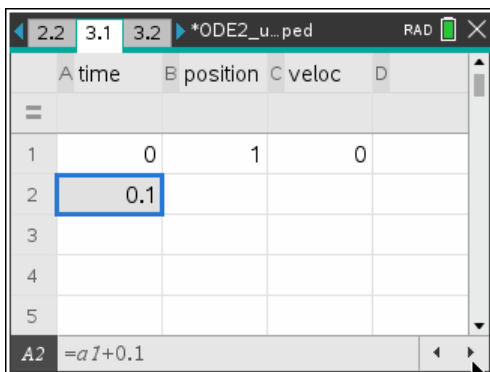

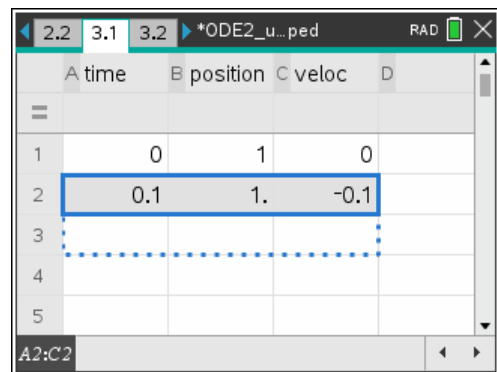*Fig.2: Entering the initial values; increasing time by $\Delta t = 0.1$*

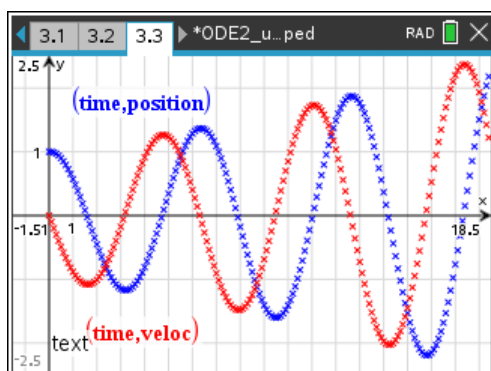*Fig.3: Iterative calculation of position and velocity*

Fig. 4: Approximation of displacement and velocity of the oscillations of a spring using the Euler method

This iteration is performed for 200 rows resulting in the coordinates for position and velocity for the first 20 seconds. An upper bound is only limited by the hardware resources.

Fig. 4 shows the numerical approximation of the position of the oscillating mass (blue) and its velocity (red) dependent on time.

As expected, sinusoidal shaped curves appear. But the amplitude increases by time. This is not realistic – due to the method applied for the approximation. Fig. 5 will explain this.
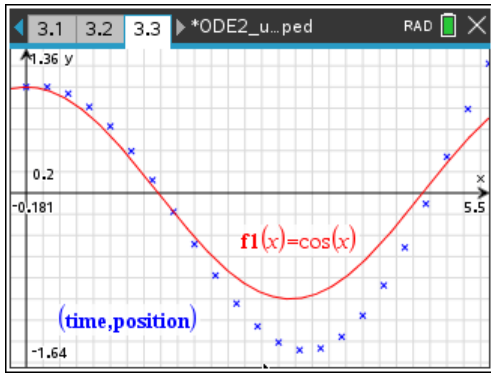
3

Fig. 5: Comparison of the exact solution and the approximated one

In Fig. 5 the exact solution of the differential equation (red) of an undamped oscillation $f1(x) = \cos(x)$ is plotted. The coordinates obtained by the Euler-method are marked by blue crosses. Compared with fig. 4 Δt was changed to 0.25.

Calculation starts with initial values x_0 and v_0 which give the ordinate of the next point $x_1 = x_0 + v_0 \cdot \Delta t$. Since the change of the position in the first interval is $v_0 \cdot \Delta t = 0$, x_1 does not change: $x_1 = x_0$.

In the following the values at the right borders (x_{i+1}, v_{i+1}) of the respective intervals are calculated from the values at the left borders (x_i, v_i). This leads to discrepancies compared with the exact solution. At the beginning are the calculated ordinates greater than those of the exact solution. And this leads consequently to higher velocities, too. The next consequence is that the displacements become larger by the time, which can be observed in fig. 5.

It is obvious that the approximation goes worse when a larger time interval Δt is chosen. A significant reduction of the time step interval leads at one hand to an improved approximation but at the other hand to a larger amount of data. It is also possible to calculate the slope for calculation of the next point not at the begin of the respective interval but in its mid. This yields an enhanced approximation. There are numerous other methods which lead to sufficient approximations. average velocity in the intervals. A description of more methods for numerical solutions of differential equations would go too far in our teaching unit intended for secondary II students.

Paragraph 2.1 introduces thoroughly to the simple Euler-iterations by means of spreadsheets. Here the students have the chance to learn the use the iterative calculation of displacement and velocity by means of spreadsheets from scratch.

2.2 Application of the menu-controlled TI-Nspire CAS-tool

TI-Nspire CX CAS enables to perform the calculations presented in section 2.1 by means of a menu-driven application That application needs simply the differential equations of the displacement and the velocity, their initial conditions and the parameters concerning the iteration. Using such a "Black Box" is acceptable, because the underlying concept has been worked out with the students in the section before).

Here again undamped oscillations shall be investigated first, in order to compare with the result shown in 2.1.

Firstly, the 1st order differential equation for the position yields:

$$\dot{x} = v. \tag{6}$$

Secondly, another 1st order differential equation for velocity v can be derived from the law of force $m \cdot \ddot{x} = -k \cdot x$ considering that $\ddot{x} = \dot{v}$:

$$\dot{v} = -\frac{k}{m} \cdot x. \tag{7}$$

4

Equations (6) and (7) are the base for the approximation of the difference quotients, which are used like in paragraph 2.1 for calculating tables iteratively. The differential equations can be entered directly. However, the names for the occurring variables are predefined. They differ from the notation known from physics but they must be used. So, renaming is necessary. Position variable x is replaced by $y1$ and velocity v by $y2$. Additionally, time t is named x and the derivative with respect to time t must be written as $y1´$ and not as $\dot{y}1$. Parameter values are chosen as above: $\Delta t = 0.1$, $x_0 = 1$, $v_0 = 0$ and $\dfrac{k}{m} = 1$.
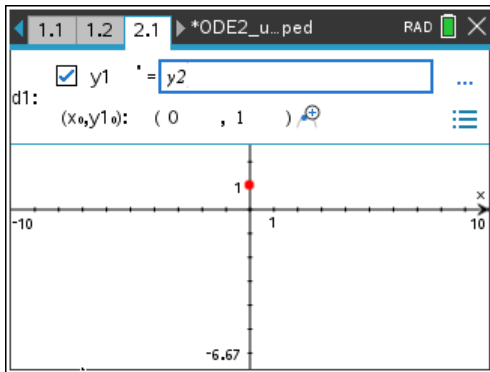
Fig. 6: Entering differential equation d1

For entering the differential equation the *Graph Entry/Edit* Menu in a *Graphs*-Application is chosen and the option *Diff Eq.* is selected. Now equation (5) and the initial conditions for $y1$ can be entered using the syntax of the calculator (see fig. 6).
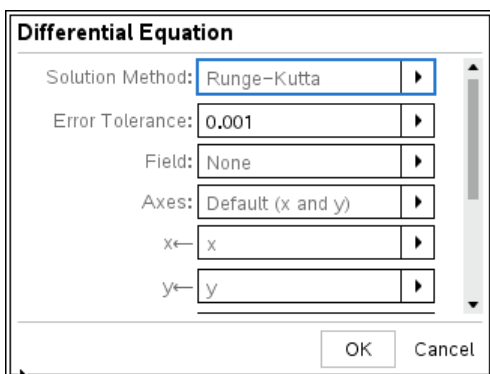

Fig. 7: Setting the parameters for the approximation

Afterwards the 2nd differential equation (6) and its initial conditions are entered (see fig. 7). The parameters for the numerical procedure must be entered as shown in fig. 7. As seen earlier that the Newton method is too inaccurate, the much more accurate Runge-Kutta method is taken. For displaying the displacement as a function of time, as axes are chosen x (time) and y (displacement). Further down in this menu the temporal interval for the numerical approximation must be defined.
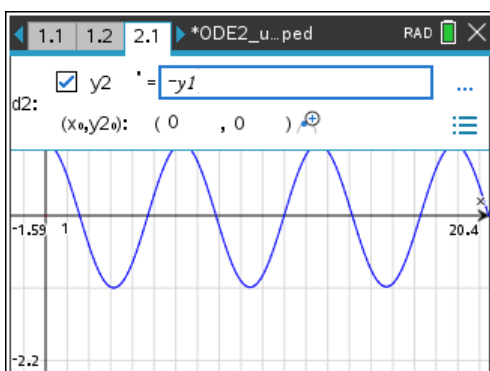

Fig. 8: Entering the DE d2 and the representation of the displacement

Fig. 8 presents the result of this approximation. In contrary to the result of the Euler method, the amplitude of this graph – again sinusoidal – seems to be constant.
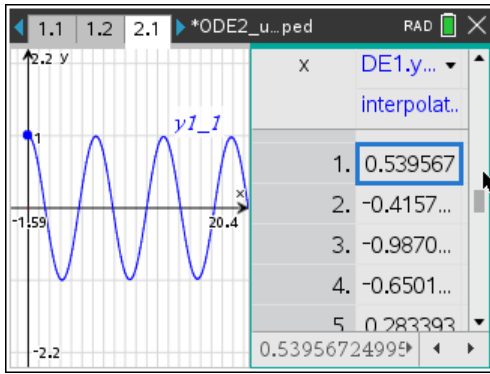
Fig. 9: Table for the positions and its graph

Fig. 9 demonstrates to students, that the menu-controlled procedure generates as well at first a list of values of time, displacement and velocity by iterations. These data are displayed then. Keypress Ctrl + T displays both, graph and list. (see fig. 9).

The resulting graphs in fig. 8 and fig.10 can be modeled by trigonometric functions. When for instance cos *x* is drawn in the same graph, *both graphs coincide exactly*
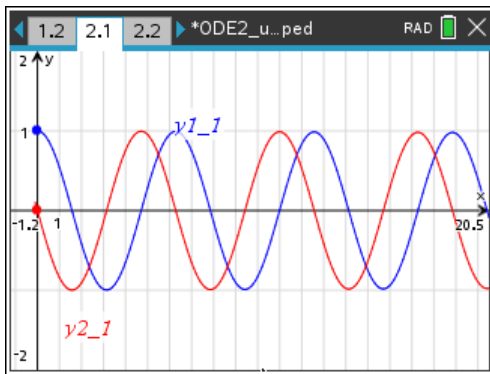

Fig. 10: Displacement and velocity

Both the displacement *y1* and the velocity *y2* have been calculated as it is seen in fig. 10. Due to the initial conditions and the assumption $\frac{k}{m} = 1$, what means $\omega = 1 \frac{rad}{s}$, two phase-shifted sine waves with equal amplitudes appear.

Phase relation of both graphs can be recognized in a phase diagram (fig. 11), where *y2* is drawn as function of *y1*. The phase diagram gives a circle only when the amplitudes of both waves are equal and the phase shift between both is $\frac{\pi}{2}$.
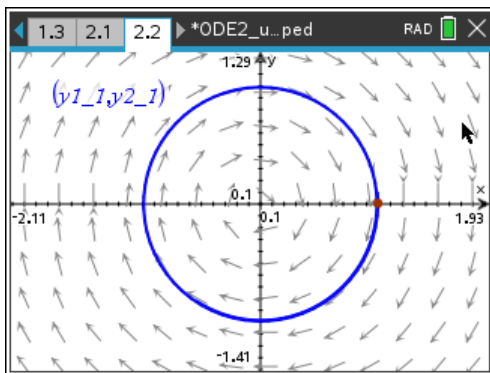

Fig. 11: Phase diagram y2 vs y1

Students can reproduce such diagrams, when they draw two sine functions against each other and vary the phase shift between 0 and 2π.

This diagram shows the direction vectors of the motion in several points. We can recognize that the diagram is passed through clockwise starting at point (0|1).

3. Examples from Physics Lessons

Paragraphs 2.1 and 2.2 served as introduction into computer assisted numerical solution of differential equations. This procedure is an important tool in physics. As it has demonstrated before, this tool can also easily used in physics lessons. The procedures enable to answers questions, which cannot be treated without computer assistance. Among others there exist physical problems without analytic solutions in closed form or problems which cannot be solved with the mathematical knowledge of secondary II level.

Students should proceed as follows:

- At first, they develop the law of force of a motion and deduce then the respective 2nd order differential equation.
- Thereafter they replace this differential equation by two 1st order differential equations for position and velocity which enable a numerical solution.
- They enter the differential equations into the CAS-calculator and obtain the graphic representation of position and velocity as function of time.
- They have also access to the calculated values given in a table.

Two examples shall be demonstrated in the next section.
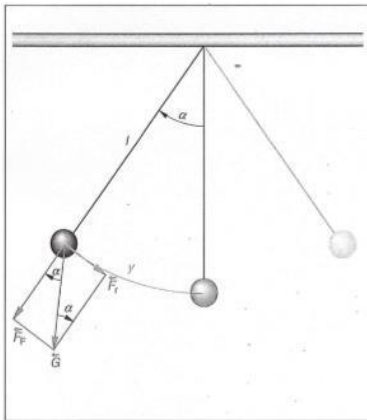
3.1 Pendulum Swinging

Fig. 12: Forces acting on a rod pendulum

Gravity pendulums generally don't execute undamped (harmonic) oscillations. Only for small swings the motion can be considered approximatively as an undamped oscillation.

In order to work out the acting forces, an idealized model is assumed, the so-called mathematical pendulum. It consists of a point-shaped mass fixed on a massless rod moving without friction on a pivot (fig. 12).

For a rotation around the pivot point after being displayed by the angle $\alpha(t)$ repulsive forces $\vec{F}_t$ are acting which are directed tangent to the circular arc. This force can be calculated according fig. 12 and because the gravity force $\vec{G}$ can be substituted by $m \cdot g$, it results in

$$F_t(\alpha(t)) = -m \cdot g \cdot \sin(\alpha(t)), \qquad (8)$$

where m is the mass of the body and g the acceleration of gravity. In the following α is always $\alpha(t)$.

From equation (8) Thus, it can be seen that there is no proportional relation between repulsive force and the displaced angle. Hence, the swing of the pendulum is not harmonic oscillation.

The respective differential equation can be derived from the above law of force. Due to Newton's 2nd axiom the left side of (8) is replaced by $m \cdot a_t$. The acceleration a_t of a body moving along a circle with the radius l is $l \cdot \ddot{\alpha}$. Division (8) by $l \cdot m$ results in

$$\ddot{\alpha} = -\frac{g}{l}\sin(\alpha). \qquad (9)$$

By introduction of an additional variable – angle velocity $\omega = \dot{\alpha}$ – the system of two 1st order differential equations can be deduced from the 2nd order differential equation:

$$\dot{\alpha} = \omega \text{ and } \dot{\omega} = -\frac{g}{l}\sin(\alpha). \tag{10}$$

Using the syntax of the calculator this must be entered as:

$$y' = y2 \text{ and } y2' = -\frac{g}{l}\sin(y1). \tag{11}$$

For convenience $\frac{g}{l} = 1 \cdot \frac{rad^2}{s^2}$ is chosen and the initial velocity is set to $0 \cdot \frac{m}{s}$.

In order to provide variable initial conditions for y1, a slider is used to vary the values of y1(0) from the interval $[0; \pi]$.

System (11) is entered in the calculator. Both displacement (black) and angle velocity (red) as well are displayed dependent on time.

The following figures 13 and 14 give a first impression of the run of displacement and velocity dependent on the respective initial value of the displacement. The scales on the axes remain unchanged in order that changes in the magnitude of displacement, velocity and frequency become clear.
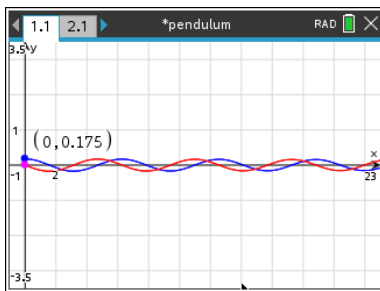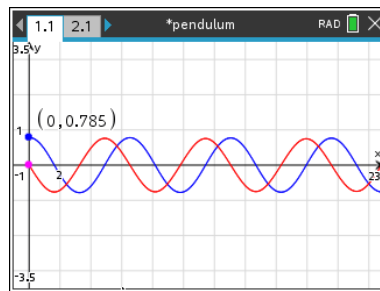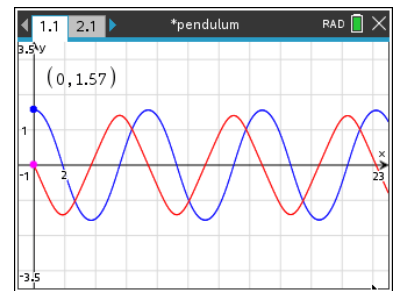

Fig. 13:   10° ~ 0.175 rad   45° ~ 0.785 rad   90° ~ 1.57 rad
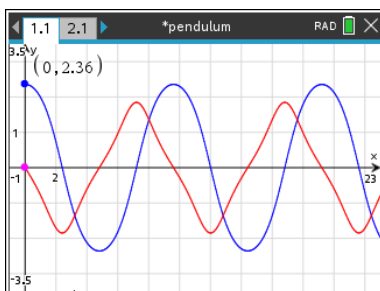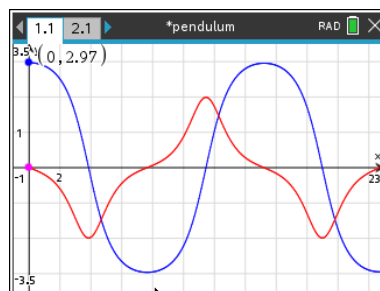
Fig. 14:   135° ~ 2.36 rad   170° ~ 2.79 rad   179.90° ~ 3.14 rad
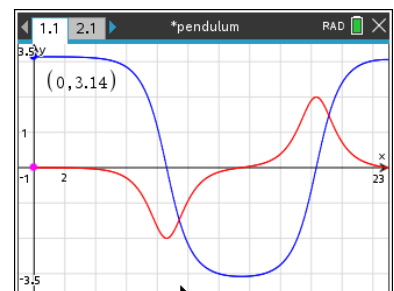
As long as displacement at the beginning is small (see 1st picture in fig. 13), the graphs of *y1* and *y2* are sinusoidal. In this range the motion shows all indications of a harmonic oscillation. If the displacement increases, the deviation from harmonic oscillation becomes more and more significant. Moreover, the frequency of the oscillation changes with the magnitude of the initial displacement of the pendulum.

The acting forces, causing the pendulum motion are described by equation (8). In contrary to the linear motion law (3) the forces are restricted by the factor sin(α). This enforces a slowdown of the motion and an enlargement of the oscillation time. Because in particular the repulsive forces at angles above 90° decrease – instead of increasing like in undamped oscillations – there are significant deviations from sinusoidal graphs.
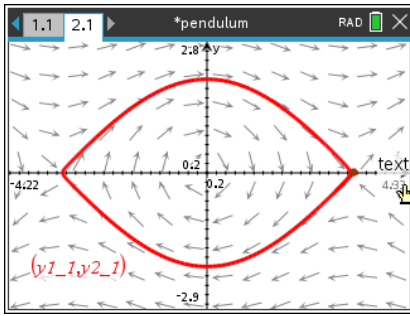
Fig. 15: Phase diagram;
initial value 179.9° ~ 3.14 rad

Instead of investigating the temporal behavior of $y1$ and $y2$ it is also possible to look at the phase diagram.

Fig. 15 shows the graph of $y2$ dependent on $y1$ for an initial displacement of 179.9° (fig. 14, last screen). In the presentation motion starts at the red point (maximum displacement). The direction arrows indicate that the curve is run through clockwise. Comparison with fig. 11 shows clearly the difference to a harmonic (undamped) oscillation).

3.2 Damped and Enforced Oscillations

The following investigation shall help to understand the process of the Tacoma-Bridge collapse. Obviously, the bridge executes vibrations under influence of the wind. It is proposed to investigate enforced oscillations on a simple model at first. It makes sense to watch and study motions in reality before mathematical modelling the procedures.
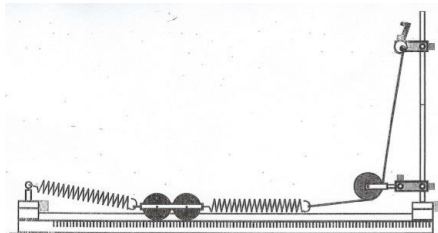

Fig. 16: Experiment to investigate
damped and enforced oscillations

As a simple experiment a cart on a track[2] has been ch⟨ which is fixed between two springs. When periodical⟨ caused by rotation of the eccentric wheel are acting o⟨ cart then it performs enforced oscillations. After own ⟨ periments the students will find out:

1. After response time the cart moves in the rhythm of the acting force.

2. For a relative small frequency range of the excitation the displacement of the cart exceeds significantly the displacement of the excitation.

Damped Oscillations

For simulating the experiment knowledge of the forces acting on the cart is necessary. At first, the repulsive force of the springs – given in equation (2) – $f(x) = -k \cdot x$ acts on the cart.

It is known that the amplitude of real oscillations decreases over time. So, besides the repulsive force another force, the friction, acts in the same direction, which is in general proportional to the velocity v of the oscillating body:

$$F_f = -r \cdot v = -r \cdot \dot{x}$$

where r is a measure for the strength of damping.

Thus, the law for the damped oscillation of the cart results in:

$$F(x(t)) = -k \cdot x > (t) - r \cdot \dot{x}(t) = m \cdot \ddot{x}(t). \tag{12}$$

In analogy to the procedure presented in paragraph 2.2 one obtains the following system of 1st order differential equations:

$$\dot{x}(t) = v(t) \text{ and } \dot{v}(t) = -\frac{k}{m} \cdot x(t) - \frac{r}{m} \cdot \dot{x}(t),$$

[2] 2013, MEKRUPHY, Mechanik 6, Erzwungene Schwingungen, p 24

9

which is written in the syntax of the calculator as follows:

$$y1' = y2 \text{ and } y2' = -\frac{k}{m} \cdot y1 - \frac{r}{m} \cdot y2. \tag{13}$$

For calculating a special solution values of the constants m, k and r are needed. The students know from physics lessons that $\frac{k}{m} = \varpi_0^2$ is the resonance frequency. Oscillation time of the eigenfrequency (fig. 16) can be measured and the angular velocity ω_0 can be determined from it. Alternatively, ω_0 can be determined from the spring constant k and the mass m of the cart.

$$\frac{k}{m} = 6.7 \frac{rad^2}{s^2}.$$

Parameter $\frac{r}{m}$, which describes the damping cannot be taken from the experimental set-up in such an easy way. Therefore, a slider b is introduced for $\frac{r}{m}$. It was set such that the time flow of the numerically calculated oscillation matches with the time flow of the displacement of the cart (fig. 16) in the best possible way. Comparison lead to $b = 0.25$. It will be defined later on.
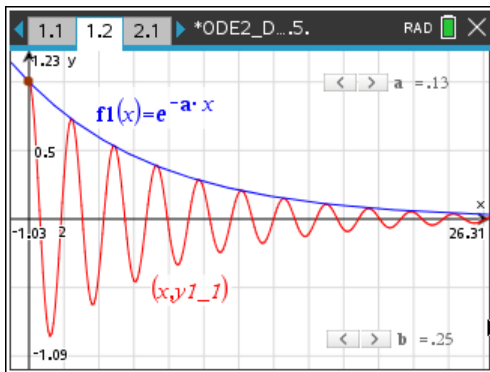

Fig. 17: Damped oscillation and exponential function

Taking the given parameters both differential equations are entered in respective menu in the *Graphs*-application. The result is displayed in fig. 17.

The calculated damped oscillation is described by a cosine-function with obviously exponential decreasing amplitudes. In order to confirm this, the graph of $f1$ was additionally plotted. The slider for variable a was adjusted so, that the graph of $f1$ touches and enclosed the relative maxima of the graph of the numerical solution. This is the case for a ~ 0.13 s^{-1}.

From these observations can be concluded that the time flow of the oscillation is given by:

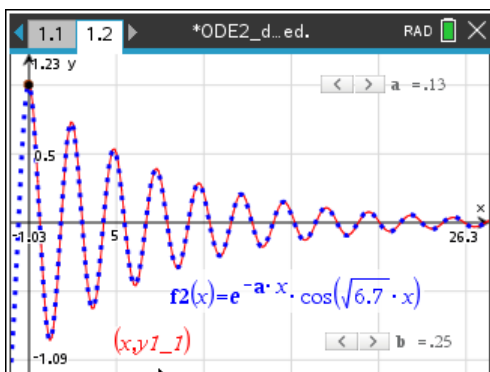$$e^{-a \cdot x} \cdot \cos(\sqrt{6.7}s^{-1} \cdot x). \tag{14}$$

Fig. 18: Modelling the graph by function f2

In fig. 18 the numerical solution (red) and the graph according to expression (14) (blue and dotted) is modelled. The algebraic model (equation 14) and the numerical approximation match excellently for $a = 0.13$ s^{-1}.

The numerical solution can be described algebraically – in the notation used in physics –by the following functional equation

$$x(t) = e^{-a \cdot t} \cdot \cos(\sqrt{6.7}s^{-1} \cdot t). \tag{15}$$

Yet the relationship between the parameters a and b, which both describe the strength of damping, is to find out. The example above shows that the value of b is approximately twice the value of a. This relation is valid for any other parameter a and b. Using the exact solution of the differential equation

one can see that the decay of the oscillations is determined by the factor $e^{-\frac{1}{2}\cdot\frac{r}{m}\cdot t}$. With students one can only suppose that $a = \frac{1}{2}\cdot b = \frac{1}{2}\cdot\frac{r}{m}$. With this assumption one can model the numerical solution by a function.

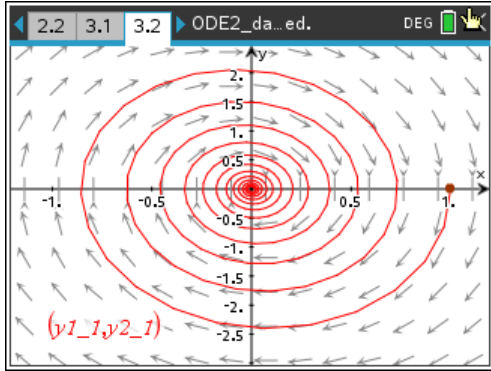$$x(t) = e^{-\frac{1}{2}\cdot\frac{r}{m}\cdot t} \cdot \cos\left(\sqrt{\frac{k}{m}}\cdot t\right). \tag{16}$$

Finally, the phase diagram of the damped oscillation is displayed (fig. 19), which presents the velocity of the swinging cart as function of its displacement. It shows clearly the exponential decrease of the oscillations.

Fig. 19: Phase diagram of a damped oscillation

Enforced Oscillations

After the detailed investigations of damped oscillations of the cart the differential equation for the motion of the cart (fig. 16) under influence of a periodical external force can be set up.

The external force for a continuous rotation of the crank is $F_0 \sin(\omega t)$, where ω the frequency of the exciting force and F_0 the amplitude of the force. Then the occurrent forces are given by:

$$m\cdot\ddot{x} = -k\cdot x(t) - r\cdot\dot{x}(t) + F_0\cdot\sin(\omega t). \tag{17}$$

The respective inhomogeneous 2nd order differential equation result in:

$$\ddot{x} = -\frac{k}{m}\cdot x(t) - \frac{r}{m}\cdot\dot{x}(t) + \frac{F_0}{m}\cdot\sin(\omega t). \tag{18}$$

For the numerical solution of equation (18) again a system of two 1st order differential equations – for the position and the velocity – is needed. Using the syntax of the calculator one obtains:

$$y1' = y2 \text{ and } y2' = -\frac{k}{m}\cdot y1 - \frac{r}{m}\cdot y2 + \frac{F_0}{m}\cdot\sin(\omega x). \tag{19}$$

Following parameters are taken:

$\frac{k}{m} = 6.7\frac{rad^2}{s^2}$, $\frac{r}{m} = 0.2\cdot\frac{1}{s}$, angular frequency $0.1\cdot\frac{rad}{s} \leq \varpi \leq 4\cdot\frac{rad}{s}$. The time interval is set on 40 s.

the amplitude of the acceleration forced by the hank is given by $\frac{F_0}{m} = 0.5\frac{1}{s^2}$. The initial values then are $y1(0) = 0.2$, $y2(0) = 0$.

In the following presentations the angular frequency is varied using a slider (variable w). Scaling of the axes is kept constant so that the differences in size of the amplitudes and frequencies are recognizable.
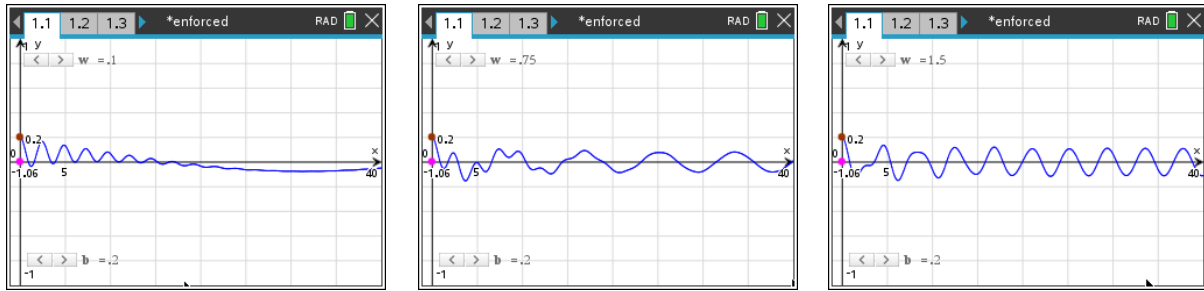
Fig. 20: Period $T \cong$ 63s $\qquad$ $T \cong$ 8.4s $\qquad$ $T \cong$ 4.2s

At the beginning of the excitation of the oscillation the displacement of the cart is displaced by the motion of the crank. As a result the cart is excited to damped oscillation with the natural frequency of the system. These oscillations are superimposed with the periodical excitation. This can be seen in fig. 20. After a transient phase of approximately 20s the damped oscillation is decayed and after that follows a stationary oscillation of the cart with the frequency of the excitation. For very small frequencies the amplitude of the cart is just as great as the displacement by the eccentric.
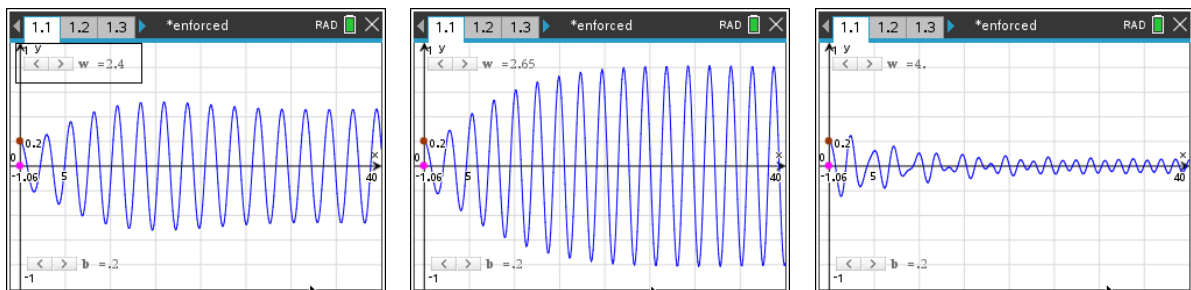

Fig. 21: Period $T \cong$ 2.62s $\qquad$ $T \cong$ 2.46s $\qquad$ $T \cong$ 1.57s

As soon as the excitation frequency approaches the resonance frequency of the clamped cart (fig. 21, pictures 1 and 2) the amplitude of the motion of the cart grows up to nearly the twentyfold of the excitation displacement. After this state of resonance the oscillation decreases again and its amplitude tends towards zero for high frequencies.

This qualitative evaluation of the numerical solution offers the students a physical explanation of the vibrations of the bridge and its collapse. From the point of physics, it is more satisfying if a quantitative statement for enforced oscillations is also possible. From the graphic representations of the solutions for various frequencies of excitation one can determine the amplitude of the stationary displacements of the cart after transient effects have been disappeared.
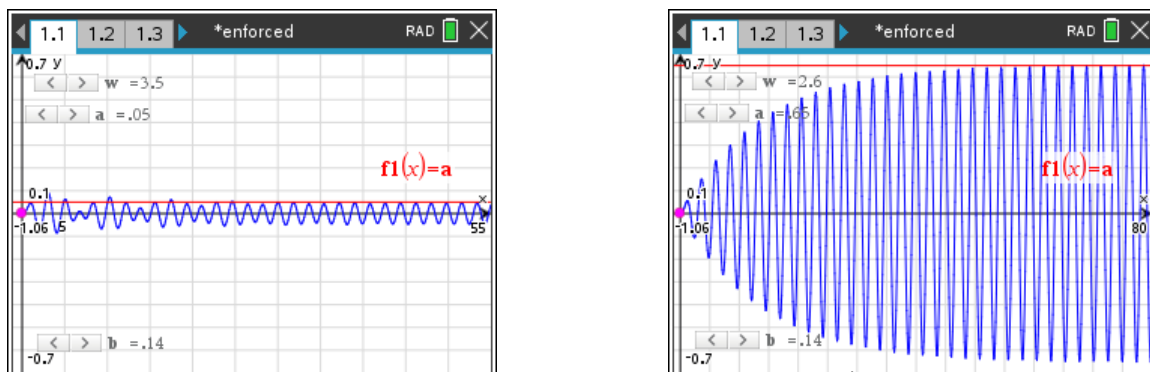

Fig. 22: Plotting f1(x) passing the maximum values of the displacements

In graphical displays of the application *Diff Eq* one cannot display the coordinates of points of graphs. In order to find the coordinates of the maxima during the steady state, the graph to $f1(x) = a$ is plotted. Parameter a is ruled by a slider. The graph of f1 is parallel to the *x-axis*. The value of a can adjust so, that the line touches and enclose the displayed maxima as shown in fig 22.

The frequencies and the respective amplitudes are measured and transferred to the TI-Nspire spreadsheet application. The measured values are divided by the value of the amplitude of the periodic excitation of the eccentric and displayed then in the third column. (This standardization makes the comparison of different systems possible.).

Fig. 23: Frequencies and amplitudes

Now the relative amplitudes can be plotted versus the frequencies (fig. 24). The graph shows very clear the resonance magnification when excitation acts with resonance frequency. In this case the amplitude of oscillation of the cart is almost twenty times greater than the maximum displacement of the excitation.
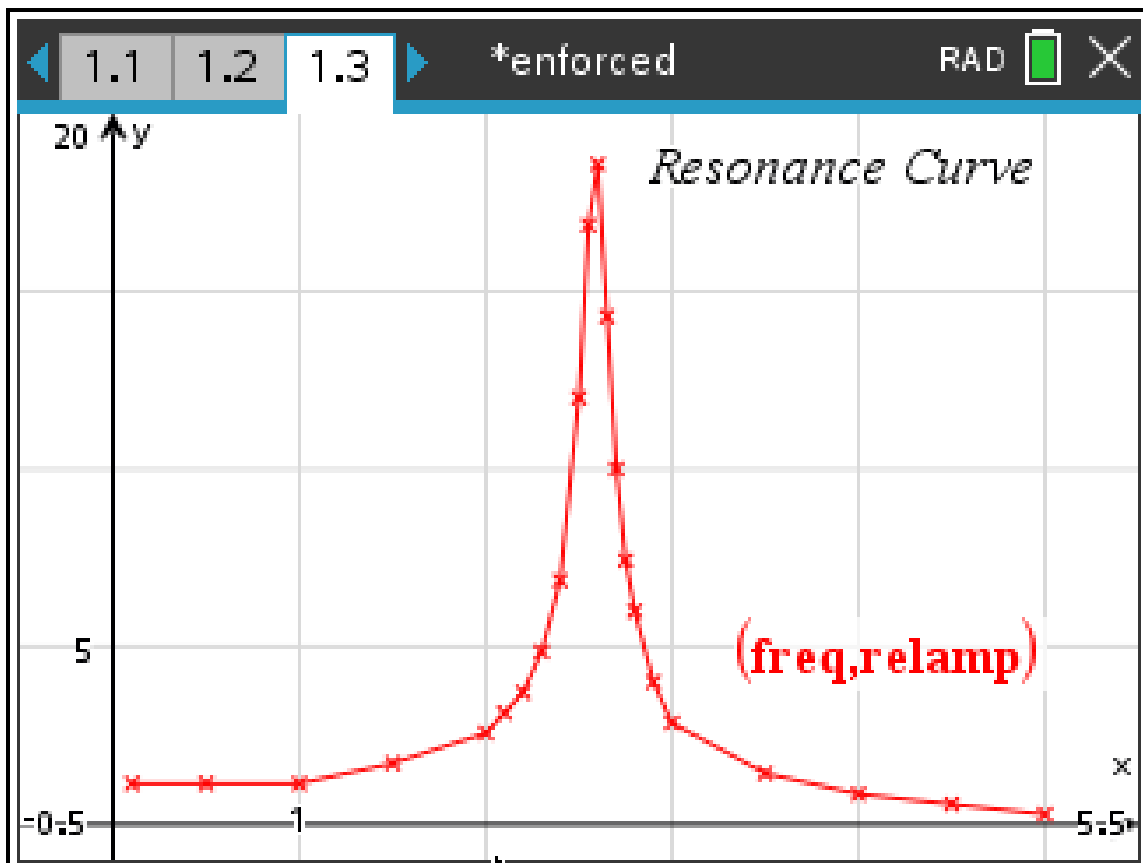

Fig. 24: Resonance curve of the model for enforced oscillation

Calculations can be continued for other values of damping, to understand the phenomenon of enforced oscillations under other conditions, too.

With this investigated model for enforced oscillations it was not possible to increase the energy input so far, that a resonance catastrophe occurs. As a little candy for holding on so long one should as a teacher let an empty wine glass split using a loudspeaker with the appropriate resonance frequency and enough sound volume. Finally, the **definition of the concept "resonance catastrophe"** found by Wikipedia is presented to the students and discussed with them:

In mechanics and construction, a **resonance disaster** describes the destruction of a building or a technical mechanism by induced vibrations at a system's resonance frequency, which causes it to oscillate. Periodic excitation optimally transfers to the system the energy of the vibration and stores it there. Because of this repeated storage and additional energy input the system swings ever more strongly, until its load limit is exceeded.

4. Conclusion

The functional description of the resonance curve can only be deduced from the exact solution of the corresponding differential equation. Students lack of the considerable skills in mathematics for such calculation.

This article on numerical solutions of differential equations has shown, that students have the ability to determine the resonance curve point by point (fig. 24) This result demonstrates, that the numerical solving of differential equations could be very helpful in physics lessons.

Email of the author: kh.keunecke@keukiel.de