Employing Technology to Visualize Complex Roots of Real Polynomials

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We show how the combination of computer algebra and geometric software allows students to locate the complex roots of real polynomials. Students use the geometric software to interactively conjecture the pattern of polynomial roots. They construct the complex roots of fourth degree polynomials from geometric features and algebraically verify that these conjectures are true.

1 INTRODUCTION

There is a well-known, simple construction that locates the complex roots of a parabola using a vertical line through the vertex intersecting a circle with radius given by the reflected parabola's intersection points with the x-axis. (See, e.g., [8].) We investigated extending this technique to higher degree polynomials. The extension became possible as a result of combining The Geometer's Sketchpad™ dynamic geometry software with Maple's™ computer algebra. We subsequently created interactive tools with Web SketchPad© for students to use to explore the connection between the geometry shown in a polynomial's graph and the location of the polynomial's complex roots. See [3] and [4].

In this paper, we provide the results of an investigation which employed a number of computer-based technologies synergistically. These technologies can now be used to enhance mathematical investigations and even open new questions to students and users.

We invite the reader to experience the dynamic documents we have constructed for students to investigate complex roots' relation to a polynomial's graphic properties. Authoring interactive tools like these are only possible with the technology now universally available. Web links to our interactive documents will be highlighted throughout this paper.

To simplify the exposition and computations, all polynomials are real and monic. Division by a constant doesn't affect the roots, this restriction is without loss of generality.

I CIRCLES AND LINES

2 QUADRATIC POLYNOMIALS

The standard technique for locating complex roots of a parabola $p(x)$ does not generalize easily. We altered the method by centering a circle with radius $R = \sqrt{p(0)}$ at the origin. Now draw a vertical line through the parabola's vertex. The intersections of the line and circle identify the complex conjugate roots of p as shown in Figure 1.

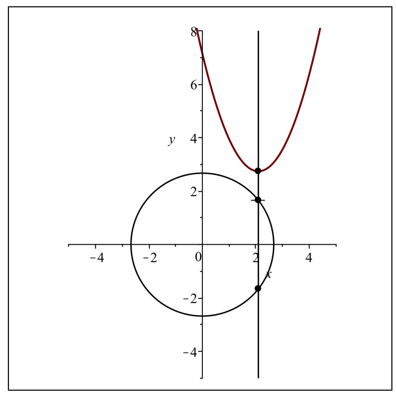

Figure 1: Constructing the Complex Roots of a Quadratic

A dynamic graphing applet for student explorations is available at: http://mathsci2.appstate.edu/TIME-2016/DFigure2Quadratic/. Users can perform the *action* of dragging complex roots in the diagram observing the *consequences* of their changes and *reflecting* on the mathematical meanings. A Maple worksheet for explorations is available at: http://mathsci2.appstate.edu/TIME2016/Maple/Explore Circles and Lines v2.0.mw. These interactive activities implement the *action-consequence-reflection* paradigm (see [1], [5], and [6], etc.).

3 CUBIC POLYNOMIALS

The standard technique for cubic polynomials $c(x)$ begins with determining the real part of the complex conjugate roots by finding the line through the real root that is also tangent to the cubic. The abscissa of the point of tangency is the real part of the complex root. We then continued our theme of circles centered at the origin with radius $R = \sqrt{-c(0)/r}$ or, when $r = 0$, use $R = \sqrt{c'(0)}$. as shown in Figure 2.

A dynamic graphing applet for student explorations is available at: http://mathsci2.appstate.edu/TIME2016/DFigure3Cubic/. For the cubic, users can perform the *action* of dragging the complex roots and/or the real root in the diagram observing the *consequences* of their changes and *reflecting* on the mathematical meanings. The Maple worksheet linked previously also explores cubic polynomials.

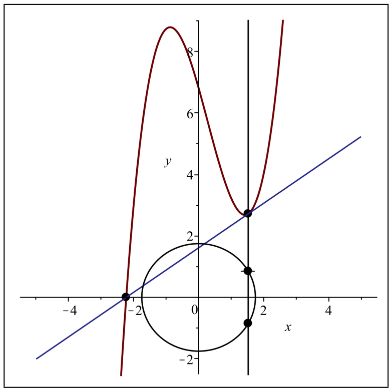

Figure 2: Constructing the Complex Roots of a Cubic

4 QUARTIC POLYNOMIALS

Quartic polynomials make a very interesting case: they can have $n = 0, 2,$ or 4 real roots (counting multiplicity) with the complement $m = 2 - \frac{1}{2}n$ complex conjugate root pairs.
Earlier graphical approaches cleverly used of surfaces (see, e.g., [7]), but were inaccessible to most secondary students. We continue our theme of locating the roots via vertical lines intersecting circles centered at the origin. However, the computations are a good deal more involved. The formulas display symmetries in the roots that become apparent from our 'circles and lines' interactive diagram. Users can quickly discover these symmetries in Figure 3

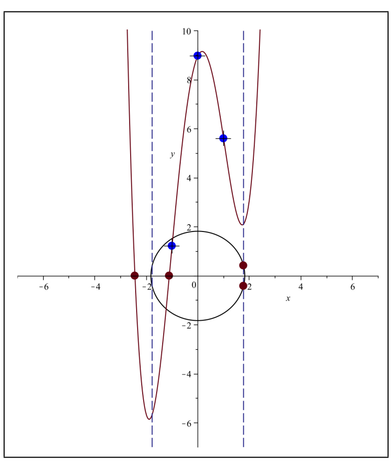

Figure 3: Symmetries of Quartic Roots

and also explore these symmetries with the Maple worksheet previously linked.
Our first interactive diagram for quartics has a pair of real roots and complex conjugate roots; it is available at http://mathsci2.appstate.edu/TIME2016/DFigure4Quartic/.

Our second interactive diagram for quartics focuses on the case of two complex conjugate root pairs. Examine http://mathsci2.appstate.edu/TIME2016/DFigure7Quartic2Complex/.
A quartic polynomial may possess a bitangent — a line tangent to the quartic at two points. A bitangent can be used to locate the real part of the complex roots. Investigate http://mathsci2.appstate.edu/TIME2016/DFigure6QuarticBitangent

5 SPECIAL HIGHER DEGREE POLYNOMIALS

Attempting to generalize the construction method for cubic polynomials led us to discover a technique for special polynomials with n real roots and a single complex pair. Suppose that

$$f(x) = (x - r)^n \left((x - a)^2 + b^2\right).$$

Locate the real part a of the complex conjugate roots by graphing the auxiliary function

$$\hat{f}(x) = f'(x) - \frac{n\, f(x)}{x - r};$$

the roots of $\hat{f}(x)$ occur at r and a. Draw a vertical line through $(a, 0)$. The circle centered at the origin with radius $R = \sqrt{|f(0)/r^n|}$ or, when $r = 0$, with radius $R = \sqrt{|f^{(n)}(0)/n!|}$ intersects the vertical line at the complex conjugate roots as shown in Figure 4. The auxiliary function is shown in blue.

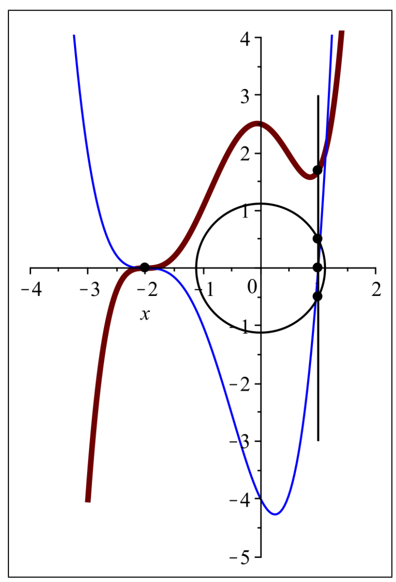

Figure 4: Special Higher Degree Polynomials

Users are invited to explore special higher degree polynomials with the interactive diagram at http://mathsci2.appstate.edu/TIME2016/DFigure8HigherDegree/.

The analysis above can be extended to more general $n + 2$ degree monic polynomials

$$f(x) = \left((x-a)^2 + b^2\right) \cdot \prod_{i=1}^{n}(x - r_i)$$

with one complex conjugate pair of roots and n real roots, not necessarily distinct, by defining the auxiliary function

$$\hat{f}(x) = f'(x) - f(x) \cdot \sum_{i=1}^{n} \frac{1}{x - r_i}.$$

II THREE POINT CONSTRUCTION OF A QUARTIC'S ROOTS

6 COMPUTING THE ROOTS OF QUARTICS

In our search for methods to identify the complex conjugate roots of a quartic, we discovered a result that allows us to compute the roots, both real and complex, and which highlights the symmetries of the roots.

Theorem (Three-Point Construction)**.** *Let $q(x)$ be a monic, reduced, real polynomial of degree 4. Choose any nonzero base point $x_0 \in \mathbb{R}$. The zeros of f can be computed from the three values $q(x_0)$, $q(0)$, and $q(-x_0)$.*

The theorem depends on two observations: For any point $x_0 \neq 0$,

$$q'(0) = \frac{1}{x_0}\left(f(x_0) - f(-x_0)\right) \qquad (1)$$

$$q''(0) = \frac{1}{x_0^2}\left(f(x_0) - 2f(0) + f(-x_0) - 2x_0^4\right) \qquad (2)$$

The computations in the proof bring us to see that the roots of the quartic are given by

$$z_{1,2} = \sqrt{A} \pm \sqrt{B} \qquad \text{and} \qquad z_{3,4} = -\sqrt{A} \pm \sqrt{C}$$

where A, B, and C were derived from the values given in equations (1) and (2). Here A will always be real, but C and B may be either real or complex; these formulas clearly show the symmetry of the roots. The complete proof appears in [2].

A printout of Maple visualization code for the theorem is available at http://mathsci2.appstate.edu/TIME2016/Maple/MapleVisualizationCode.pdf

CONCLUSION

By combining the technology available for dynamic geometry with computer algebra, we were able to investigate the rich connections between the geometric features of a polynomial's graph and the location of its complex roots. New technology, Web Sketchpad, allowed us to create interactive environments in which students can explore and uncover these relationships for themselves. The combination of capabilities of the different programs gives us extremely powerful pedagogical tools.

We also came to a deeper appreciation of the genius of Descartes, Fermat, Newton, Leibniz, and those who came before who had amazing understanding developed without the aid of interactive technological tools.

ACKNOWLEDGEMENTS

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REFERENCES

[1] W C Bauldry and W Ellis, "A CAS Approach to Understanding from Beginning Algebra to Advanced Calculus & Abstract Algebra," TIME-2008. Available at: http://rfdz.ph-noe.ac.at/fileadmin/Mathematik_Uploads/ACDCA/TIME2008/TIME08_contribs/Bauldry/Bauldry_Ellis.pdf

[2] W C Bauldry, M J Bossé, and S H Otey, "A Three Point Construction for the Roots of a Quartic Polynomial," *(submitted)*.

[3] W C Bauldry, M J Bossé, and S H Otey, "Circle Constructions of Polynomial Complex Roots," Electronic J of Math & Tech *(to appear)*.

[4] W C Bauldry, M J Bossé, and S H Otey, "Visualizing Complex Roots," *(submitted)*.

[5] T Dick, G Burrill, and G Brady, "New technologies offer new ways to engage students," NCSM Newsletter, 38 (2007).

[6] W Ellis, "Using Inquiry Questions and Action/Consequence Documents to Improve Student Understanding," MSRI - *Mathematical Circles and Olympiads*, 2004. Available at: http://jessica2.msri.org/attachments/13653/13653.ppt

[7] C Long and T Hern, "Graphing the Complex Zeros of Polynomials Using Modulus Surfaces," College Math Journal, 20, (1989), 2, 98–105.

[8] G A Yanosik, "Graphical solutions for complex roots of quadratics, cubics and quartics," *National Mathematics Magazine*, 17 (1943), 4, 147–150.

An extended bibliography is available at:
http://mathsci2.appstate.edu/TIME2016/Extended Bibliography.pdf

BIOGRAPHICAL NOTES

- Wade is Professor Emeritus of Mathematics at West Valley College and a Senior Mathematics Consultant for Texas Instruments, Inc. He currently is focusing on creating calculator microworlds implementing the *Action-Consequence-Reflection* paradigm.

- Bill is Professor of Mathematics at Appalachian State University and Associate Director of COMAP's *Mathematical Contest in Modeling* (MCM). His current focus is using Maple to enhance mathematics instruction.

- Mike is the Distinguished Professor of Mathematics Education at Appalachian State University and Director of the summer *Mathematics Education Leadership Training* program. He incorporates Geometer's Sketchpad extensively in his classes.

- Hunter is an undergraduate student finishing his mathematics degree in December, 2016; he is looking at his new possibilities.