Hypothetical Learning Trajectories that use digital technology to tackle an optimization problem

By Rubén Elizondo and Armando Hernández-Solís

CINVESTAV-IPN, Av. Instituto Politécnico Nacional 2508, San Pedro Zacatenco, 07360, Mexico City
CCH Vallejo-UNAM, Av. 100 metros s/n, Magdalena de las Salinas, 07760, Mexico City
elizondocch@gmail.com, armandhs@gmail.com

In this paper, we documented the results of the implementation of a series of activities related with an optimization problem. The objective of the paper consists in determining the dimensions of a right triangle with a maximum area, maintaining a fixed perimeter. Said activities, addressed towards 15 year old students, surge from a free interpretation of the construct "Hypothetical Learning Trajectory" (HLT) by Simon (1995), considering different representations and utilizing digital technologies. With this, we propose a route to tackle variation problems based on HLT similar to the proposed.

1 INTRODUCTION

Problem solving is an essential part in learning mathematics. Once teachers have approached and explored problems, we search for a way to pass them on to the classroom. To accomplish this, we first have to design activities with the main goal being that our students construct various learning methods. In this direction, we show a series of activities related to the following problem: "Given a 12 unit long segment, what are the dimensions of the triangle of maximum area?" To address it, we propose five sequences of activities based on an interpretation of the construct "Hypothetical Learning Trajectory" (HLT) of Simon (1995).

Among the objectives of presenting different approaches (physical, with dynamic geometry, with spreadsheet and algebraic) are the following: addressing the problem based on different representations and thereby establishing particular conditions and relations of the problem; in addition, generate, verify and/or refute conjectures; also complementing and reinforcing concepts and emerging mathematical meanings; and, of course, eventually solve the problem. In this way, we ask ourselves the following question, one which is also central to our work: How is the construction of mathematical meanings facing a problem through different representations promoted?

2 LITERATURE

Solving a problem with different approaches has always been a challenge, regardless of the school level to which it applies. Gomez et al. (2002) solved a problem in diverse ways, on multiplicative reasoning and without digital technology, with elementary students in levels 4 and 5.

Santos-Trigo (2012) proposes several problems which are solved in diverse ways, with different representations, always including digital technology. In Santos-Trigo and Reyes-Rodríguez (2016), many dynamic constructions are presented to draw an equilateral triangle given some (different) initial conditions. In Santos-Trigo and Moreno-Armella (2013), a conceptual framework based on problem solving and digital technology is proposed.

3 THEORETICAL FRAMEWORK

Simon's pedagogical proposal for teaching mathematics derives from constructivist ideas Simon (1995). The questions that arise from Simon are: how can constructivism help the reconstruction of mathematics? Or said differently, how can constructivism contribute to the development of theoretical frameworks for mathematical pedagogy? The search for frames that allow the reconstruction of mathematical pedagogy involves activities from both students and teachers. Planning is included among the activities that the teacher should carry out and within this we find what is known as a **hypothetical learning trajectory (HLT)**.

For Simon, consideration of a goal and learning activities is what is known as a hypothetical learning trajectory, which is a fundamental part of the mathematical learning cycle. The term is used to indicate the way in which learning can occur. Simon uses the term "hypothetical" because it is an assumption that the student will learn that way, which will only be known when applied. If not successful, we must modify it so that it, again, takes a hypothetical character. Thus, it will be in the learning cycle until a successful implementation of it is obtained.

A HLT has three components: the goal of learning, the learning activities and the hypothesis of learning processes, with a symbiotic relationship between the last two. Any of the three components can be modified to be implemented in the classroom, a place where social constructivism takes place, which Simon refers to.

Simon and Tzur (2004) give a more practical infrastructure for the development of the components found in the learning process, and the selection of activities or tasks. Other research around the HLT is the one proposed by Gomez and Lupiáñez (2007), in which it is said that the components of the learning process and selection of tasks require a level of design that is even more practical. This is through what is called content analysis and cognitive analysis, in which the learning goal is broken down into the contents that should be addressed during the course, and the cognitive processes required to achieve them.

A learning model that is closely related to this work is **problem solving**. Polya (1945), reflecting on his own practice and experience, proposes a general framework that describes four stages in solving a problem: understanding the problem, design of a solution plan, implementing the plan and retrospective view. Moreover, Polya discusses the role and importance of the use of heuristic methods in solving problems. Subsequently, Schoenfeld (1985) implements a research program based on the ideas of Polya with an important objective: to characterize what it means to think mathematically and document how students become successful in solving problems.

Problem solving is considered, by several countries, an important model for the study of math (Santos-Trigo, 2007; Moreno-Armella and Santos-Trigo, 2013).

The role of digital technologies in education can be analyzed from the **instrumental mediation**. One of the characteristics of humans is building tools, which, in principle, amplifies an intentional activity, whether physical or cognitive. Wertsch (1985) refers that for Vygotsky there are two types of tools: techniques (artifacts) and psychological (symbols). The tools are mediators of human activity in building concepts.

Moreno-Armella and Sriraman (2005) also distinguish between a symbolic and a material tool. On the one hand, the material tool affects human activity (this activity is mediated by the tool). On the other hand, the symbolic tool affects consciousness, the cognition of the individual, so that the tool is not only an amplifier but also a reorganizer of ideas. In addition to Moreno-Armella and Hegedus (2009), the tool is not only the physical object itself, but is the embodiment of a purpose.

Digital technologies have become mediating tools to learn, to educate, providing new forms of digital representation of a mathematical object, digital representations that involve new relationships with those that already exist. Digital representations produce a sense of material existence and these are executable.

4 RESEARCH DESIGN

This research corresponds to an exploratory-descriptive study (Hernández, Fernández & Baptista, 2003). The HLT proposed was implemented with a group of 42 students, aged between 15 and 16 years. Teams of three members were formed according to their choice. During all sessions, the participants answered the questions in the worksheets, so that those and the researcher's field notes formed the evidence of collection instruments. The following table describes briefly each of the HLT's implemented and the expected results.

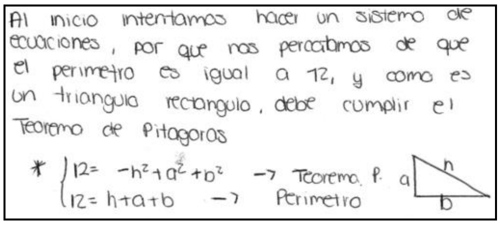
Table 1 Description of the HLT.

HLT	Brief description	Expected results
Pre-test	The problem is posed to tackle it freely.	To bring on an algebraic approach.
HLT 1 The physical exploration	The problem is physically explored with wires.	To understand that the area of the triangles varies, although its perimeter is maintained.
HLT 2 The triangle inequality	A dynamic simulation in GeoGebra is explored.	To set the conditions involved with the triangle inequality. To phrase the triangle inequality.
HLT 3 The exploration with spreadsheet	A tabular simulation is explored in spreadsheet.	To check that the area varies. To check that, for certain values, the problem does not make sense. To strengthen the conjecture that the isosceles right triangle is the triangle of maximum area.
HLT 4 The dynamic construction	The dynamic geometric construction is carried out.	To look at all triangles satisfying the conditions.
HLT 5 The algebraic exploration	It is carried out the algebraic process.	To obtain the exact solutions and compare them with previous approaches.
Final impressions	Questions are posed to gather final impressions.	To gather the final impressions and conclusions.

The HLT proposed were implemented in three sessions of two hours each. Session 1, Pre-test and HLT 1; session 2, HLT 1 (point cloud), HLT 2 and HLT 3, session 3. HLT 4, HLT 5; final impressions.

5 DESCRIPTION OF THE INFORMATION COLLECTED

We will mention only the work done by one team. Initially, the members of this team, evidence 1, attempt to establish a system of equations to tackle the problem (Pre-test). However, they could not progress with this approach because one of the equations was not posed correctly.


Al inicio intentamos hacer un sistema de ecuaciones, por que nos percatamos de que el perimetro es igual a 12, y como es un triangulo rectangulo, debe cumplir el Teorema de Pitagoras

$$* \begin{cases} 12 = -h^2 + a^2 + b^2 & \to \text{Teorema P.} \\ 12 = h + a + b & \to \text{Perimetro} \end{cases}$$

Evidence 1 Equations established in pre-test.

Following the instructions of the HLT 1, students measured the right triangles built manually, recording the data in a table (Evidence 2).

Base	Altura	Área
4	3	6
4.5	2.5	5.6
3.25	3.75	6.09
2	4.75	4.75
5	1.5	3.75
1	5.5	2.75

Evidence 2 The data in a table.

Based on the table, these students note that "the area [of the right triangles] varies depending on the size of the sides" and it is important to highlight the following observation: "The greater the difference between the legs, the smaller the area is" (Evidence 3). In addition, they conjecture that "the maximum area is achieved when the legs are equal or when the difference between the measurements of the legs is lower". Also, they give a first numerical approximation of the solution (Evidence 4).

No, notamos que el área varía dependiendo el tamaño de los lados. Mientras mayor sea la diferencia entre los catetos, el área es más chica.

Evidence 3 "The area varies depending on the size of the sides".

Si, el área máxima se logra cuando la medida de los catetos es igual o cuando la diferencia entre las medidas de los catetos es menor

3.5 x 3.5 a= 6.125

Evidence 4 First numerical approximation.

Their reasoning about the possibility of representing the measurements of the right triangle of minimum area is important, they mention that: "No [it is not possible to represent it] because it tends to infinity; when the difference between the legs is greater, the triangle reduces its area" (Evidence 5).

No, ya que tiende a infinito, porque cuando la diferencia de los catetos es mayor el triángulo reduce su área.

Evidence 5 "When the difference between the legs is greater, the triangle reduces its area".

At the start of the second session, as previously indicated, the point cloud generated by the measures of the 14 participating teams was displayed. The cloud of points is shown in the figure below (Figure 1).

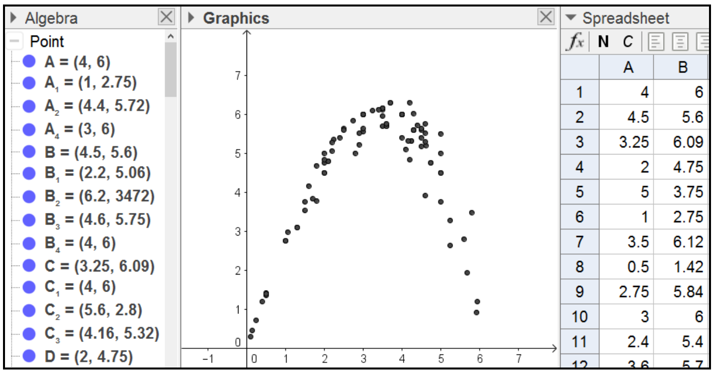


Figure 1 The point cloud.

The members of this team not only identify that the abscissas of these points correspond to the measurements of the base of the triangles and the ordinates correspond to the measurements of the area, but also identify that these are, respectively, the independent and dependent variable (Evidence 6).

En el eje de las x es la medida de la base de los triángulos (variable independiente) y en el eje de las y la medida del área (variable dependiente)

Evidence 6 "The independent and dependent variable".

On physical exploration (HLT 1), this team believes that it is always possible to construct triangles with perimeter of 12 units. However, after the HLT 2 was implemented, they understand the triangle inequality arguing that "when one side is greater than the sum of the other two sides, vertices do not intersect and no triangle is formed" (Evidence 7). In addition, "the sum of two sides must be greater than the remaining side" (Evidence 8).

Porque cuando un lado es mayor que la suma de los otros dos los vértices no se intersectan y no se forma el triángulo.

Evidence 7 Conjecture about the triangle inequality.

La suma de dos de sus lados debe ser mayor al lado restante.

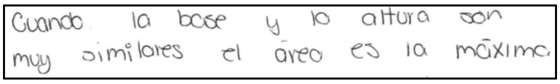
Evidence 8 The triangle inequality stated by this team.

At the end of Session 2, HLT 3 was implemented. In this HLT, students look at the table built in the spreadsheet in which, modifying the value called "increment", the values of base, height, hypotenuse, perimeter and area, were displayed; all these measures were in function of the value of the right triangle base (Figure 2).

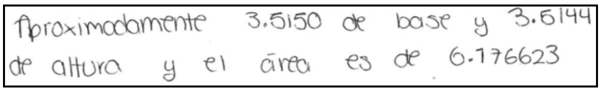
	A	B	C	D	E	F
1	Base	Altura	Hipotenusa	Perímetro	Área	Incremento
2	3,510000	3,5194	4,9706	12,0000	6,176607773852	0,001
3	3,511000	3,5184	4,9706	12,0000	6,176613735422	
4	3,512000	3,5174	4,9706	12,0000	6,176618284637	
5	3,513000	3,5164	4,9706	12,0000	6,176621420997	
6	3,514000	3,5154	4,9706	12,0000	6,176623144002	
7	3,515000	3,5144	4,9706	12,0000	6,176623453153	
8	3,516000	3,5134	4,9706	12,0000	6,176622347949	

Figure 2 The table in the spreadsheet.

Based on the tabular representation, the team states that "when the base and height are very similar, the area is at its maximum", reinforcing the conjecture mentioned during the implementation of the HLT 1 (Evidence 9). Also, they reinforce their first numerical approximation, indicating that the dimensions of the base, height and area of the triangle of maximum area are 3.5150, 3.5144 and 6.176623, respectively (Evidence 10).



Evidence 9 Reinforcing the initial conjecture.



Evidence 10 Second numerical approximation.

In the third session, participants carried out the dynamic construction following the instructions provided in the worksheets (HLT 4). Also, following the teacher's instructions, they built an outline of the area function. The following image shows the construction and the outline of the area function (**¡Error! No se encuentra el origen de la referencia.**).

6 CONCLUSIONS

Based on the previous description, we believe that the expected results shown in Table 1 were obtained. Particularly, we highlight the following conclusions related to the implementation of the HLT proposed, answering thus the question posed that guides this work:

- Participants establish particular conditions and relations of the problem.
- From exploring different representations, participants generate, validate and/or refute conjectures.
- Participants complement, appropriate, reinforce and/or build different mathematical concepts and meanings.

In addition, as a result of the implementation of these HLT, we think it is timely profound reflection on the relevance of current curricula.

As it is described throughout this work, with the HLT proposed there are different representations organized as follows: a physical exploration, a dynamic exploration with GeoGebra and spreadsheet, and an algebraic exploration. Thus, we believe that we have traced a route to tackling problems of variation.

REFERENCES

Gómez, C., Rosseau, C., Steinthorsdottir, O., Valentine, C., Wagner, L. and Wiles, P. (2002). Multiplicative Reasoning: Developing Student´s Shared Meanings. In eds. Litwiller, B., Bright, G. *Making Sense of Fractions, Ratios, and Proportions*. pp 213-223. NCTM, USA.

Gómez, P. and Lupiáñez, J. L. (2007). Trayectorias hipotéticas de aprendizaje en la formación inicial de profesores de matemáticas de secundaria, *PNA*, **1**(2), 79-98.

Hernández, R., Fernández, C. and Baptista, P. (2003). *Metodología de la investigación*. Third edition. McGraw-Hill/Interamericana Editores S.A., Mexico.

Moreno-Armella, L. and Sriraman, B. (2005). Structural stability and dynamic geometry: Some ideas on situated proofs, *International Reviews on Mathematical Education*, **37**(3), 130-139.

Moreno-Armella, L. and Hegedus, S. (2009). Co-action with digital technologies. *ZDM Mathematics Education*, (41), 505-519.

Moreno-Armella, L. and Santos-Trigo, M. (2013). Introduction to international perspectives on problem solving research in mathematics education, *The Mathematics Enthusiast*, **10**(1), Article 2.

Polya, G. (1945). How to solve it. Princeton University Press, New York.

Santos-Trigo, M. (2007). *Resolución de problemas matemáticos. Fundamentos cognitivos.* Trillas, Mexico.

Santos-Trigo, M. (2012). *El estudio de fenómenos de variación y el empleo de herramientas digitales.* eBook available for download on iPad with iBooks or computer with iTunes.

Santos-Trigo, M. and Moreno-Armella, L. (2013). Sobre la construcción de un marco conceptual en la resolución de problemas que incorpore el uso de herramientas computacionales. In Rojano, M.T. (Coord.). *Las tecnologías digitales en la enseñanza de las matemáticas.* Trillas, Mexico.

Santos-Trigo, M. and Reyes-Rodríguez, A. (2016). The use of digital technology in finding multiple paths to solve and extend an equilateral triangle task. *International Journal of Mathematical Education in Science and Technology*. **47**(1), 58-81.

Shoenfeld, A. (1985). *Mathematical problem solving*. Academic, New York.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective, *Journal for Research in Mathematics Education*, **26**(2), 114-145.

Simon, M. A. and Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory, *Mathematical Thinking and Learning*. **6**(2), 91-104.

Wertsch, J. V. (1985). *Vygotsky and the social formation of mind*. University Press, Cambridge, Massachusetts, London: Harvard.