HEAT TRANSFER IN A ONE DIMENSIONAL DOMAIN OF VARIABLE CROSS-SECTIONS

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Abstract.

The method of approximating solutions of partial differential equations with variable coefficients is studied. This is done by considering heat flow through a one dimensional model, with variable cross-sections. Two cases are considered. The first one involves approximation of variable coefficient quadratically and by direct integration. This case is studied using a conic domain. The second case approximates the variable coefficient quadratically and by step functions. The solution of the problem in each case is expressed using the Green's function, and the results are compared.

Keywords: Partial differential equations (PDE's), variable coefficients, eigenvalues, eigenfunctions, boundary conditions, Green's function.

1. INTRODUCTION

In this paper the method of approximating solutions of PDE's with variable coefficients is based on the study of heat equation [1].

\[
c(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ k(x)A(x) \frac{\partial u}{\partial x} \right] + A(x)f(x,t). \quad (1)
\]

The heat equation under study is considered with a variable cross section area \( A(x) \). In this case \( A(x) = p (ax + \beta)^2 \) is the cross sectional area of the domain, we set \( f(x,t) = 0 \). \( c(x) = c \) is the heat capacity, \( \rho(x) = \rho \) is the density of the domain and \( k(x) = k \) is the thermal conductivity of the domain. We suppose that the coefficients \( c, \rho \) and \( k \) are constants for this investigation. Equation (1) is considered with boundary conditions of the first kind described by \( u(0,t) = 0 \) and \( u(l,t) = 0 \), and initial condition of the form \( u(l,0) = g(x) \).

This problem describes the process of heat transfer in an axisymmetric body with cross section \( A(x) \) oriented so that the \( x \)-axis lies along the axis of the body. The case where

\[
A(x) = p (ax + \beta)^2 \equiv pa^2 \left( x + \frac{\beta}{\alpha} \right)^2,
\]

where \( \gamma = \frac{\beta}{\alpha} \) corresponds to linear dependence of boundary equation. This corresponds to the conic shape of the considered domain. It is possible to find an exact solution expressed by Green’s function.

The linear function \( ax + \beta \) can be approximated by a step function. Physically this
corresponds to a cylinder consisting of \( N \) sections of constant cross sections \( A_j (j = 1, 2, \ldots, N) \). The case for \( N > 2 \) was considered by Fedotov et al. [2], [3]. The Green’s function can be obtained subject to that the solution satisfies the boundary conditions at the junctions. The continuity of the solution at the junctions is described as follows

\[
    u_j(l_j, t) = u_{j+1}(l_j, t),
\]

and the continuity heat flow is given by

\[
    A_j u'(l_j, t) = A_{j+1} u'(l_{j+1}, t). \quad (j = 1, \ldots, N-1)
\]

The initial condition in our example is defined by

\[
    g(x) = x(4-x).
\]

The results can be considered as approximation of the solution for variable cross section \( A(x) \). See figure 2. The results of both methods will be compared and the solutions at the end of the paper show approximate similarity of the results.

2. Model of a one dimensional domain with variable cross-sections governing the PDE and boundary conditions.

The model of a conic domain of variable cross-sections of \( N \) sections is illustrated by figure 1 below.

![Figure 1](image)

The solution to equation (1) is sought in the form

\[
    u(x, t) = \sum_{n=1}^{\infty} b_n X_n(x) e^{-\lambda_n t},
\]

where

\[
    X_n(x) = \sin \frac{\lambda_n x}{x + \gamma}, \quad (n = 1, 2, \ldots)
\]

was obtained from the method of separation of variables to equation (1) with application of the boundary conditions of the first kind.

It is clear that the system (2) is orthogonal with weight \((x + \gamma)^2\). It follows from equation
(2) that

\[ \lambda_n = \frac{n\pi}{l} \]

and this formula gives the eigenvalues. An example is considered where \( l = 4 \) and \( \gamma = 1 \). The eigenvalues obtained for this example are shown in section 3.

The solution of the problem is given by the following equation:

\[
\sum_n \frac{1}{\Phi_n} \int_0^l A(\xi) X_n(x) X_n(\xi) g(\xi) e^{-\lambda_n \xi} d\xi
\]

Where

\[
\Phi_n^2 = \int_0^l A(\xi) X_n^2(\xi) d\xi = \int_0^l (x + \gamma)^2 \sin^2 \lambda_n x dx = \frac{l}{2}
\]

is the norm squared, and

\[
g(x) = \sum_{n=1}^\infty b_n \left[ \sin \frac{n\pi x}{l} \left( \frac{1}{x + \gamma} \right) \right]
\]

is the initial function. The coefficient \( b_n \) is described as follows

\[
b_n = \frac{2}{l} \int_0^l A(\xi) X_n(\xi) g(\xi) d\xi.
\]

The solution given by equation (3) can also be represented by

\[
u(x, t) = \int_0^l G(x, \xi, t) g(\xi) d\xi,
\]

where

\[
G(x, \xi, t) = \sum_{n=1}^\infty \frac{2}{l} A(\xi) X_n(x) X_n(\xi) e^{-\lambda_n \xi}
\]

is the Green's function.

**Approximating by using step functions.**

The domain for this case is illustrated by figure 3 below.

![Figure 3](image-url)
The solutions is also sought in the form

\[ u(x, t) = \sum_{m=1}^{\infty} b_m X_m(x) e^{-\lambda x}, \]

where in this case

\[ X_m(x) = \sum_{j=1}^{N} X_j^{(m)} \theta_j(x), \]
\[ \theta_j(x) = H(x - l_{j-1}) - H(x - l_j) \]

is the Heaviside function, and

\[ X_j^{(m)}(x) = c_{2j-1} \cos(\lambda x) + c_{2j} \sin(\lambda x) \]

was obtained as it was described under the conic domain case, with

\[ b_m = \frac{1}{\mathcal{Q}_m} \int_{l}^{l_N} X_m(\xi) g(\xi) d(\xi). \]

and \( g(x) \) is the initial condition described as follows

\[ g(x) = \sum_{n=1}^{\infty} b_n X_n(x). \]

Application of the boundary conditions at the endpoints and at the junctions to equation (4), results with an \((N \times N)\) block matrix, which is helpful in determining the eigenvalues. The solution of the problem is given by the following equation

\[ u(x, t) = \sum_{m=1}^{\infty} \frac{1}{\mathcal{Q}_m} \int_{l}^{l_N} A(\xi) X_m(x) X_m(\xi) g(\xi) e^{-\lambda x} d(\xi) \]

where the weight function

\[ A(x) = \begin{cases} 
A_1 & l_0 \leq x \leq l_1 \\
A_2 & l_1 \leq x \leq l_2 \\
\vdots & \vdots \\
A_{n-1} & l_{N-2} \leq x \leq l_{N-1} \\
A_n & l_{N-1} \leq x \leq l_N 
\end{cases} \]

and the norm squared is described as follows

\[ \mathcal{Q}_m^2 = \int_{l}^{l_N} A(\xi) X_m^2(\xi) d\xi \quad (m = n). \]

The solution is also given in the form
\[ u(x,t) = \int_0^\infty G(x,\xi,t)g(\xi)d\xi, \]

where

\[ G(x,\xi,t) = \sum_{m=1}^\infty \frac{1}{\phi_m^2} A(\xi) X_m(x) X_m(\xi)e^{-\lambda_m \xi}, \]

is the Green's function.

3. Results

3.1. The conic domain case

First five eigenvalues of the smooth functions (conic) case, for an example where \( l = 4 \), are given by the following table:

<table>
<thead>
<tr>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \lambda_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{2} = 0.78540 )</td>
<td>( \frac{\pi}{4} = 1.5708 )</td>
<td>( \frac{2\pi}{3} = 2.09437 )</td>
<td>( \frac{\pi}{3} = 3.1416 )</td>
<td>( \frac{\pi}{4} = 3.9270 )</td>
</tr>
</tbody>
</table>

First five eigenfunctions corresponding to the eigenvalues in table 1 are given in the following table.

<table>
<thead>
<tr>
<th>( X_1(\alpha) )</th>
<th>( X_2(\alpha) )</th>
<th>( X_3(\alpha) )</th>
<th>( X_4(\alpha) )</th>
<th>( X_5(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \frac{\pi \alpha}{(\alpha+1)} )</td>
<td>( \sin \frac{\pi \alpha}{(\alpha+1)} )</td>
<td>( \sin \frac{\pi \alpha}{(\alpha+1)} )</td>
<td>( \sin \frac{\pi \alpha}{(\alpha+1)} )</td>
<td>( \sin \frac{\pi \alpha}{(\alpha+1)} )</td>
</tr>
</tbody>
</table>

Graphs of the smooth eigenfunctions in table 2 are represented by figure 4 (Derive used).
The following figure gives the analytical solution of the problem (Derive used).

3.2. The stepped domain case

To obtain the eigenvalues for the stepped case, we also consider an example with \( l_N = 4 \). For this example an \((8\times8)\) matrix was obtained from the following system of equations.

\[
\begin{align*}
&c_1 \cos \lambda.0 + c_2 \sin \lambda.0 = 0 \Rightarrow c_1 = 0 \\
&c_1 \cos \lambda.l_1 + c_2 \sin \lambda.l_1 - c_3 \cos \lambda.l_1 - c_4 \sin \lambda.l_1 = 0 \\
&A_1 \lambda[-c_1 \sin \lambda.l_1 + c_2 \cos \lambda.l_1] - A_2 \lambda[-c_3 \sin \lambda.l_1 + c_4 \cos \lambda.l_1] = 0 \\
&c_3 \cos \lambda.l_2 + c_4 \sin \lambda.l_2 - c_5 \cos \lambda.l_2 - c_6 \sin \lambda.l_2 = 0 \\
&A_2 \lambda[-c_3 \sin \lambda.l_2 + c_4 \cos \lambda.l_2] - A_3 \lambda[-c_5 \sin \lambda.l_2 + c_6 \cos \lambda.l_2] = 0 \\
&c_5 \cos \lambda.l_3 + c_6 \sin \lambda.l_3 - c_7 \cos \lambda.l_3 - c_8 \sin \lambda.l_3 = 0 \\
&A_3 \lambda[-c_5 \sin \lambda.l_3 + c_6 \cos \lambda.l_3] - A_4 \lambda[-c_7 \sin \lambda.l_3 + c_8 \cos \lambda.l_3] = 0 \\
&c_7 \cos \lambda.l_4 + c_8 \sin \lambda.l_4 = 0
\end{align*}
\]

(5)

The variable area for the domain with variable cross sections of four sections are described as follows:

\[
A(x) = \begin{cases} 
A_1 = \left(\frac{1}{2}+1\right)^2 = 2.25 & l_0 \leq x \leq l_1 \\
A_2 = \left(\frac{3}{2}+1\right)^2 = 6.25 & l_1 \leq x \leq l_2 \\
A_3 = \left(\frac{5}{2}+1\right)^2 = 12.25 & l_2 \leq x \leq l_3 \\
A_4 = \left(\frac{7}{2}+1\right)^2 = 20.25 & l_3 \leq x \leq l_4
\end{cases}
\]

The following table shows the first five eigenvalues obtained from solution of the transcendental system (5), using the method of finding roots by Fedotov et al. [5].
The continuous eigenfunctions, with discontinuous derivatives due to approximating using non smooth functions, corresponding to the eigenvalues in Table 4 are given by the table that follows:

Table 3: Eigenvalues
<table>
<thead>
<tr>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \lambda_5 )</th>
<th>( \lambda_6 )</th>
<th>( \lambda_7 )</th>
<th>( \lambda_8 )</th>
<th>( \lambda_9 )</th>
<th>( \lambda_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7374</td>
<td>1.5708</td>
<td>2.9542</td>
<td>3.14159</td>
<td>3.929</td>
<td>4.71239</td>
<td>5.4952</td>
<td>6.28319</td>
<td>7.0708</td>
<td>7.85938</td>
</tr>
</tbody>
</table>

The following table shows the values of the coefficients \( c_1, c_2, \ldots, c_8 \).

Table 4: Eigenfunctions

- \( X_1(\omega) = c_1 \cos(0.7374\omega) + c_2 \sin(0.7374\omega) \)
- \( X_2(\omega) = c_2 \cos(1.5708\omega) + c_3 \sin(1.5708\omega) \)
- \( X_3(\omega) = c_4 \cos(2.9542\omega) + c_5 \sin(2.9542\omega) \)
- \( X_4(\omega) = c_6 \cos(3.14159\omega) + c_7 \sin(3.14159\omega) \)

The following table shows the values of the coefficients \( c_1, c_2, \ldots, c_8 \).

Table 5: Coefficients values
<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
<th>( c_4 )</th>
<th>( c_5 )</th>
<th>( c_6 )</th>
<th>( c_7 )</th>
<th>( c_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>0</td>
<td>1</td>
<td>0.32</td>
<td>0.68128</td>
<td>0.16198</td>
<td>0.68065</td>
<td>-0.00411</td>
<td>0.5126</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.8102</td>
<td>0</td>
<td>0.8102</td>
<td>0</td>
<td>0.5102</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>0</td>
<td>1</td>
<td>-0.32</td>
<td>0.68128</td>
<td>-0.16198</td>
<td>0.68065</td>
<td>0.00411</td>
<td>0.5126</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.36</td>
<td>0</td>
<td>0.18167</td>
<td>0</td>
<td>0.1111</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>0</td>
<td>1</td>
<td>0.32</td>
<td>0.68128</td>
<td>0.16198</td>
<td>0.68065</td>
<td>-0.00411</td>
<td>0.5126</td>
</tr>
</tbody>
</table>

Solutions of the eigenfunctions in Table 4 are represented by Figure 7 that follows (Mathcad used):

The following figure gives the solution of the problem (Mathcad used). Notice again the jump of the derivatives at the points of junctions. This is a property of eigenfunctions.
4. Conclusion

In this paper we studied approximation of solutions of partial differential equations in a domain of varying area. Two cases were considered. The first one involved a conic domain where the equation under study was approximated by smooth eigenfunctions. The second case involved a stepped domain and here the equation was approximated by non-smooth eigenfunctions. Figures 4 and 7 are closely related, and this shows that the methods of approximating solutions of PDE’s give similar results. Different kinds of boundary conditions may be considered, for example, boundary conditions of the second and third kinds [4]. But we did not present these boundary conditions in this paper for the sake of simplicity. This concept can be extended to domains of more complicated shapes. An example of such a shape is illustrated by the following diagrams:

A domain of an arbitrary shape can be approximated either using linear functions or step functions. Consider figures 8(a) and 8(b).

References