A Discreet Compartmental Model for Lead Metabolism in the Human Body

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Abstract

A real-life example of a mathematical technique, employing a so-called transfer matrix, is developed for the compartmental analysis of lead metabolism in the human body. The technique is demonstrated with the aid of Bert’s four-compartment biokinetic model. The results produced by this time-discrete approach correspond almost perfectly with those of a continuous-time method. The powerful calculation tools of the Computer Algebra System (CAS) "Scientific Workplace" are employed to illustrate the results using tables and graphs.

1 Introduction

Holtzman at the Argonne National Laboratory was one of the pioneers who applied numerical models of compartmental analysis to predict lead levels over time in different parts of the human body. In 1973 Rabinowitz, Wetherill & Kopple ([4], 1973) published an article in which they applied compartmental analysis to a three-compartment model for lead metabolism in the human body. This was followed by more complicated models, such as those of Batchelet, Brand and Steiner ([1], 1979), Bert as reviewed in U.S. EPA ([5], 2001), and Leggett ([3], 1993). In this article a mathematical technique, employing a so-called transfer matrix is developed and demonstrated with the aid of Bert’s four-compartment biokinetic model. Results obtained by this time-discrete approach are compared with those obtained by a continuous-time method.

2 Bert’s four-compartment model

Dainty ([2], 2001) defines a compartment as a theoretical construct that may well combine material from several different physical spaces. It is an amount of material that acts as though it is well mixed and kinetically homogeneous. Further, a compartmental model consists of a finite number of compartments with specified interconnections between them, which represent the flux of material that is transported from one location to another. By defining a compartmental model for lead in this way, we can reduce a complex metabolic system into a small number of pathways and compartments. The outputs from such a model include the quantity of lead stored in certain body pools, the rate and extent to which lead moves from one pool to another and how long it will remain in the body before being excreted. Figure 1 is a schematic diagram of the systemic model, showing the compartments and directions of movement of lead among these compartments. As shown, there are four main compartments where lead distributes, namely, blood (#1), cortical bone (#2), trabecular bone (#3), and
soft tissue (#4). Although numbered 5 and 6 in the diagram, the lungs and gastrointestinal tract are not considered as compartments as they are simply the channels by which lead enters the system.

Figure 1: A four-compartment model.

The rate at which lead is inhaled into the lungs is $\alpha \text{ mg/day}$. Bert ([5], 2001) established that a portion $p = 0.35$ of this lead is absorbed into the blood. The rate at which lead enters the blood via the respiratory system is thus

$$y_{15} = 0.35\alpha, \quad \text{mg/day.} \quad (1)$$

Lead in food and drink enters the body via the gastrointestinal tract. There is also a translocation of lead from the soft tissue to the digestive system in bile, saliva and other gastric secretions. It is assumed that the rate of transfer of lead from the soft tissue to the digestive tract is directly proportional to the amount of lead $x_4(t)$ in the soft tissue. That is,

$$y_{64} = \tau_{64} x_4, \quad \text{mg/day,} \quad (2)$$

where $y_{64}$ is the transfer rate in mg/day from compartment 4 (the soft tissue) to system 6 (the gastrointestinal tract). The symbol $\tau_{64}$ is the transfer coefficient from soft tissue to the digestive tract, and is the fraction of lead in compartment 4 that is translocated to system 6 per day. If $\beta \text{ mg}$ is the daily intake of lead by mouth and $q = 0.08$ (according to Bert ([5], 2001)) the portion of lead $\beta + y_{64}$
that is absorbed by the blood, then the rate at which lead is translocated from the digestive system to the blood is

\[ y_{16} = 0.08 (\beta + y_{64}) = 0.08 (\beta + \tau_{64} x_4), \text{ } \mu g/\text{day}. \] (3)

The pathways with directions of transfer of lead in Figure 1 indicate the interconnection between compartments. Associated with each pathway and direction is a constant, \( \tau \), which denotes the transfer rate of lead from compartment \( j \) to compartment \( i \). The constant rate, \( \tau \), is called the transfer coefficient of lead from compartment \( j \) to compartment \( i \) and is the fraction of the amount \( x_j (t) \) of lead in compartment \( j \) that is translocated to compartment \( i \) per day. So, the rate of lead transfer from compartment \( j \) to compartment \( i \) is given by

\[ y_{ij} = \tau_{ij} x_j, \text{ } \mu g/\text{day}. \] (4)

Denoting the environment by 0 and using values found in the aforementioned publication, U.S. EPA ([5], 2001), we construct the following table of values for \( \tau \) for a normal healthy human adult male.

<table>
<thead>
<tr>
<th></th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 0 )</td>
<td>0.021</td>
<td></td>
<td></td>
<td>0.00292</td>
</tr>
<tr>
<td>( i = 1 )</td>
<td>0.0000325</td>
<td>0.00229</td>
<td>0.00235</td>
<td></td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>0.00578</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( i = 3 )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( i = 4 )</td>
<td>0.001835</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( i = 6 )</td>
<td></td>
<td></td>
<td></td>
<td>0.01143</td>
</tr>
</tbody>
</table>

Note that it is assumed that the transfer rate for lead from compartment \( j \) to \( i \) is directly proportional to the amount \( x_j (t) \) of lead in compartment \( j \) and that the constant of proportionality, \( \tau_{ij} \), is a daily fraction of \( x_j \).

### 3 The transfer matrix (a discrete model)

In this article the custom in many textbooks is followed, namely to denote a column vector (\( n \times 1 \) matrix) by a small letter in **boldface** and an \( m \times n \) matrix by CAPITALS (not in boldface).

Let the entry \( x_i (t) \) in the \( 4 \times 1 \) column vector \( \mathbf{x} \),

\[ \mathbf{x} (t) = \begin{pmatrix} x_1 (t) \\ x_2 (t) \\ x_3 (t) \\ x_4 (t) \end{pmatrix}, \quad \mathbf{x} (t + \Delta t) = \begin{pmatrix} x_1 (t + \Delta t) \\ x_2 (t + \Delta t) \\ x_3 (t + \Delta t) \\ x_4 (t + \Delta t) \end{pmatrix}, \]

represent the amount of lead (in \( \mu g \)) in compartment \( i \) at time \( t \). The column vector \( \mathbf{x} (t) \) specifies the state of the system at time \( t \). The \( 4 \times 1 \) column vector \( \mathbf{x} (t + \Delta t) \) is the state of the system \( \Delta t \) units of time later. Assume that a fixed fraction \( \tau_{ij} \) of the lead in compartment \( j \) is passed to compartment \( i \) every \( \Delta t \) units of time. Note that it is assumed that the system changes only
at times \( t + \Delta t, t + 2\Delta t, \ldots, t + n\Delta t, \ldots \). The calculations of \( x_i(t + \Delta t) \) for the four compartments are as follows:

\[
\begin{align*}
x_1(t + \Delta t) &= \left\{ \begin{array}{l}
x_1(t) + \left[ \tau_{12} x_2(t) + \tau_{13} x_3(t) + \tau_{14} x_4(t) + y_{15} + y_{16} \right] \\
- \left[ \tau_{01} x_1(t) + \tau_{21} x_1(t) + \tau_{31} x_1(t) + \tau_{41} x_1(t) \right]
\end{array} \right.
\end{align*}
\]

Use Equations (1) and (3) to obtain

\[
x_1(t + \Delta t) = \left\{ \begin{array}{l}
1 - (\tau_{01} + \tau_{21} + \tau_{31} + \tau_{41}) x_1(t) + \tau_{12} x_2(t) + \tau_{13} x_3(t) \\
+ (\tau_{14} + 0.08\gamma_{64}) x_4(t) + 0.35\alpha + 0.08\beta
\end{array} \right.
\]

If \( \tau_{11} = 1 - (\tau_{01} + \tau_{21} + \tau_{31} + \tau_{41}) \), then \( \tau_{11} \) is merely the fraction of the lead content of compartment 1 that remains in 1, and so

\[
x_1(t + \Delta t) = \left\{ \begin{array}{l}
\tau_{11} x_1(t) + \tau_{12} x_2(t) + \tau_{13} x_3(t) \\
+ (\tau_{14} + 0.08\gamma_{64}) x_4(t) + 0.35\alpha + 0.08\beta
\end{array} \right.
\]

where \( \tau_{22} = 1 - \tau_{12} \) is the fraction of the lead content of compartment 2 that remains in 2.

\[
x_2(t + \Delta t) = x_2(t) + \tau_{21} x_1(t) - \tau_{12} x_2(t)
\]

\[
x_2(t + \Delta t) = (1 - \tau_{12}) x_2(t) + \tau_{21} x_1(t)
\]

where \( \tau_{33} = 1 - \tau_{13} \) is the fraction of the lead content of compartment 3 that remains in 3.

\[
x_4(t + \Delta t) = x_4(t) + \tau_{41} x_1(t) - \left[ \tau_{04} x_4(t) + \tau_{14} x_4(t) + \tau_{64} x_4(t) \right]
\]

\[
x_4(t + \Delta t) = [1 - (\tau_{04} + \tau_{14} + \tau_{64})] x_4(t) + \tau_{41} x_1(t)
\]

where \( \tau_{44} = 1 - \tau_{04} + \tau_{14} + \tau_{64} \) is the fraction of the lead content of compartment 4 that remains in 4.

The last four equations, Equations (5) to (8), can be written as follows in matrix form:

\[
x(t + \Delta t) = Tx(t) + b,
\]

where

\[
T = \begin{pmatrix}
\tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} + 0.08\gamma_{64} \\
\tau_{21} & \tau_{22} & 0 & 0 \\
\tau_{31} & 0 & \tau_{33} & 0 \\
\tau_{41} & 0 & 0 & \tau_{44}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0.968985 & 0.0000325 & 0.00229 & 0.0032644 \\
0.00578 & 0.999675 & 0 & 0 \\
0.0024 & 0 & 0.99771 & 0 \\
0.001835 & 0 & 0 & 0.9833
\end{pmatrix}
\]

and

\[
b = \begin{pmatrix}
0.35\alpha + 0.08\beta & 0 & 0 & 0
\end{pmatrix}^T.
\]
The matrix $T$ is referred to as the **transfer matrix**. It follows from Equation (9) that

$$
x(t + 2\Delta t) = T x(t + \Delta t) + b = T^2 x(t) + T b + b,
$$

$$
x(t + 3\Delta t) = T x(t + 2\Delta t) + b = T^3 x(t) + T^2 b + T b + b
$$

$$
\vdots
$$

$$
x(t + n\Delta t) = T^n x(t) + T^{n-1} b + T^{n-2} b + \ldots + T^2 b + T b + b.
$$

Rearranging and grouping terms gives

$$
x(t + n\Delta t) = T^n x(t) + (I + T + T^2 + \ldots + T^{n-2} + T^{n-1}) b,
$$

(12)

where $I$ is the $4 \times 4$ identity (unit) matrix. Let

$$
S = I + T + T^2 + \ldots + T^{n-2} + T^{n-1}.
$$

(13)

Premultiply by $T$ to obtain

$$
TS = T + T^2 + T^3 + \ldots + T^{n-2} + T^{n-1} + T^n.
$$

(14)

Subtract Equation (14) from Equation (13) to obtain

$$
(I - T) S = I - T^n.
$$

If $(I - T)$ is nonsingular, then $(I - T)^{-1}$ exists and premultiplication by $(I - T)^{-1}$ is possible. So,

$$
S = (I - T)^{-1} (I - T^n).
$$

Using this result, Equation (12) can be re-expressed as

$$
x(t + n\Delta t) = T^n x(t) + (I - T)^{-1} (I - T^n) b.
$$

Let $t = 0$ and $\Delta t = 1$ day, then

$$
x(n) = T^n x(0) + (I - T)^{-1} (I - T^n) b,
$$

(15)

where $x(0)$ is the initial state of the system at time $t = 0$ and $n$ is the number of days, starting to count from $t = 0$. Since the determinant $|I - T| \neq 0$, $(I - T)^{-1}$ can be calculated as follows:

$$
(I - T)^{-1} = \begin{bmatrix}
44.49129 & 44.49129 & 44.49129 & 8.696849 \\
7912.606 & 38681.84 & 7912.606 & 1546.701 \\
46.62843 & 46.62843 & 483.3097 & 9.114602 \\
4.888714 & 4.888714 & 4.888714 & 60.83585 \end{bmatrix}.
$$

(16)

## 4 A system of ODEs (continuous model)

The flow of lead through the human body is based on the basic **mass balance law**

$$
x'_i(t) = \frac{dx_i}{dt} = \sum_j y_{ij} - \sum_j y_{ji} \quad (\mu g/\text{day}),
$$

(17)
where \( x'_i(t) \) is the rate at which the mass of the lead in compartment \( i \) changes with time; \( y_{ij} \) is the time rate at which lead is transferred from compartment \( j \) to compartment \( i \); and \( y_{ji} \) is the time rate at which lead is transferred from compartment \( i \) to compartment \( j \). Applying this law to Bert’s four-compartment model yields the following four coupled linear first-order ordinary differential equations:

\[
\begin{align*}
x'_1(t) &= \tau_{12}x_2(t) + \tau_{13}x_3(t) + [\tau_{14} + 0.08\tau_{64}]x_4(t) + 0.35\alpha + 0.08\beta \tag{18} \\
x'_2(t) &= \tau_{21}x_1(t) - \tau_{12}x_2(t) \tag{19} \\
x'_3(t) &= \tau_{31}x_1(t) - \tau_{13}x_3(t) \tag{20} \\
x'_4(t) &= \tau_{41}x_1(t) - [\tau_{04} + \tau_{14} + \tau_{64}]x_4(t). \tag{21}
\end{align*}
\]

If the values for the transfer coefficients, \( \tau_{ij} \), in Table 1 are substituted into Equations (18) to (21), these equations can be written in matrix form as

\[
x'(t) = Ax(t) + b, \tag{22}
\]

where

\[
x' = \begin{pmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \\ x'_4(t) \end{pmatrix}, \quad x = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}, \quad b = \begin{pmatrix} 0.35\alpha + 0.08\beta \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

and

\[
A = \begin{pmatrix} -0.031015 & 0.0000325 & 0.00229 & 0.0032644 \\ 0.00578 & -0.0000325 & 0 & 0 \\ 0.00240 & 0 & -0.00229 & 0 \\ 0.001835 & 0 & 0 & -0.0167 \end{pmatrix}. \tag{23}
\]

Matrix \( A \) with constant entries is referred to as the coefficient matrix and column vector \( b \) as the input.

## 5 A lead environment

Rabinowitz et al. ([4], 1973) tested a healthy 53-year-old man, weighing 70 kg, whose diet contained, on average, 367 µg/day Pb. It was estimated that the testee inhaled, on average, 49 µg/day Pb. Substituting

\[ \alpha = 49\mu \text{ g/day} \quad \text{and} \quad \beta = 367\mu \text{ g/day} \]

into Equation (11), gives

\[
b = \begin{pmatrix} 46.51 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T. \tag{24}
\]

Consider a healthy adult male who, at time \( t = 0 \), had no lead stored in his body, that is,

\[ x(0) = 0. \]
With this initial condition, Equation (15) becomes

\[ x(n) = (I - T)^{-1} (I - T^n) \mathbf{b}. \]  

Equation (25) predicts the lead content of each compartment, after \( n \) days of lead intake.

Now, consider the nonhomogeneous system (22). Using standard methods for the solution of the system, the following equations are obtained:

\[
\begin{aligned}
x_1(t)_{\text{Blood}} &= \begin{cases} 
-1423.633e^{-0.03161015t} - 72.83207e^{-0.01630369t} \\
+147.924e^{-0.002097818t} - 424.9006e^{-0.00002583925t} 
\end{cases} \\
+2069.29 \\

x_2(t)_{\text{Cortical}} &= \begin{cases} 
260.583e^{-0.03161015t} + 25.87207e^{-0.01630369t} \\
+413.9801e^{-0.002097818t} \\
-3.687158 \times 10^5 e^{-0.00002583925t} + 3.680153 \times 10^5 
\end{cases} \\

x_3(t)_{\text{Trabecular}} &= \begin{cases} 
116.5315e^{-0.03161015t} + 12.4733e^{-0.01630369t} \\
-1847.3e^{-0.002097818t} - 450.3925e^{-0.00002583925t} \\
+2168.688 
\end{cases} \\

x_4(t)_{\text{Tissue}} &= \begin{cases} 
175.2073e^{-0.03161015t} - 337.2319e^{-0.01630369t} \\
+18.58904e^{-0.002097818t} \\
-46.76049e^{-0.00002583925t} + 227.3741 
\end{cases}
\]

The functions (26) to (29) are plotted in Figure 2 below.

![Figure 2: Lead levels during lead intake (continuous model)](image)

The graph for blood in Figure 2 shows that during lead intake, the rate at which lead in the blood increases is rapid at the beginning but slows down with time, and that the lead level in the blood has practically reached equilibrium after 200 days. In other words, after 200 days of lead intake, the rate at which lead is passed from the other compartments to the blood nearly equals the rate at which lead is passed from the blood to the other compartments. During lead intake, the lead level in the soft tissue increases at a much lower rate than that in the blood and also reaches an equilibrium state after 200 days. After 200 days,
blood and tissue levels approach equilibrium, but bone lead soars, especially in cortical bone. These trends of lead levels in the blood, the soft tissue and the skeleton are confirmed by published results.

Equation (25) which has been derived for the discreet model produces Table 2 below:

<table>
<thead>
<tr>
<th>$t$ (days)</th>
<th>$x_1$ (µg)</th>
<th>$x_2$ (µg)</th>
<th>$x_3$ (µg)</th>
<th>$x_4$ (µg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>405 (400)</td>
<td>11 (12)</td>
<td>5 (5)</td>
<td>3 (4)</td>
</tr>
<tr>
<td>20</td>
<td>701 (694)</td>
<td>43 (44)</td>
<td>18 (18)</td>
<td>12 (13)</td>
</tr>
<tr>
<td>30</td>
<td>918 (910)</td>
<td>89 (91)</td>
<td>36 (37)</td>
<td>24 (24)</td>
</tr>
<tr>
<td>40</td>
<td>1077 (1069)</td>
<td>147 (148)</td>
<td>59 (60)</td>
<td>37 (37)</td>
</tr>
<tr>
<td>50</td>
<td>1194 (1186)</td>
<td>212 (213)</td>
<td>85 (85)</td>
<td>51 (51)</td>
</tr>
<tr>
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<td>1280 (1274)</td>
<td>283 (185)</td>
<td>112 (112)</td>
<td>64 (64)</td>
</tr>
<tr>
<td>70</td>
<td>1344 (1338)</td>
<td>359 (360)</td>
<td>141 (141)</td>
<td>76 (76)</td>
</tr>
<tr>
<td>80</td>
<td>1392 (1387)</td>
<td>438 (439)</td>
<td>170 (170)</td>
<td>88 (88)</td>
</tr>
<tr>
<td>90</td>
<td>1427 (1423)</td>
<td>519 (520)</td>
<td>199 (200)</td>
<td>98 (98)</td>
</tr>
<tr>
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<td>1454 (1451)</td>
<td>602 (603)</td>
<td>229 (229)</td>
<td>108 (107)</td>
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<tr>
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<td>687 (687)</td>
<td>259 (259)</td>
<td>116 (115)</td>
</tr>
<tr>
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<td>1490 (1488)</td>
<td>772 (772)</td>
<td>288 (288)</td>
<td>123 (123)</td>
</tr>
<tr>
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<td>1503 (1501)</td>
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<td>317 (317)</td>
<td>129 (129)</td>
</tr>
<tr>
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<td>945 (945)</td>
<td>346 (345)</td>
<td>135 (135)</td>
</tr>
<tr>
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<td>140 (140)</td>
</tr>
<tr>
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<td>1120 (1120)</td>
<td>402 (401)</td>
<td>144 (144)</td>
</tr>
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<td>1296 (1296)</td>
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<td>151 (151)</td>
</tr>
<tr>
<td>190</td>
<td>1541 (1540)</td>
<td>1385 (1385)</td>
<td>482 (481)</td>
<td>154 (154)</td>
</tr>
<tr>
<td>200</td>
<td>1544 (1544)</td>
<td>1474 (1474)</td>
<td>507 (507)</td>
<td>156 (156)</td>
</tr>
</tbody>
</table>

The figures in Table 2 are rounded off to the nearest µg and those within parentheses are the calculations from Equations (26) to (29).

The graphs in Figure 3 depict the results in Table 2, generated by Equation
The graphs displayed in Figure 3 are almost identical to those displayed in Figure 2.

6 A nonlead environment

Suppose that after 200 days of lead intake the testee moves to a nonlead environment, that is, $\alpha = 0$ and $\beta = 0$. The question now is: How will the lead levels in the compartments change with time in a lead-free environment after the 200 days of lead intake? The problem can be stated as follows: Determine $x(t)$ if

$$x(0) = \begin{pmatrix} 1544.4 & 1473.5 & 507.5 & 156.3 \end{pmatrix}^T.$$  

For this situation, Equations (15) and (22) respectively become

$$x(n) = T^n x(0) \quad (30)$$

and

$$x'(t) = Ax(t) \quad (31)$$

since $b = 0$. Using Equation (30), lead levels in the four compartments are calculated for every 10th day, starting from the moment the testee stops taking in lead. Table 3 summarizes the calculations to the nearest $\mu g$, and Figure 4 is a graphic representation of the data. The figures within parentheses in Table 3 are the calculations from Equations (32) to (35) below.

Table 3: Lead levels during isolation from lead

<table>
<thead>
<tr>
<th>$t$ (days)</th>
<th>$x_1$ ($\mu g$)</th>
<th>$x_2$ ($\mu g$)</th>
<th>$x_3$ ($\mu g$)</th>
<th>$x_4$ ($\mu g$)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1474 (1474)</td>
<td>508 (508)</td>
<td>156 (156)</td>
</tr>
<tr>
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<td>1142 (1147)</td>
<td>1551 (1550)</td>
<td>528 (527)</td>
<td>155 (154)</td>
</tr>
<tr>
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<td>849 (857)</td>
<td>1609 (1607)</td>
<td>540 (539)</td>
<td>148 (147)</td>
</tr>
<tr>
<td>30</td>
<td>635 (643)</td>
<td>1651 (1650)</td>
<td>545 (544)</td>
<td>138 (137)</td>
</tr>
<tr>
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<td>478 (487)</td>
<td>1683 (1682)</td>
<td>546 (545)</td>
<td>126 (126)</td>
</tr>
<tr>
<td>50</td>
<td>364 (371)</td>
<td>1707 (1706)</td>
<td>544 (543)</td>
<td>114 (113)</td>
</tr>
<tr>
<td>60</td>
<td>280 (286)</td>
<td>1725 (1724)</td>
<td>539 (538)</td>
<td>102 (102)</td>
</tr>
<tr>
<td>70</td>
<td>218 (224)</td>
<td>1739 (1738)</td>
<td>533 (532)</td>
<td>90 (90)</td>
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<td>401 (401)</td>
<td>13 (13)</td>
</tr>
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</table>
The exact solution to the system of linear first-order ODEs in Equation (31) is

\[
x_1(t) \begin{cases} 
\text{Blood} & = & 1421.076e^{-0.031610t} + 70.03822e^{-0.016304t} \\
+50.68859e^{-0.0020978t} + 2.190083e^{-0.000025839t} 
\end{cases} 
\tag{32}
\]

\[
x_2(t) \begin{cases} 
\text{Cortical} & = & -260.1150e^{-0.031610t} - 24.87961e^{-0.016304t} \\
-141.8571e^{-0.0020978t} + 1900.487e^{-0.000025839t} 
\end{cases} 
\tag{33}
\]

\[
x_3(t) \begin{cases} 
\text{Trabecular} & = & -116.3221e^{-0.031610t} - 11.99482e^{-0.016304t} \\
+633.0078e^{-0.0020978t} + 2.321477e^{-0.000025839t} 
\end{cases} 
\tag{34}
\]

\[
x_4(t) \begin{cases} 
\text{Tissue} & = & -174.8926e^{-0.031610t} + 324.2956e^{-0.016304t} \\
+6.36984e^{-0.0020978t} + 0.2410196e^{-0.000025839t} 
\end{cases} 
\tag{35}
\]

Figure 4: Lead levels in a nonlead environment (discrete model)

Figure 4 shows that as soon as the testee stops taking in lead, his blood lead immediately decreases rapidly, while the lead in his soft tissue decreases slowly. Lead in trabecular bone still builds up during the first 40 days because of the presence of lead in his blood. After about 40 days, the lead in trabecular bone starts to decrease slowly. Even when lead levels in the other compartments are decreasing, the lead in cortical bone continues to increase for a long time. Equation (33) calculates that only after three years does the lead level in cortical bone start to drop. After 10 years there is still a considerable amount of lead in cortical bone. Bone lead acts as an endogenous source of lead, and by continuous release to other tissues it can represent a health risk long after exposure has ended.
REFERENCES


