CONTS

Three-fold activities for discovering conceptual connections within the cognitive neighborhood of a mathematical topic

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Abstract

New technologies provide an efficient tool for widening the mathematical landscape discovered by the students through traditional ways of learning. Compound activities, including numerous techniques, should be developed by educators, in order to enhance the epistemic value of the learning process and enlarge the student's knowledge of the internal connections within the cognitive neighborhood of learned topics. We propose a canvas for a multi-task activity, involving hand-work, CAS assisted computations and related websurfing.

1 Introduction.

In a traditional course, when a student gives his/her homework, there is little chance to discover whether the work has been made following a purely technical path (we mean applications of theorems and methods taught in classroom), or if the student make a personal effort to have a more profound insight into the proposed assignment. Generally, "what you see is what is there"; the student did not try to embed the homework into a braoder frame. An added value can be given via adequate questions and remarks made by the educator. Now suppose that the student is acquainted with various sources of knowledge (such as those mentioned above): the learning process becomes much more comprehensive. It includes the acquisition of the main notions and needed techniques via direct lecturing and practicing under the supervision of educators, together with an enlarging of his/her mathematical world through his/her own research. According to the student's level, this research is either autonomous or driven by the educator's indications. Nevertheless teacher's intervention will be more rare than in the traditional way.

A few years ago, Cuoco and Goldenberg wrote:

> New technology poses challenges to mathematics educators. How should the mathematics curriculum change to best make use of this new technology? Often computers are used badly, as a sort of electronic flash card, which does not make good use of the capabilities of either the computer or the learner. However, computers can be used to help students develop mathematical habits of mind and construct mathematical ideas.([Cuoco and al. 1996])

It happens that even this level of use is not achieved, for various reasons. Among them:

- Despite the expanding availability of new tools, a great number of teachers still convey Mathematics in a traditional way, with ex-cathedra lectures and technical computations. The *pragmatic value* of the teaching is perhaps obtained, but the *epistemic value* generally not (see [Artigue 2002] page 246).
- Educators are not always well-trained to include technological tools in their teaching (we do not discuss that issue here).
- Students manifest a lack of interest for Mathematics, even when they learn a scientific curriculum, and consider Mathematics as a list of receipes for solving specific problems, i.e. only the pragmatic value, at a low level, seems to them worth of an effort.

Thereafter, Cuoco and Goldenberg claimed:

> The mathematics curriculum must be restructured to include activities that allow students to experiment and build models to help explain mathematical ideas and concepts. Technology can be used most effectively to help students gather data, and test, modify, and reject or accept conjectures as they think about these mathematical concepts and experience mathematical research.([Cuoco and al. 1996])

Among the newly available technological tools, we find CAS and the World-wide-web. Therefore new activities are needed, involving their usage, along with "older" techniques, and aimed at the following achievements:

1. Stimulate students' curiosity for interlaced techniques, using more than one of the newly available technologies.
2. Make Mathematics more attractive, and show them as a living field of knowledge by discovering new tracks.
3. Discover links between apparently different fields (in a traditional curriculum, courses are taught as separate topics, often without bridges between them).
4. Last, but not least, traditional libraries offer very small appeal to the average student. Searching the WWW makes him/her more fond of looking for references and documents relevant to his/her learning domain.

For this last point in particular, a well-driven search of the WWW leads to new perspectives on old topics and helps to discover on-going research and interactive mathematical processes. The student is not passive, he/she can influence the teaching process, changing the pace of learning, opening connections, discovering new horizons.

This construction of a "compound cognitive process" fits Artigue's point of view: the paper and pencil part of the work, together with a possible CAS assisted part provide both efficient mathematical practice and conceptual insight into the mathematics involved in the problem under consideration (see [Artigue 2002] p. 246). A well-driven web-search can add a supplementary value to the solution, and generally makes the learning process more efficient, by providing a broader perspective on the problem, its solution and what we could call *the cognitive neighborhood* of the pair problem-solution. To this neighborhood belong domains in Mathematics related to the problem under consideration; the relation can be either obvious from start or be discovered during the student's autonomous work. The example we develop in section 2 shows combinatorial identities belonging to the cognitive neighborhood of some parametric integrals, two topics generally taught in independent courses.

Parametric integrals are often the central topic of exercises leading to induction formulas and/or closed formulas (see [Glaister 2003], [Dana-Picard 2003a]). As a consequence of these formulas, various properties of the sequence of integrals are derived.

Such an exercise can be proposed as a stand-alone task; its aim is training fundamental mathematical skills at a higher level of abstraction, with more advanced examples. Recall that parametric problems are important in Physics. The frame of the exercise can be broaden and becomes what we will call a *compound activity*; for example:

1. Consider other mathematical objects "looking like" the mathematical object of study. Such a task is generally built by the teacher, i.e. in this context, the teacher is active and creative; the student reproduces the teacher's working steps.

2. Incite the students to search for related material, in particular using the WWW. Links to neighboring mathematical topics, to "real-world" situations can be discovered. Here the student is more autonomous and can develop more initiative. Actually, both the educator and the student are creative.

In section 2 we present a family of definite integrals depending on two parameters. The computations are not straightforward: we show the different steps without the technicalities. For details, see [Dana-Picard 2003c]. The role of the two parameters are very different, and general conclusions are looked for. Using all the tools at their disposition (books, CAS, Internet connection), the students can participate in a constructing activity.

2 An infinite sequence of parametric integrals.

Let a be a positive real number. For any non negative integer n, consider the following definite integral

$$I_n = \int_0^a x^n \sqrt{a^2 - x^2}\, dx. \tag{1}$$

The first two integrals of the sequence are easy to compute; we have:

$$I_0 = \int_0^a \sqrt{a^2 - x^2}\, dx = \left[\frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2}\right]_0^a = \frac{1}{2} a^2 \arcsin 1 = \frac{a^2\, \pi}{4}, \tag{2}$$

and, by an integration by parts,

$$I_1 = \int_0^a x \sqrt{a^2 - x^2}\, dx = \left[-\frac{1}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{a^3}{3}. \tag{3}$$

Now let us compute an induction formula for the sequence (I_n). An integration by parts, with

$$u(x) = x^n \qquad \text{and} \qquad v(x) = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a},$$

leads to the following result:

$$I_n = \frac{1}{2} a^{n+2}\, \frac{\pi}{2} - \frac{n}{2}\, I_n - \frac{1}{2} a^2 n \int_0^a x^{n-1} \arcsin \frac{x}{a}\, dx.$$

Denote

$$K_n = \int_0^a x^{n-1} \arcsin \frac{x}{a}\, dx.$$

Thus, the following identity holds:

$$\left(1 + \frac{n}{2}\right)\, I_n = \frac{\pi}{4} a^{n+2} - \frac{1}{2} a^2 n\, K_n. \tag{4}$$

Now we compute K_n. By an integration by parts, with

$$u_1(x) = x^{n-1} \qquad \text{and} \qquad v_1(x) = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2},$$

we obtain the relation

$$nK_n = a^n\, \frac{\pi}{2} - (n-1)\, I_{n-2}. \tag{5}$$

We substitute the right-hand side of Equation (5) into Equation (4). Finally, the following relation holds:

$$I_n = \frac{a^2\, (n-1)}{n+2}\, I_{n-2}. \tag{6}$$

3 Interpretation.

The relation (6) defines a recurrence with a step equal to 2; therefore we need to distinguish two subsequences, according to whether the index is even or odd. This justifies a posteriori the previous computation of the first two integrals in the given sequence.

First, suppose that n is even; we denote $n = 2p$, $p \in \mathbb{N}$. Equation (6) becomes:

$$I_{2p} = \frac{a^2\ (2p-1)}{2(p+1)}\ I_{2p-2}. \tag{7}$$

On the one hand, telescopic methods as in [Dana-Picard 2003a] provide a closed form for I_{2p}, namely

$$I_{2p} = \left(\frac{a}{2}\right)^{2p+2} \frac{(2p)!}{p!\ (p+1)!}\ \pi, \tag{8}$$

On the other hand, the following data can be discovered by a quick web-search. The numbers

$$C(p) = \frac{(2p)!}{p!\ (p+1)!}$$

are called *Catalan numbers*. The interested reader will look at [Cossali 2003], [On-Line Encyclopedia 2003] and [Dickau 1996]. They can also be defined as the terms of the sequence of integers given by

$$\begin{cases} a_1 = 1 \\ a_p = \frac{2(2p-1)}{p+1} a_{p-1} \end{cases}.$$

They have numerous combinatorial occurences. Among them

- counting triangulations of a convex $(n+2)-$gon into n triangles by $n-1$ diagonals which do not intersect at interior points;
- counting binary parenthesizations of a string of $n+1$ letters (Schröder's problem).
- the number of paths of length $2n$ through an $n \times n$ grid that do not rise above the main diagonal.

See [Dickau 1996] or [MacTutor 2003], or make your own web-search, for more details.

Our computations show that for every natural p, the integral (I_{2p}) is the product of $a^2\pi/2$ by the corresponding Catalan number; for $a = 2$, the coefficient of π is the Catalan number itself.

We found here an integral presentation for Catalan numbers: for any positive integer p,

$$C(p) = \frac{1}{\pi} \int_0^2 x^{2p} \sqrt{4 - x^2}\ dx. \tag{9}$$

Actually, another integral presentation for Catalan numbers is already known (see [On-Line Encyclopedia 2003]), namely:

$$C(n) = \int_0^4 x^n \sqrt{\frac{4-x}{x}}\, dx. \tag{10}$$

Our presentation can be obtained from this one by substitution; a proof of this can be an additional task.

Finally, we should note that considering the subsequence (I_{2p+1}) shows also an occurence of Catalan numbers, but they appear multiplied by a non trivial coefficient. We leave these computations to the reader.

4 Computer assisted activities.

Maybe the computation by hand of the induction relation and of the closed form for the given integrals is beyond the abilities of a student. In [Dana-Picard 2003b] we described such a situation, giving examples where the usage of a Computer Algebra System can help to fill the gap. We think that here the situation is different.

If we leave to the student the choice of which commands to use, his/her decision could be to enter directly the parametric form of the integral, no matter which CAS is used. Then he/she would substitute special values for the parameter a and compute I_n for small values of the parameter n.

For some CAS, this is feasible, despite the fact that a general pattern does not appear: the given integral is left as it is, not transformed into another equivalent expression. The successive substitutions give separate answers without a visible general pattern. A web-search, in particular through ad-hoc interactive sites like [On-Line Encyclopedia 2003] or [MacTutor 2003], provides sometimes a remedy to this problem, by enabling the student to find either previous work on the same topic, or a pathway into further inquiry. The following frame could be accurate:

i. Using a CAS, compute I_n, for n equal to $0, 1, 2, 3, \ldots, 10$.

ii. Look for a pattern in the output. Here two distinct patterns appear.

iii. Connect to the site [On-Line Encyclopedia 2003]; enter the values of the coefficient of π in the first few terms of the even-index subsequence. This provides a conjecture for a general formula for I_{2p}. In this example, the web-answer is unique; in other cases, there can be multiple propositions. Further work is then needed in order to make a decision.

iv. With the CAS check the conjecture for greater values of the parameter.

v. Re-do these four steps for odd indices, after suitable modification.

Of course, such a process does not provide a proof of an explicit formula, only some kind of conviction is afforded.

With another CAS, a pattern may appear, involving the Gamma function (this function generalizes the factorial to non integer positive numbers; see [Thomas' Calculus] page 605; it is generally taught only in an advanced course).

$$I_n = \frac{a^{n+2}\sqrt{\pi}\,\Gamma((n+1)/2)}{4\,\Gamma(2+n/2)} \tag{11}$$

In such a case, the hope to bypass the lack of knowledge using the CAS is deceived: the student replaced his/her problem by a problem still worse from his/her point of view: he/she cannot understand the actual meaning of the output, the CAS is used as a blackbox and the pedagogical aspect of the work is totally lost. No conceptual understanding is afforded; we claimed already in [Steiner and Dana-Picard 2004] that the usage of such "high level" commands (here "high level function" could be more appropriate) does not help to build conceptual understanding. Or maybe the educator can catch this opportunity to reverse the trend, by giving a definition of the Gamma function and showing its first properties; this is part of the educator's building of the theoretical discourse accompanying the technique. The "bad problem" becomes a motivating example for further discovery. Nevertheless, the educator must pay attention to the danger inherent to the multiplication of the goals of an activity: none of them is totally achieved. In our opinion, the usage of commands whose output involves the Gamma function should be postponed to a later task, after the present one has been fully performed. In other words, the cognitive neighborhood of a given topic can be very large, even too large for the student to be able to find a reasonable pathway for an exhaustive exploration. The educator must make the appropriate choices: which topic is at a "reasonable distance" from the main topic under consideration, and which one is too far away at that time?

At this point, we should emphasize the fact, already mentioned in the previous paragraph, that the theoretical discourse for instrumented techniques required in [Artigue 2002] is intimately connected to the choice of the CAS (for example, shall we explain the Gamma function or not?). Moreover, in our activity model, the discourse has to include a subdiscourse aimed to master ways of web-searching to broaden mathematical horizons, and not getting lost in this huge amount of more or less relevant sources of information.

Nevertheless we do not see computation of our parametric integrals using a computer as a hopeless endeavour. At least, the discovery of Catalan numbers is a good topic for a computerized activity, mixing hand-work, CAS assisted computations and WWW-search. Let us propose a canvas:

1. Compute by hand the two first integrals I_0 and I_1 for general a. If needed, i.e. here when the student does not master trigonometric substitution, a primitive of $x \mapsto \sqrt{a^2 - x^2}$ can be computed using either a CAS or a table of integrals. Here an educator could prefer the second tool, because of the risk of appearance of the Gamma function. Why should we not present both tools and ask students to discuss their choice of the tool?

2. The computational skills acquired in the previous step enable the student to compute the induction formula connecting I_n with I_{n-2}. This task is

impossible with a CAS; we are beyond the parametric problems described in [Artigue 2002], p. 267.

3. Now the educator can lead the students in one of the directions of the previous canvas.

5 Conclusion.

The frame of the author's courses is fixed by the institution where he teaches; the added value of extra tasks, not officially present in the syllabus but given as pilots, is received by most students as a "plus" in their education. After some adaptation process, their reaction is very positive and, for example, the best results of their explorations are dispatched among their peers, generally via the electronic forum of the class.

The solution of an "old" problem with new techniques has always a great mathematical value and a pedagogical interest. Therefore the introduction into the curriculum of compound activities, including traditional ways of doing mathematics together with the most up-to-date technologies, is important. As we claimed, the widening of the mathematical landscape provided by new technological tools reinforces the students' will for a deeper understanding of what they learn and stimulates them to further learning. A few days ago, a former student, whose name is Dor, came to the author's office, asking for extra mathematical material on a certain topic and for personal help. He said: "I learnt this material, but I still want more profound insight into what these objects are". In Dor's words, this means: it is not sufficient in my eyes to know only what has been taught, I wish to understand more profoundly the nature of the mathematical objects under study, and the connections between them. In our words, Dor wishes to explore the cognitive neighborhood of his topic.

Another advantage of this teaching-learning process is that the student is self-teaching, at least part of the time. The learning process is composed of a synchronous part (as in a traditional process) and an asynchronous part (mostly the exploration of the problem's cognitive neighborhood), the importance of this asynchronous work being emphasized.

Our enthusiasm for this kind of mixed learning process must yet be tempered. The example of a compound mathematical activity that we described here shows an application of Lagrange's claim in [Lagrange 2000] p. 27. The coordination of new techniques with the traditional ones, will not change in a miraculous way the learning process. With new technological tools, some results will be obtained more quickly, but such a compound activitiy demands profound reflexion from the educator, and "demands from the students time and efforts for their passage towards theory". "The difficulties encountered when implementing new praxeologies should not be underestimated".

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