Real numbers representations and charts

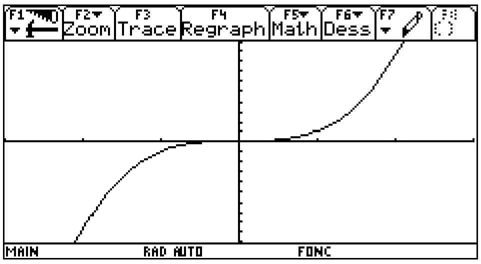
A mathematical investigation carried out with K12 students in 2001

The (mathematical) crime
Picture 1

A mathematics teacher in his class in full confidence in the computer plots the chart of :

$$x \mapsto x^{\pi}$$

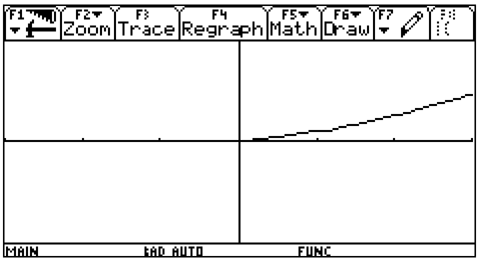
Here is what he obtains:



Picture 2 : the good news

This is a curse related to π number?
New test with this time:

$$x \mapsto x^{\sqrt{2}}$$



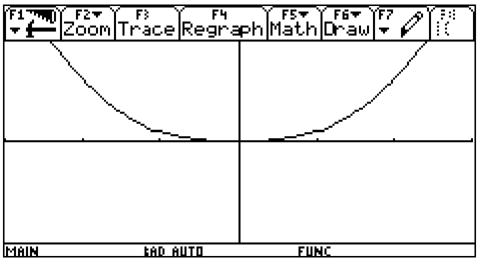
Our teacher regains confidence

Picture 3: New question

It was thus π … The teacher, reassured, can continue his lesson.
But, at the back of the classroom, a student, curious and dissatisfied, tests another function:

$$x \mapsto x^{\sqrt{7}}$$

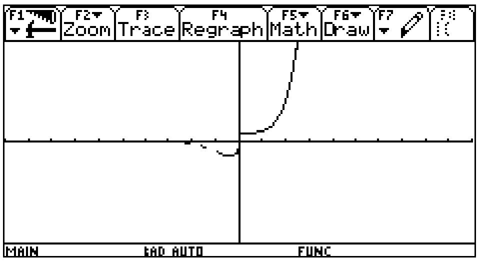
CONTS



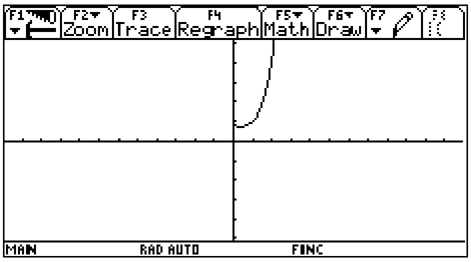
Picture 4 : Another class, others students

The same year, at the same time, in another class, two students plot the function $x \mapsto x^x$
But they did not choose the same "Zoom".

Zoom : Standard



Zoom : Decimal

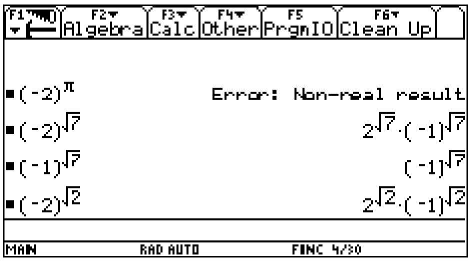


The beginning of the investigation

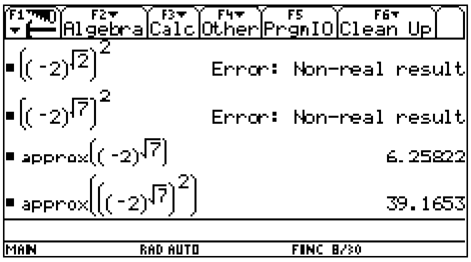
Embarrassed, the teacher calls for help from an investigator in computer "bugs".
The two students also make contact with this "specialist"…
He rejects the idea of a "bug", or a programming error.
Our man directs himself immediately towards the search for a mathematical reason… Yes, but which one???

Calculative investigation

Some small things should initially be checked and already some surprises

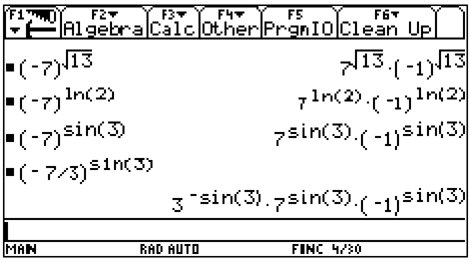


And even more surprises

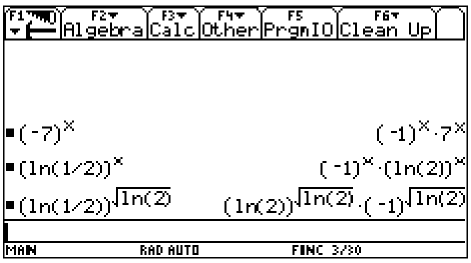


First conclusions

The calculator observes rules of purely algebraic transformation when real powers are used for which there can be a doubt about the nature of the result.

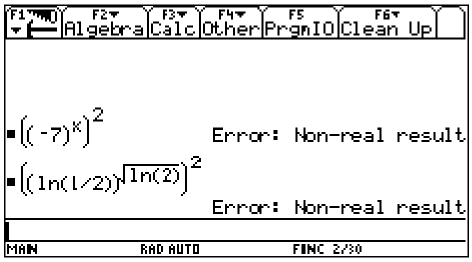


It does not examine if the writing is correct.

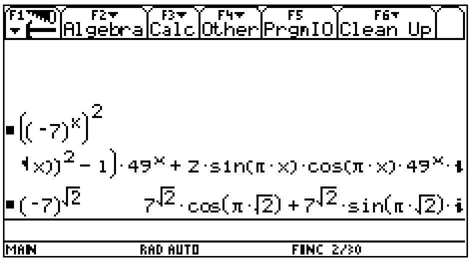


We see here a little more clearly

The application of the square forces the evaluation: whereas the calculator knows the result is not a real number.

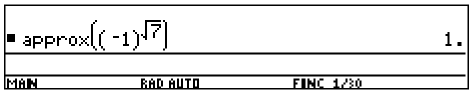


Moreover, in complex format: rectangular, we obtain

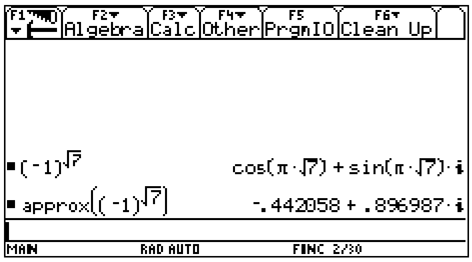


A clue

The passage in approximate mode highlighted a problematic situation:

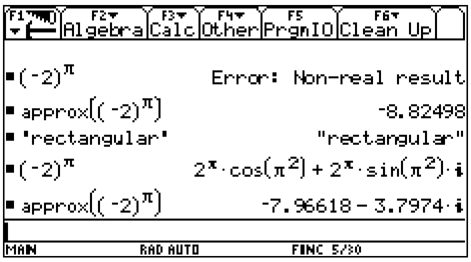


However if we take the complex format: rectangular, we find:
It seems well that the calculator "knows" that the number $(-1)^{\sqrt{7}}$ is a complex number, but in approximate mode, it regards it as a real number

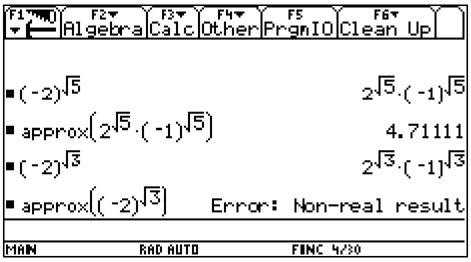


Let us go further in our research

We met an aberrant arc for x^{π}. We must thus have an identical situation to that of $x^{\sqrt{7}}$.
Which is true.

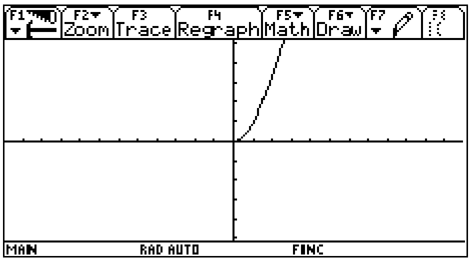


This observation allows forecasts.
Let us examine the two examples opposite:

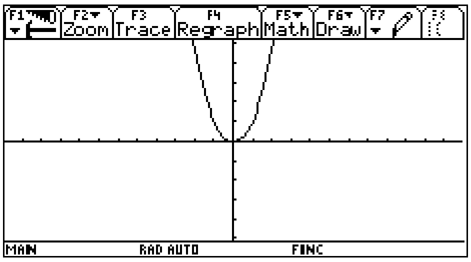


A small verification

On the basis of our first result, we can think that $x \mapsto x^{\sqrt{3}}$ will be represented correctly, but that $x \mapsto x^{\sqrt{5}}$ will be represented like an even function.
Which is true…



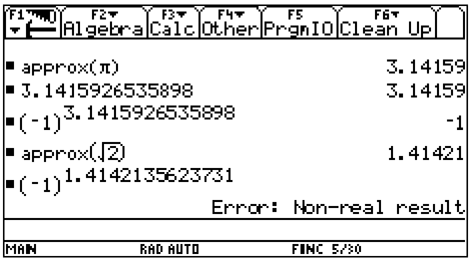
$x \mapsto x^{\sqrt{3}}$



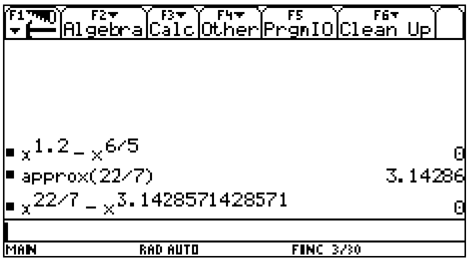
$x \mapsto x^{\sqrt{5}}$

Now let's return to x^{π}

In approximate mode, π is replaced by a decimal number, $\sqrt{2}$ also.



We must thus ask ourselves what a power with a decimal exponent corresponds to: the calculator "answers" by an equality for any x which shows that the representation in memory of a power with a decimal exponent is **a power with a rational exponent**.



We are advancing…

To plot a curve, the calculator works in approximate mode, therefore for x^{π}, π is replaced by a decimal number.
But to calculate the power, this decimal number is replaced by a fraction of integers, which we can suppose is irreducible…
It remains to examine the powers whose exponent is rational.

Take a glance at this : $x^{p/q}$

If p and q are two positive integers such as GCD(p, q) = 1, we have three cases :

1) p and q are odd
2) p is even, q is odd
3) p is odd, q is even.

We can always write : $x^{p/q} = (x^p)^{1/q}$
If q is odd, $x \longrightarrow x^q$ is a bijection over IR, and thus $a^{1/q}$ exists in IR for every real number a.

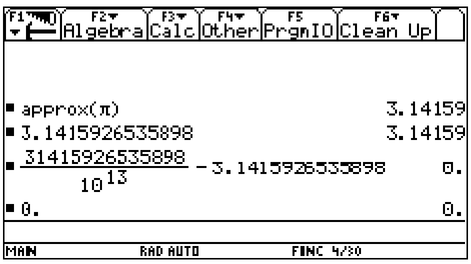
Small assessment on the investigation

We can thus think that the calculator replaces π or $\sqrt{5}$ by a rational number whose denominator is odd, and replaces $\sqrt{2}$ or $\sqrt{3}$ by a rational number whose denominator is even.
We have really advanced…

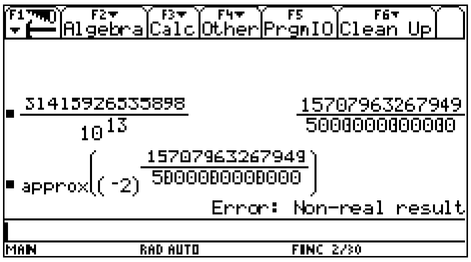
And, now, in search of the rational number

Let us examine the case of x^{π}.

Initially we can think that π is replaced by $\dfrac{31415926535898}{10^{13}}$

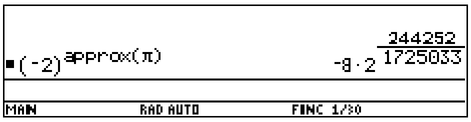


However, that does not work: while simplifying, we obtain an irreducible fraction with an even denominator

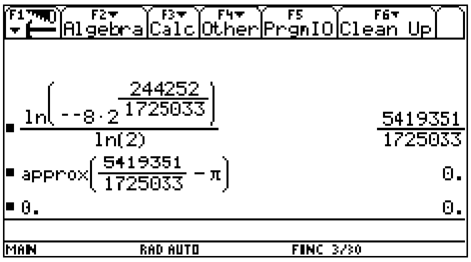


An idea…

If our assumptions are right, to carry out the calculation of $(-2)^{\pi}$ in approximate mode, the calculator will go to a rational form.



It is good. And now, let us use formal calculation.



Here is the rational approximation used by the calculator for π.

$$\pi \approx \frac{5419351}{1725033}$$

How and why does the calculator use this rational expression as approximate value of π?

To approach a real number by a rational number, it is a known problem of which a method of resolution is

The continued fractions

Continued fractions

$$3{,}141592653598 = 3 + 0{,}141592653598 = 3 + \frac{1}{\dfrac{1}{0{,}141592653598}}$$

Which gives :

$$3{,}141592653598 \approx 3 + \frac{1}{7{,}0625133059307}$$

We thus have

$$3{,}141592653598 \approx 3+\frac{1}{7}\left(=\frac{22}{7}\right)$$

Then

$$3{,}141592653598 \approx 3+\cfrac{1}{7+\cfrac{1}{\cfrac{1}{0{,}0625133059307}}} \approx 3+\cfrac{1}{7+\cfrac{1}{15{,}99594406774}} \approx 3+\cfrac{1}{7+\cfrac{1}{15}}\left(=\frac{333}{106}\right)$$

We can write :

$$\pi \approx q_1+\cfrac{1}{q_2+\cfrac{1}{q_3+0{,}9959...}}$$

with $q_1 = 3$, $q_2 = 7$, $q_3 = 15$…

If we name $\frac{a_1}{b_1},\frac{a_2}{b_2},\frac{a_3}{b_3}...\frac{a_n}{b_n}...$ the successive continued fractions, we have here :

$$a_1=3, b_1=1, a_2=22, b_2=7, a_3=333, b_3=106$$

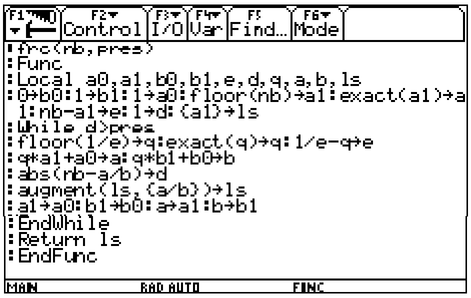
If we write $x = \pi$, and $e_{n+1}=\frac{1}{e_n}-q_{n+1}$ with $e_1 = x - \text{ipart}(x)$

We can show recursively that

$$q_{n+1}=ipart\left(\frac{1}{e_n}\right) \;; a_{n+1}=q_{n+1}\times a_n+a_{n-1} \;; b_{n+1}=q_{n+1}\times b_n+b_{n-1}$$

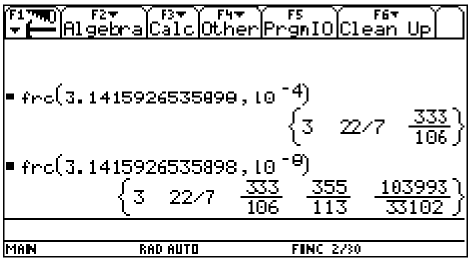
with $a_0 = 0$ et $b_0 = 1$.

We translate these results by a program on the calculator which returns a list of fractions approaching a number (nb) with a given precision (pres)

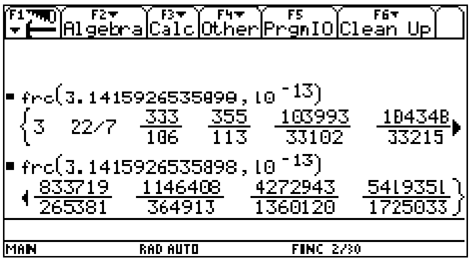


Uses of the program

We have for example



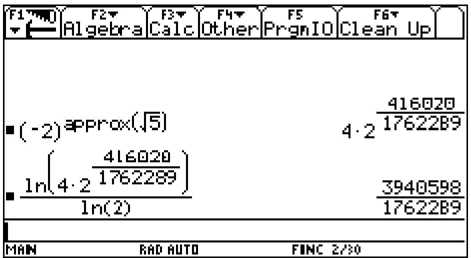
But also with more precision :



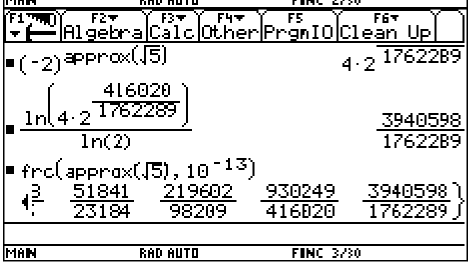
We find again the rational number used by the calculator.

Verification of the hypothesis

The case of √5 is taken again.
We have:



We also have :

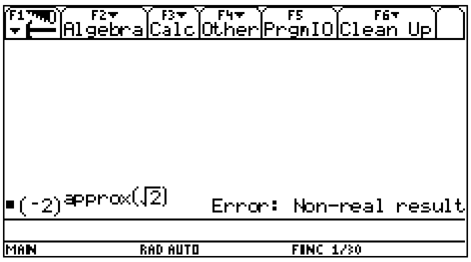


The result is convincing

And with the root of 2

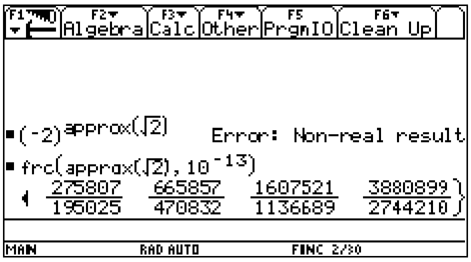
There are no anomalies in the curve of $f(x) = x^{\sqrt{2}}$.

So this will not give us any information;



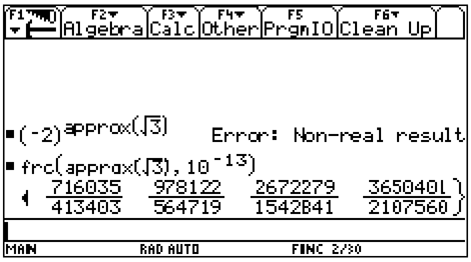
Let's try the program :

The last fraction obtained has an even denominator … which is what we expected



Let 's try with √3 or √7

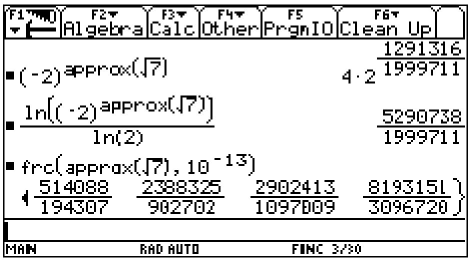
Confirmations of our results are found for √3 but…



… not for √7.

Why ?

We can suppose it is a problem of precision, our program is using the approximate values of the calculator…



More precision, please

The calculator has reached its limits.
It is necessary to push the research to software in which we can choose the precision. Let us take Derive…
We write the same program with two small alternatives:
APPEND instead of AUGMENT and
LOOP instead of WHILE

With Derive

```
PROG(
    a0 := 1,a1 := FLOOR(nb),b0 := 0,b1 := 1,e := nb - a1,d := 1,ls := [a1],
LOOP(
        IF(d < pres, RETURN ls),
        q := FLOOR(1/e),e := 1/e - q,a := q·a1 + a0,b := q·b1 + b0,
        d := ABS(nb - a/b),ls := APPEND(ls, [a/b]),
        a0 := a1,b0 := b1,a1 := a,b1 := b)
         )
```

A surprising result

```
frc(2.64575131106459059, 10^-13)
```

$$\left[2,\ 3,\ \frac{5}{2},\ \frac{8}{3},\ \frac{37}{14},\ \frac{45}{17},\ \frac{82}{31},\ \frac{127}{48},\ \frac{590}{223},\ \frac{717}{271},\ \frac{1307}{494},\ \frac{2024}{765},\ \frac{9403}{3554},\ \frac{11427}{4319},\ \frac{20830}{7873},\ \frac{32257}{12192},\ \frac{149858}{56641},\ \frac{182115}{68833},\ \frac{331973}{125474},\ \frac{514088}{194307},\ \frac{2388325}{902702},\ \frac{2902413}{1097009},\ \frac{5290738}{1999711},\ \frac{8193151}{3096720}\right]$$

This number missing from our list on the calculator…
We can check that it is the only one.

An interpretation… to follow…

We can think the reason for this "incident" is a particular situation of one of the numbers q_n, combined with the limited precision of the calculator
To be continued…
But a new enigma: why does the calculator choose the one that is precisely missing, whereas it is not the last number …

Who's following who?

Derive will still give the solution: the sequence is alternating around its limit: the minor term is here before the last and the result is thus coherent. It is thus well for a problem of precision of calculation that the calculator stopped.

```
u := frc(2.64575131106459059, 10^-13)

VECTOR(SIGN(u_(k+1) - u_k), k, 1, DIM(u) - 1)

[1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1,
 -1, 1]
```

And in the general case

In the general case, the sequence of the continued fractions has the same property. We can easily see it in the first terms of the sequence with the preceding conventions. Thus :

$$\frac{a_1}{b_1} = q_1, \frac{a_2}{b_2} = q_1 + \frac{1}{q_2}, \frac{a_3}{b_3} = q_1 + \frac{1}{q_2 + \frac{1}{q_3}}, \frac{a_4}{b_4} = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{q_4}}}$$

We have $\frac{a_1}{b_1} \le \frac{a_2}{b_2}$

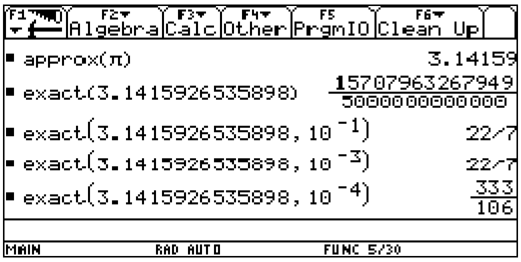
But too $q_2 \le q_2 + \frac{1}{q_3}$ thus $\frac{a_3}{b_3} \le \frac{a_2}{b_2}$.

We also have $q_3 \le q_3 + \frac{1}{q_4}$ donc $\frac{a_4}{b_4} \le \frac{a_3}{b_3}$.

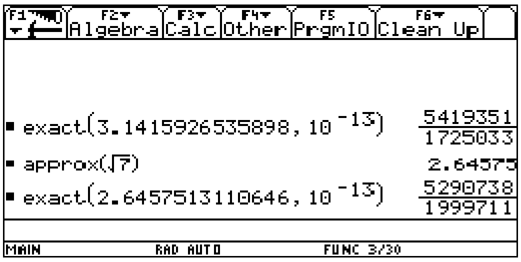
And so on…

The function exact

The calculator has a function badly described by TI: the function EXACT.



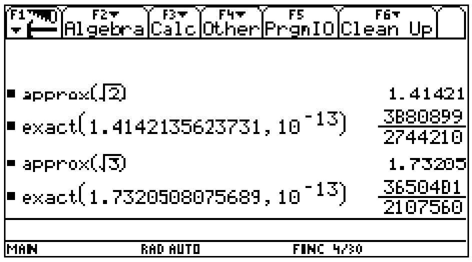
The results on the screen on the right are saying something to us.
And if we go further:



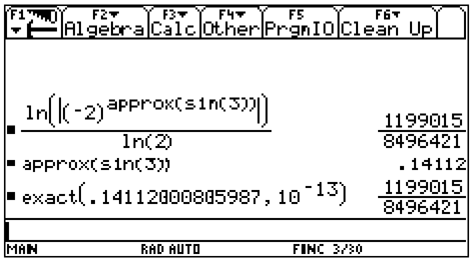
It seems the function EXACT returns the best continued fraction which approaches a real number with a given precision.

And with the other values…

There are apparently new confirmations



We can test with a completely new number



And if we looked at x^x

We could see that the curve of this function was very different according to the choice from the "Zoom".
In the "decimal Zoom", it did not seem to have a problem. In the "standard Zoom", parasitic points or arcs appear.

We are now able to understand the differences observed and to interpret them.

x^x with decimal Zoom

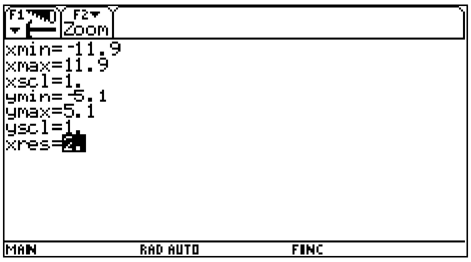
In this case, the situation can appear easy. The values of x are decimal numbers from the form k*0.1 with k integer number. Thus x^x is replaced by:

$$\left(\frac{k}{10}\right)^{k/10}$$

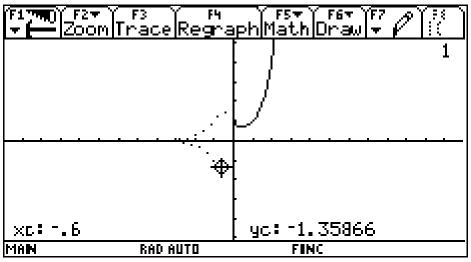
If k is a negative number, and if k is an even number, the fraction is simplified and we will have a problematic point. If k is odd, calculation cannot be done.

x^x and the choice of the resolution

Our conclusion results in thinking that the resolution selected on the calculator is important. A quick check shows that the drawing without "problem" was with resolution 2.



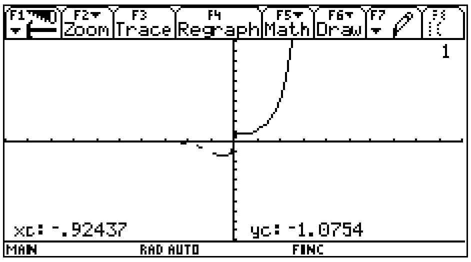
But with resolution 1, there are parasitic points which correspond to the even values of k.

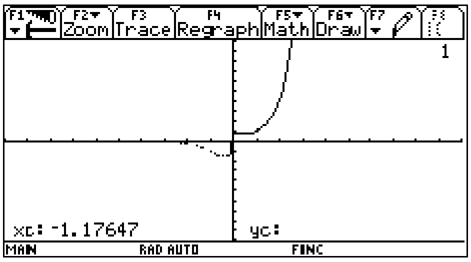


x^x *with standard zoom*

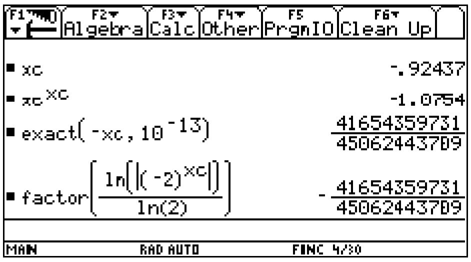
In this case, everything is more complicated: the points calculated by the calculator have a rational approximation by continued fraction with an even denominator or an odd denominator and that done much difference …

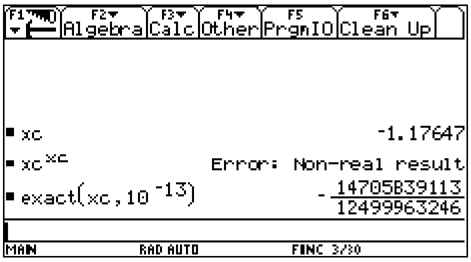
Let us look at these two cases .





The two selected points

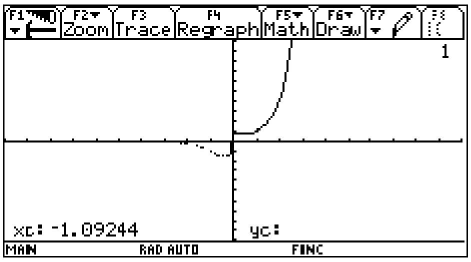


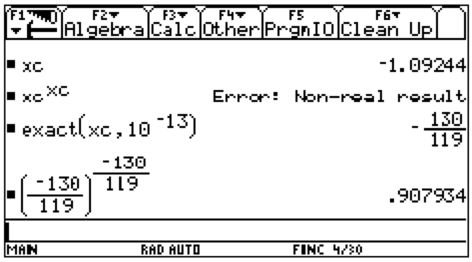


The calculations seem to confirm our assumptions in both cases.

But nothing is easy…

Indeed, we could have made a worse choice as shown in the following example:

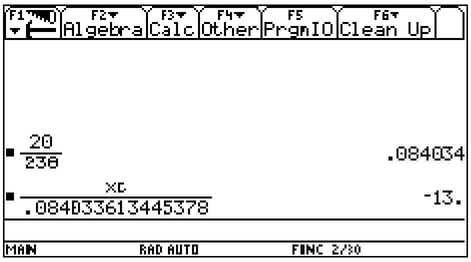


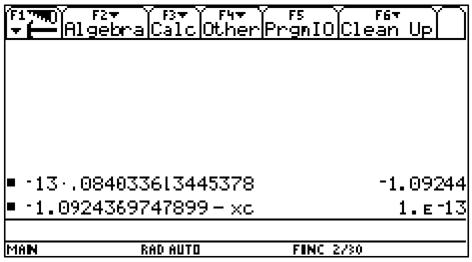


That goes much worse.
A denominator is found odd which is not coherent

What is this point?

The screen of the calculator contains 238 calculated pixels.
In standard zoom, the X-coordinates are between -10 and 10.
We can calculate "more precisely" the value of xc.
We find a difference which shows the difficulty to know the value used by the calculator





Help from Derive

If, by example, the calculator has "taken" the value 1.092436974789 for xc, Derive give as continued fractions, the last being smallest and more precise than the preceding one.
Why EXACT this value does not return?
New question to be continued…

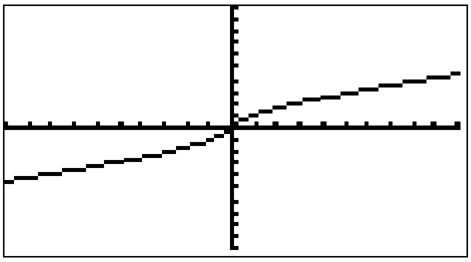
$$\mathrm{frc}(1.092436974789,\ 10^{-13})$$

$$\left[1,\ \frac{11}{10},\ \frac{12}{11},\ \frac{59}{54},\ \frac{71}{65},\ \frac{130}{119},\ \frac{10022357481}{9174311848}\right]$$

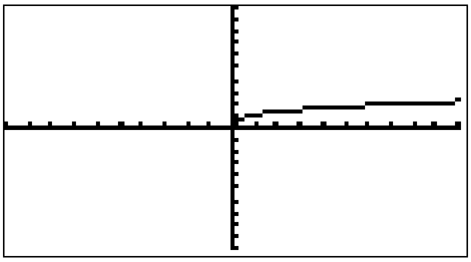
$$\frac{10022357481}{9174311848} - \frac{130}{119} = -\frac{1}{1091743109912}$$

And if we took another calculator…

Let us look at what occurs with TI-84.
First, we plot the curves of the function $x \mapsto x^A$



with A = 67/103

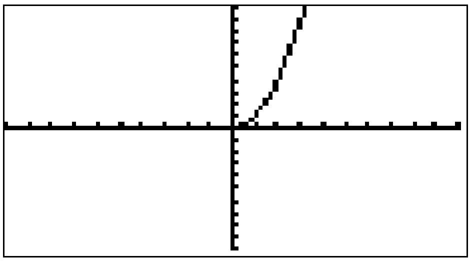


and with A = 67/208

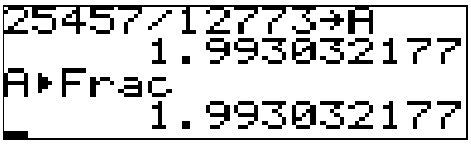
The results are those until we wait (odd or even denominators)

The troubles start

I we take now A = 23457/12773, The curve is « incomplete »



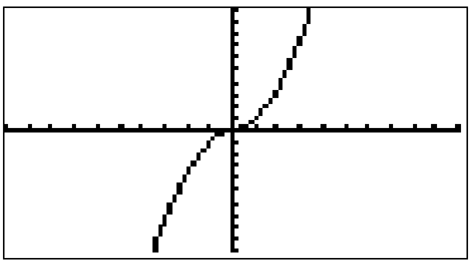
TI-84 has "a small" function EXACT : the instruction FRAC.
Let us look at what it gives here.



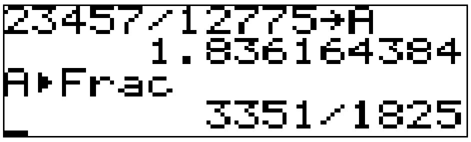
It does not modify anything.

The small difference

A small change in the value of A and that works well again: A = 23457/12775

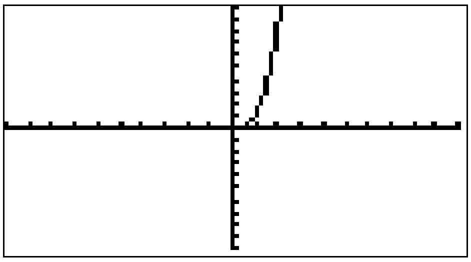


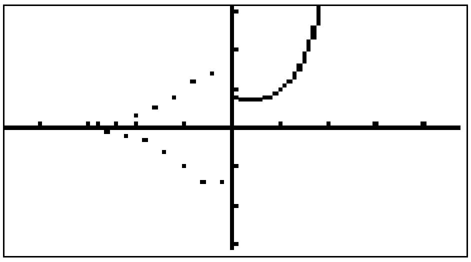
And for FRAC…



Brief conclusion for TI-84

Like Voyage 200, TI-84 plots arc for the negative X-coordinates only if it be able to write the exponent like a rational number with odd denominator, what avoids problems for x^{π} (left screen) but not for x^{x} with decimal zoom… (right screen)





And with Derive…

Derive has been an invaluable assistant, by the possibility that it offers to define the precision of the results.
But does it avoid the problems involved in the power functions?
On the graphic point of view, we can think that it is the case. The curves correspond so that the professors of mathematics wait, but in the field of calculation, we still meet some surprises…

Derive and the power functions

Precision := Approximate

PrecisionDigits := 19

$$(-2)^{\pi} = -7.966178303885685737 - 3.797398698989756366 \cdot i$$

PrecisionDigits := 18

$$(-2)^{\pi} = -7.96617830388568573 - 3.79739869898975636 \cdot i$$

Until here, everything is OK.

But of the problems appear.
First, here is how $(-2)^{\pi}$ becomes a positive real number

CONTS

```
PrecisionDigits := 18

Branch := Real
```

$$(-2)^{\pi} = 8.8249778270762876$$

Then a negative real number and even a complex number:

```
PrecisionDigits := 20
```

$$(-2)^{\pi} = -8.8249778270762876238$$

```
PrecisionDigits := 21
```

$$(-2)^{\pi} = -7.96617830388568573823 - 3.79739869898975636583 \cdot i$$

In exact mode, the calculator had a "normal" behavior, but Derive is more surprising.

```
Precision := Exact

Notation := Rational
```

$$(-2)^{\pi} = 2^{\pi}$$

We must use the complex numbers to find something of more usual

```
Branch := Principal
```

$$(-2)^{\pi} = 2^{\pi} \cdot e^{\pi^2 \cdot i}$$

And if Derive worked like the calculators…

We are in Exact mode with an accuracy of 20 digits. The obtained screens are speaking about themselves

$$(-2)^{APPROX(\pi)} = -2^{21053343141/6701487259} \cdot (-1)^{948881364/6701487259}$$

$$frc(APPROX(\pi), 10^{-20})$$

$$\left[3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \frac{312689}{99532}, \frac{833719}{265381}, \frac{1146408}{364913}, \frac{4272943}{1360120}, \frac{5419351}{1725033}, \frac{80143857}{25510582}, \frac{165707065}{52746197}, \frac{245850922}{78256779}, \frac{411557987}{131002976}, \frac{1068966896}{340262731}, \frac{2549491779}{811528438}, \frac{6167950454}{1963319607}, \frac{21053343141}{6701487259}\right]$$

If the precision is modified:

PrecisionDigits := 18

$$(-2)^{APPROX(\pi)} = -2^{6167950454/1963319607} \cdot (-1)^{277991633/1963319607}$$

$$frc(APPROX(\pi), 10^{-18})$$

$$\left[3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \frac{312689}{99532}, \frac{833719}{265381}, \frac{1146408}{364913}, \frac{4272943}{1360120}, \frac{5419351}{1725033}, \frac{80143857}{25510582}, \frac{165707065}{52746197}, \frac{245850922}{78256779}, \frac{411557987}{131002976}, \frac{1068966896}{340262731}, \frac{2549491779}{811528438}\right]$$

We are in front of a small problem
We must go a little further to find the approximation rational. Question of precision ???

$$\text{frc}(\text{APPROX}(\pi),\ 10^{-19})$$

$$\left[3,\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{103993}{33102},\ \frac{104348}{33215},\ \frac{208341}{66317},\ \frac{312689}{99532},\ \frac{833719}{265381},\ \frac{1146408}{364913},\ \frac{4272943}{1360120},\ \frac{5419351}{1725033},\ \frac{80143857}{25510582},\ \frac{165707065}{52746197},\ \frac{245850922}{78256779},\ \frac{411557987}{131002976},\ \frac{1068966896}{340262731},\ \frac{2549491779}{811528438},\ \frac{6167950454}{1963319607}\right]$$

But Derive is more explicit than the calculator:

PrecisionDigits := 19

$$(-2)^{\text{APPROX}(\pi)} = -\,2^{14885392687/4738167652}\cdot(-1)^{670889731/4738167652}$$

PrecisionDigits := 18

$$(-2)^{\text{APPROX}(\pi)} = -\,2^{6167950454/1963319607}\cdot(-1)^{277991633/1963319607}$$

PrecisionDigits := 20

$$(-2)^{\text{APPROX}(\pi)} = -\,2^{21053343141/6701487259}\cdot(-1)^{948881364/6701487259}$$

Now, we understand why we have found a complex number with an accuracy of 19, a positive real number with an accuracy of 18 and a negative real number with an accuracy of 20.

Provisional end…

Here is " the end" of this mathematical investigation, carried out with the active collaboration of students, investigation which led everyone to a better comprehension of problematic situations met on a calculator or a computer. But it also allowed and especially to meet unexpected mathematical concepts.
Ultimately, everyone understood that confidence that we must make with an calculation instrument must be measured. But, and it is the most important point, now students and teacher do not speak any more of the limits supposed about these instruments. The understood that these limits are those which imposes mathematics itself.