Mathematical Arguments and their formalization within a Dynamic World.

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The Deductive Method in Mathematics

Pre-Euclidean Greek geometry evolved as a model of physical space. An important example is constituted by Erathostenes' method to measure the Earth. It appears that these examples, in which it is not possible to establish empirically their validity, were instrumental to develop the deductive method as a validation tool. This way it was possible to reach more than plausible information on the structure of physical space *without touching it*. It is understandable that developing this viewpoint might have been a first step towards the abstraction in geometry in Euclid's hands. Representing the space on a bidimensional surface and obtaining valid results on its structure, showed that those string of symbols and drawings were able to capture the essence of space. Perhaps this is what centuries later led Galileo to write that the book of nature was written in the language of mathematics and that, to understand that book, one had to learn mathematics. Today, no doubt, the persistence of this conception is clearly established as the main validation criterion in mathematics.

The establishment of deductive method within the body of Greek mathematics took some a considerable time but, when finally it was embodied in the axiomatic geometry of Euclid's ***Elements***, its fate was sealed. Axiomatics is a very compact method to save mathematical knowledge. Deduction within the system and from the first principles, is the key to make the seeds to sprout.

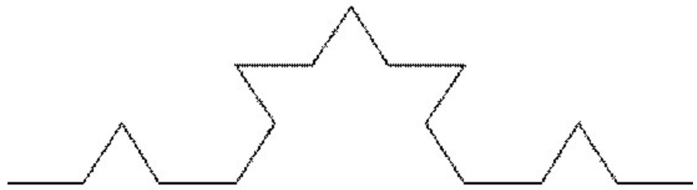
During the 17th century, mathematicians considered that Greek attachment to axiomatics was exagerated. As a consequence, deductive method led room to a more intense inductive period. The flow of discovery was taken as a proof that induction was the correct way to develop mathematics. But later, as has been always the case, the need to organize the bulk of discoveries led again to the consideration of deductive method. During the 19th century, mainly in the hands of Cauchy, Bolzano and Weierstrass, this return to the logical organization of mathematics, gave birth to what has been called the *Arithmetization of Calculus*.

Weiertrass viewpoint opened the door to the analytic proof of continuous non-differentiable functions. This result was instrumental to abandon visual intuition as a secure guide in the development of mathematics. But not all were happy with the new state of affairs. In 1904, the Swedish mathematician Helge Von Koch (1870-1924), published a paper in which he disapproved the exceedingly analytic approach followed by Weierstrass.

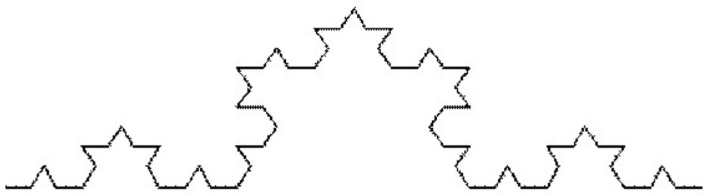
> **Until Weierstrass constructed a continuous function not differentiable at any value of its argument it was widely believed in the scientific community that every continuous curve had a well determined tangent...Even though the example of Weierstrass has corrected this misconception once and for all, it seems to me that his example is not satisfactory from the geometrical point of view since the function is defined by an analytic expression that hides the geometrical nature of the corresponding curve...**
> **This is why I have asked myself —and I believe that this question is of importance also as a didactic point in analysis and geometry— whether one could find a curve without tangents for which the geometrical aspect is in agreement with the facts**.

Von Koch geometrical approach to this problem (i.e. the existence of a continuous non-differentiable function) was genuinely geometrical. Today it is an icon in the world of fractals. It is simple to understand the construction process employed by Von Koch from the following figures:

Stage 1



Stage 2



Stage 3

The Italian mathematician Ernesto Cesaro (1859-1906) recognized the fractal nature of this curve and wrote, (in the *Atti d. R. Accademia d. Scienze d. Napoli,* 2, XII, number 15):

> **"It is this similarity between the whole and its parts, even infinitesimal ones, that makes us consider this curve of von Koch as a line truly marvelous among all. If it were gifted with life, it would not be possible to destroy it without anihilating it whole, for it would be continually reborn from the depths of its triangles, just as life in the universe is**."

The need to take into account the dual role of induction and deduction, of discovery and proof, in mathematics has been gradually accepted. For instance in the Introduction to his book *Geometry and Imagination* (written in collaboration with Cohn-Vossen) Hilbert expressed:

> **In mathematics as in any other scientific research, we find two tendencies present. On the one hand, the tendency towards *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency towards *intuitive understanding* fosters a more inmediate grasp of the objects one studies, a live *rapport* with them, so to speak, which**

stresses the concrete meaning of their realtions...it is still as true today as it ever was that *intuitive* understanding plays a major role in geometry.

Courant and Robbins, in their classic book *What is Mathematics?* Called attention towards the risks that mathematics can run if inadvertently, the balance between inductive and deductive thinking is broken:

There seems to be a great danger in the prevailing overemphasis on the deductive-postulational character of mathematics. True, the element of constructive invention, of directing and motivating intuition... remains the core of any mathematical achievement, even in the most abstract fields. If the crystallized deductive form is the goal, intuition and construction are at least the driving forces.

More recently, Lakatos has insisted on this perspective. In his book *Proofs and Refutations* he unfolds the power of analogy, of systematic experimentation, during the process of mathematical discovery that leads to a mathematical theorem. Lakatos considers that(op. cit. P. 142):

The deductivist style hides the struggle, hides the adventure. The whole story vanishes, the succesive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility.

As it has already been mentioned, the mathematical pendulum oscilates from inductive approaches to deductive ones, along the historical development of the discipline. Like if it were a natural law.

Gauss, used to say that (Bailey & Borwein, 2001, p.52 in *Mathematics Unlimited 2001 and Beyond*, edited by Björn Engquist and Wilfried Schmid. Springer, 2000): *I have the result but I do not yet know how to get it.* Besides, he considered that to obtain the result a period of *systematic experimentation* was necessary. There is no doubt then, that Gauss made a clear distinction between *mathematical experiment* and *proof*.

Nowadays, the computer (the tool that “speaks mathematics” in Lynn Steen able expression) is responsible for the new face of this old tension. In 1976, when Appel and Haken proved the Four Color Theorem using a computer in a crucial way, they were far from imagining the angry reaction of many members of the mathematical community. That was not a proof acording to the classical definition. This was not the case of using a computer to help the mathematician in her quest for truth. On the contrary, cognition up to a considerable amount had been transferred to a machine. The computer appeared as a cognitive partner, on equal terms, with the humans. The challenge cast by this new partner could not be ignored: The Gauss' *mathematical experiments* turned into a new kind, thanks to the computer. Since then, the role of the computer in mathematical investigation has increased, but this does not mean that its role is accepted by all. This is a very delicate matter that has to be thought with the utmost care as it incolves deep epistemological questions. To give a flavor of the tensions introduced into mathematics by the computer, let us remind some excerpts from the letter adreesed to Erathostenes, written by Archimedes in order to introduce his newly found *Mechanical Method* to obtain, among other results, his formula for the volume of the sphere, (see *Fractals for the Classroom* vol.1, pp. 6. Peitgen, Jürgen y Saupe, Sringer-Verlag, 1992, New York):

> **Certain things became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is without any previous knowledge.**

If we replace the red expression by the word "computer" we obtain the modern viewpoint of many mathematicians with respect to the computer. That is, the compueter is at most a tool to discover, never to proof.

Is this a mistake? That is, putting the computer aside from the realm of proof. Of course it is not, but this must not lead us into the belief that this should be always so, in strict terms.

In these days, numerical algorithms have been designed that allow the computation of a numerical answer with a precision beyond one hundred thousand decimal figures (Bailey&Borwein, 2000, p. 53, op.cit.). Then one can ask oneself if we are not entering a new era wherein the previous relationships between exploration and justification are becoming to be changed in qualitative terms. To deal with this kind of question one must practise extreme prudence. First we must elaborate on the relations between the computer and human cognition.

On computational tools and environments

The virtual, computational versions of mathematical objects produce the sensation of material existence, given the possibility of changing them where they exist, that is, on the screen. Students' growing familiarization with computational tools allows these tools to be transformed into mathematical *instruments* (Guin & Trouche, *The Complex Process of Converting Tools into mathematical Instruments...*, in Int. J. of Computers for Mathematical Learning, **3** 195-227, 1999) in the sense that computational resources are gradually incorporated into the student's activity. We suggest, then, that exploring with computational tools eventually allows students to realize how the mediational role of these tools helps them reorganize their problem-solving strategies. For example, when secondary school students were asked to explore the relationships between the inscribed angle in an arc and the corresponding central angle, we saw two behaviors in the classroom: students remained immobilized by the question (we think this is because they are not able to mobilize their expressive resources) or, when they had computational resources at their disposal (for example, Dynamic Geometry software, as Cabri for instance), they were led to draw up comparative tables between angles and to eventually realize that the central angle is "nearly double" the inscribed angle in the same arc (see Moreno & Block, **Democratic Access to Powerful Mathematics in a Developing Country**, pp.301-321, in L. English (ed.) *Handbook of International Research in Mathematics Education*, L. Erlbaum, 2002). The students' strategy, taking the inscribed angle from the central angle is possible thanks to the expressive power the students acquire through the computational tools. In the absence of these, it is not feasible for students to carry out the numerical comparison between the angles and to establish a conjecture, nor are they capable of producing a formulation associated with their explorations and express it in the language of the computational medium in which they are working.

The computing environment is an ***abstraction domain*** which can be understood as a scenario in which students can make it possible for their informal ideas to begin coordinating with their more formalized ideas on a subject. An abstraction domain supplies the tools so that exploration may be linked to formalization. In the example of dynamic geometry, we can put it this way: *The exploration of drawings and of their properties gives rise to the recognition of a system of geometric relationships, which in the final analysis constitute the "**geometric object**."* This abstract object that rises out of such exploration is still "linked" to the environment: The student can talk of its general properties but use the language, the means of expression, supplied by the environment.

One of the aims of research in this field is to understand how technology implementation should be conducted. We know that the first stage could entail working within the framework of a preestablished curriculum. Successful innovations should be able to "erode" traditional curricula, however, at that point, it becomes fundamental to understand the nature of knowledge of students that emerges from their interactions with those mediating tools. Working with computational tools in school media leads us to face the work from two different angles: as ***amplifying*** tools and as ***cognitive reconceptualizing*** tools. These amplification and reconceptualization processes can be illustrated in the following way: The amplification process is similar to the function of a magnifying glass. Through this lens, we can enlarge objects visible at first sight. Magnification does not change the structure of the objects that are being observed, however, on the other hand, the reorganization process can be compared to the act of seeing through a microscope. The microscope allows us to observe what is not visible at first sight and, therefore, to enter a new plane of reality. In this way, the possibility of studying something new and of accessing new knowledge arises.

Computing environments provide a window for studying the evolving conceptions of students and teachers. Graphing tools, for instance, produce a shift of attention from symbolic expressions to graphic representations. Representations are tools for understanding and mediating the way in which knowledge is constructed.

Our didactic work with computational tools led us to consider the phenomenology one can observe on the screens of calculators and computers. The screen is a space controlled from the keyboard, but that control is one of action at a distance. The desire to interact with virtual objects living on the screen provides a motivation for struggling with the complexities of a computational environment.

Computational representations are ***executable*** representations, and there is an attribute of executable representations on which we want to cast light: They serve to *externalize* certain cognitive functions that formerly were executed only by people. That is the case, for instance, with the graphing of functions. During the time that passes while the graph is being drawn on the screen, the student observes the characteristics of the function that are reflected in its construction. We suggest, therefore, that the student has the opportunity to transform the graph into an object of knowledge. This is similar to what the Greeks did with writing. They used the writing system not only as an external memory but also as a device to produce texts on which to reflect.

Explorations within an abstraction domain facilitate the understanding of the character *situated* in the propositions and the situatedness of its proofs. ***Situated proofs*** refer to the understanding and articulation of processes within the context in which they have been explored. Let us explain: At first, students might make some observations situated within the computational environment they are exploring, and they could be able to express their observations by means of the tools and activities devised in that environment. That is the case, for instance, when the students try to invalidate (e.g., by dragging) a property of a geometric figure and they are unable to do so. A situated proof is the result of a systematic exploration within an (computational) environment. It could be used to build a bridge between situated knowledge and some kind of formalization.

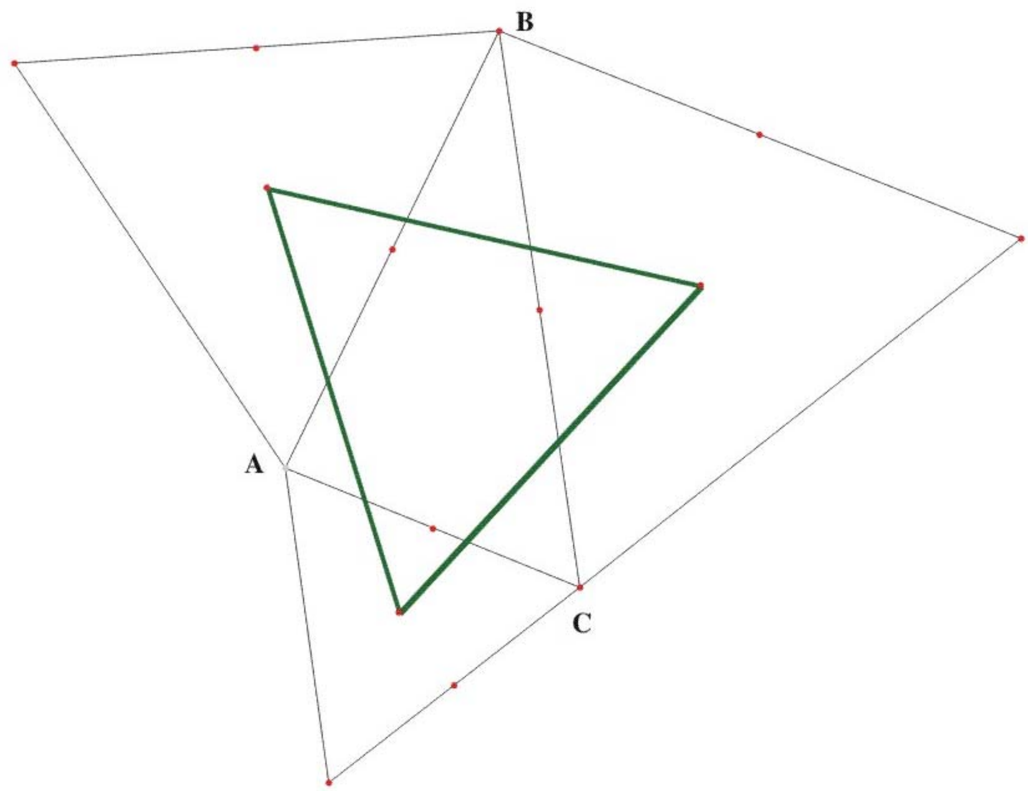
With computer explorations, we can associate the notion of a ***situated theorem***, when the tools employed become visible as part of the expression. As Noss and Hoyles explained (Noss R. and Hoyles, C. *Windows on Mathematical Meanings:Learning Cultures and Computers*. Dordrecht: Kluwer. 1996), while discussing related ideas, students can generate and articulate relationships that are general to the computational environment in which they are working. This means students can develop an ability to state general propositions in the language of the environment (i.e.: the can develop a sense of *situated abstraction*). We can say that these computational environments derive their educational power from their ability to manipulate and externalize abstract ideas.

Formal reasoning within a computational environment

The use of technology offers great potential for students to search for invariants and to propose corresponding conjectures. We have illustrated how students build dynamic environments to represent problems that eventually lead them to propose conjectures and *prove* them using the tools of the environment. Thus, the software becomes a tool for students to look for and document the behavior of objects and relationships and explore their structural nature. Our last example has to see with *features of mathematical proof* privileged via the use of technology. The mathematical discussion involving this example was much more subtle and students (teachers included) had difficulties when trying to understand it. But it was to introduce mathematical proofs within a computational environment. In a sense we can interpret this part of our work as a *teaching* experiment. At first we discussed the notion of macro construction within the Cabri dynamic environment. After a while, it became clear for the students that a geometric object built usinga macro was a genuine geometric object living in the Cabri universe. This way we could answer the question: To what extent mathematical arguments or ways to approach problems within a Cabriworld vary from the traditional approaches with paper and pencil?
We all know how controversial can be to discuss the place of computers and calculators in the field of mathematical proofs. And how important it is for students to understand what a proof is.

Can we prove a geometry theorem using Dynamic Geometry? We want to illustrate how this can be made feasible. Let us study Napoleon theorem:

Given an arbitrary triangle, construct on each side the corresponding outer equilateral triangle. Then the triangle that results by joining the centroids of these three triangles is always an equilateral triangle.



The thick triangle is the Napoleon triangle corresponding to triangle ABC. Let us recall how we proceed with this construction. We built two Cabri-macros:

(i) Given two vertices, the first macro determines the third one so that we have an equilateral triangle.

(ii) The other macro produces the centroid of a given triangle.

Then, after playing with the construction trying to "destroy" the equilateral triangle (Napoleon's triangle) one "has to accept" the validity of the proposition. This is a natural approach if we are exploring a (possible) theorem within a dynamic environment. That is, we try to make sure that the claim made is an invariant with respect to dragging. But we can go farther than that: we can give a proof within the Cabri world. We mean, a situated proof.

In fact, we design a macro that enables us to construct equilateral triangles and each time we use it, the result is a genuine equilateral triangle. Likewise, as we have a macro that determines the centroid of a triangle, each time we use it, the result is the genuine centroid of the given triangle. Taking this into account, we realize that when we point out a vertex of the Napoleon triangle we can read the question: “what object?” We can answer “Napoleon” or “centroid” and that means that the vertices of the Napoleon always coincide with the centroids. We know then, that the Napoleon triangle is always equilateral. It is important to remark that this kind of reasoning takes us beyond the perceptual level: this is precisely the case when we intend, for instance, with paper and pencil, to prove a geometrical assertion. Working within a computational environment forces us to adopt a different strategy: we have to resort to the nature of the mediating tools we have at our disposal. Of course, we cannot loose sight of the internal mathematical universe residing in the innards of the computer.

Final remarks

Explorations within a computational environment eventually allow students to generate and articulate relationships that are general in the environment in which they are working. Those relationships which encapsulate general statements have been called *situated abstractions*, precisely because they are bound into the medium in which they are expressed (Noss&Hoyles, 1996). What we have introduced in the last section is a kind of proof we could call *situated proof.* In a sense,, every proof is situated but emphasizing the situatedness while working within a computational environment pays an extra bonus. In our study, whose goal was to explore how students “proved” a mathematical proposition within a computationsl environment, we worked with 17-18 years olds, trained in dynamic geometry. For the development of the activities, teams of two or three students were formed. In this, as in other related cases, students became aware of invariants and they could express the relevant ideas *but only within the expressive medium* made feasible by the calculator.

Acknowledgement. The authors want to thank Texas Instruments for all the support during the writing of this paper.