Why do we use Theorem in Calculus?

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As teachers of mathematics, we understand how theorem and proof provide the underpinnings of the complex processes that form calculus techniques. However, the students who study calculus often view calculus as consisting chiefly of processes and some quantitative results, independent of and unrelated to the axioms and theorems underlying the results. This paper presents my opinions and some evidence as to why we do and should emphasize theorem in the teaching of calculus.

A story I am fond of retelling is getting to know the businessman husband of a friend of mine. When he discovered that I am a mathematician, he exclaimed that he recalled the Fundamental Theorem of Calculus from his college studies some 20 years earlier. “Why that is how you can get a speeding ticket on a toll road just from your clocked ticket-in and ticket-out”, he proudly exclaimed. Although he has the names of the Mean Value Theorem and the Fundamental Theorem confused, the power of the underlying mathematics remained with this friend long after our manipulative analytical techniques had worn off. It is my opinion that a sense of the strength and power of mathematics should be a goal of any calculus course, and using theorem is an excellent way to develop this sense.

Of course, when we as mathematicians teach the Fundamental Theorem of Calculus, we can’t help saying “ain’t it great?” We marvel at the two-sided link of the differential and integral calculus. But just that marvelous synchronicity isn’t a sufficient reason to teach calculus to the hundreds of thousands of students who encounter it each year. We have to choose what to teach and how to teach it according to the needs and learning objectives of our students.

In order to examine our curriculum and determine the essential elements of a successful calculus course, we must consider each of the interacting components: environment and audience, course outcomes, required content, synthesizing the content delivery with other goals (such as developing communication skills), and assessing understanding. To begin, an understanding of our classroom audience is essential to the success of any mathematics course. “Audience” is a misnomer, since of course we really aim to have “participants” or “learners”. We should be designing our calculus curriculum for the majority of our students. Who these students are varies from school to school, and differs between the experience of high school calculus students and those enrolled in college. The budding mathematicians in the calculus course are, alas, few and far between. We should hope that a well-designed curriculum will help them to continue successfully on their chosen path while ably serving a larger clientele. In this paper, we will assume that the student who is prepared for calculus is reasonably skilled at algebra, trigonometry, and the rudiments of function; and is enrolled for a variety of reasons.

Let us investigate the reasons for young students to choose to enroll in calculus, and determine how a study of calculus can help those students develop skills necessary for their chosen field as well as enhance their education in general. In turn, examining the needs of our client disciplines can help us to modify course objectives. I teach at a science-oriented liberal arts college, so many of the students will actually use calculus tools for advanced studies in their undergraduate science career. For example, they are expected to model and evaluate definite integrals, perform integration by parts, estimate series, or choose appropriate curves to fit data. It is true that students often need to be reminded at that later date of the actual calculus manipulations.

However, we hope that they do have the necessary habits of mind that allow them to understand the nature of the mathematical problem involved and sufficient facility with the material to research and implement correct solution methods. In my opinion, spending class time on the study of theorem in calculus will help these students develop those habits of mind.

While chemists, physicists, computer scientists and engineers will definitely use calculus again in their undergraduate work, at my institution we also have many prospective medical doctors and other liberal arts students who enroll in calculus. In addition, some universities require calculus for entrance into highly competitive specialized programs. How can we effectively serve these students? The liberal arts environment can help to structure the curriculum so that these students achieve liberal arts educational objectives. In a mathematics course, these students can develop reasoning skills, understand the concept of rigor, implement problem solving strategies, and communicate understanding of technical matters. The examples of the study of theorem below will illustrate how we can make sure these students develop such skills in calculus.

When designing the mathematics curriculum, we often have a content list in mind. However, there are skills and development objectives within each mathematics course that extend beyond familiarity with content and preparation for a next course. From general mathematics objectives, we can distill calculus objectives. We approach topics via the "rule of 4" so as to ensure the student can address problems in both mathematics and other disciplines from multiple avenues. We hope to instill good number sense and develop both logic skills and reasoning. Students do need some drill exercises so that they can accurately use the tools to solve problems. Through this work they should develop an appreciation for the idea of using the right tool and having a basic skill set. With all these objectives in mind, there are certainly multiple goals for each teaching session. Using the study of theorem is more suited to some of these goals than others. Finding the right blend of skill, drill, theory and common sense to develop a successful student is the ultimate goal of the teacher who hopes to do more than just cover 5 chapters of content in the time allotted. Finally, we hope the student will get a good sense of how calculus fits together—the big picture. Let us see how the study of theorem can help this happen.

The Intermediate Value Theorem is particularly important in the development of young mathematics thinkers. This is one of the first theorems that students encounter of the form "If p, then q." In preparatory coursework for calculus, most theorems are of the form "p if and only if q" or restatements, replacing equal items for equal items. Think of the Factor Theorem, for example, which equates roots of polynomials with factors. Or consider the Pythagorean identities in elementary trigonometry, which restate the Pythagorean Theorem. But the Intermediate Value Theorem requires the student to use Modus Ponens to make inferences about the values between the endpoints of a continuous curve. Furthermore, it is an existence theorem that is not constructive, so the inference about "c" is difficult for students to grasp. A graphing technology is useful for illustrating the theorem. We begin with just the endpoints plotted, a colored band spanning the y-values in the range. Then a particular function can be graphed, and it is great if the function values actually exceed the range of the band. The guarantee of intermediate values is by no means a restriction on the possible y-values of the function, which requires students to grapple with the distinction between a universal statement and a particular example. The theorem can be used to further develop the idea of inverse images of functions, since the intermediate value "N" implies the existence of at least one preimage "c", although there may be several.

Figures 1 and 2 below demonstrate how a graphic illustration of the Intermediate Value Theorem can help assist student understanding. Figure 1 shows the premise of the theorem for the domain $[a,b] = [-3,3]$. The shaded band represents the intermediate values guaranteed to be achieved by the theorem. Figure 2 includes a particular function which satisfies the hypothesis of the theorem.

This is a nice choice of function, since additional y-values are achieved, beyond those guaranteed by the conclusion of the theorem. Additional images to include in a student discussion should incorporate graphs where the functions have discontinuities, to demonstrate the necessity of the hypothesis and the fact that *some*, but not necessarily all, y-values will be excluded if the hypothesis is not valid.

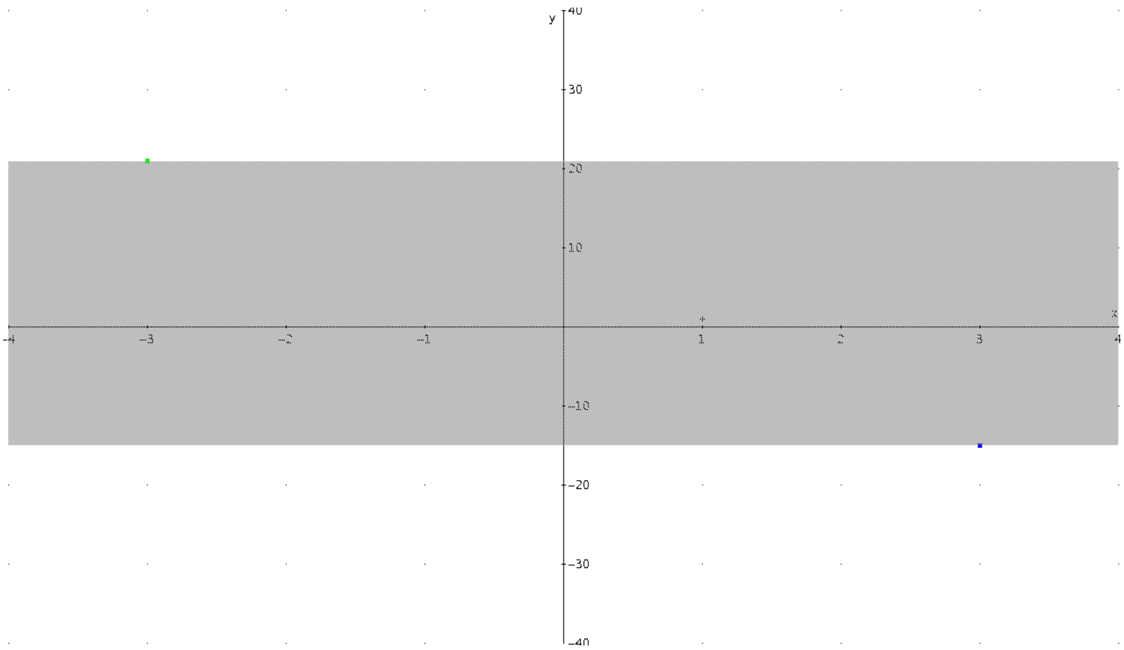

Figure1: The Intermediate Value Theorem

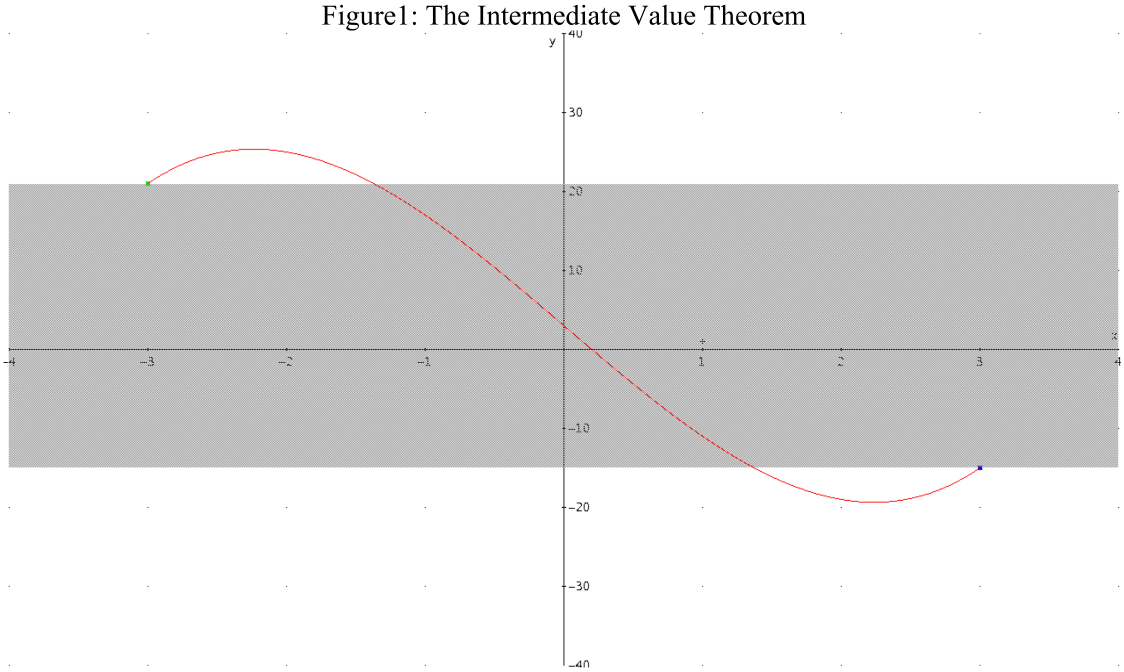

Figure 2: A function which satisfies the hypothesis

The Intermediate Value Theorem has some nice direct applications that are relevant to technologically savvy students. They are accustomed to graphing functions on their calculators, and a discussion of the way the calculator is creating this approximate image is appropriate. A quick zoom will reveal that the calculator is limited by its pixels and that line segments are not always the best way to "connect the dots". That the calculator dots are often connected across discontinuities (such as vertical asymptotes) serves to illustrate the assumptions inherent in the calculator plotting code: the program assumes that the function is continuous on a closed domain. Similarly, the classic bisection method of finding roots uses the Intermediate Value Theorem to infer the relative location of the root. This illustrates what is going on in the calculator's black box when students use handheld technology to compute a root or an intersection.

As part of the assessment of student understanding of the theorem, I usually ask the student to first explain why the Intermediate Value Theorem may be applied to the graph in Figure 3. Then I either ask the student to provide a list of all intermediate values guaranteed by the theorem or I select an appropriate intermediate value and ask the student to locate the corresponding input "c". What I find is that a thorough discussion of relevant graphs allows students to identify the intermediate values and find "c" with a fair degree of success. The point where students experience difficulty is in explaining why the theorem is applicable in the first place. A typical answer to this question of why the theorem applies is "because I could find the "c"." Discussion of graphs where the theorem does NOT apply is helpful, but the Modus Ponens nature of the theorem is difficult for students to grapple with. So we have several wonderful reasons to use the theorem in the classroom: it illustrates what is going on in their calculator, we help them develop logical understanding in formal implication statements, and we help them comprehend the need to understand the hypothesis component of a theorem.

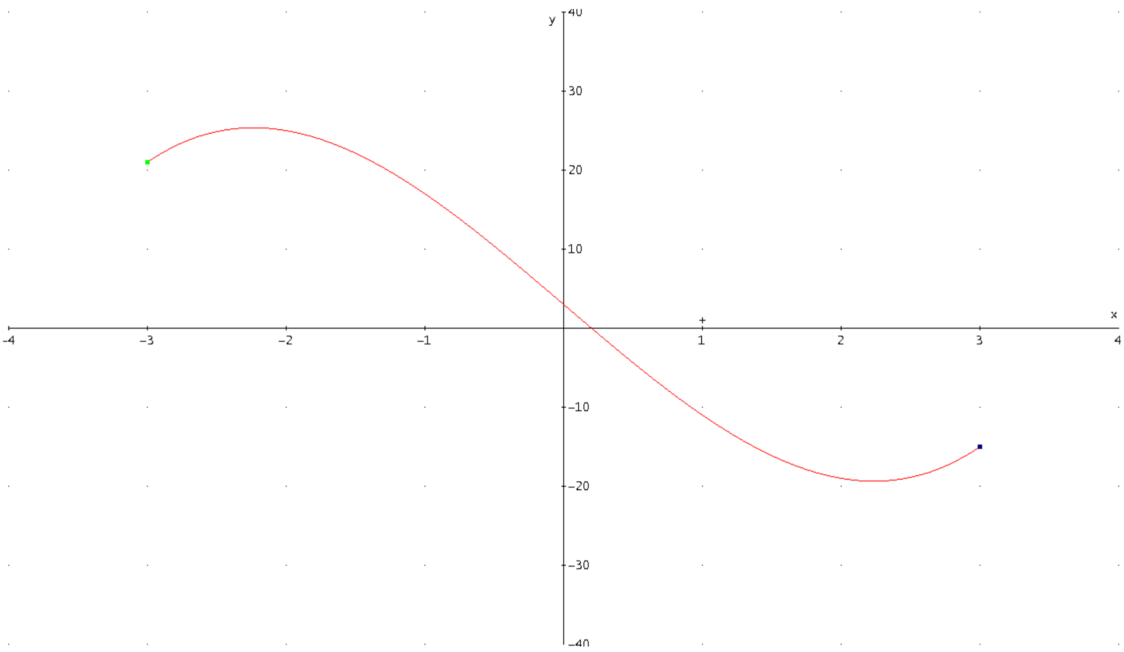

Figure 3: Intermediate Value Theorem Assessment

On [-3,3], what is the range of intermediate values guaranteed by the theorem?

Having experienced some student confusion as to the hypothesis and conclusions of the Intermediate Value Theorem, successfully introducing the Mean Value Theorem is the instructor's next big challenge. However, the students who have spent some time studying the logic behind the Intermediate Value Theorem are better prepared to handle the logic of another "if—then" implication statement. The Mean Value Theorem is a powerful application of the differential calculus, in its meaning and applications. Presented as a graphical theorem (a secant and tangent line are parallel) together with its analytical meaning (the average rate of change is achieved as an instantaneous rate), this theorem offers a wonderful opportunity to tie graphics and analytical understanding together. Its power is revealed in applications. We can finally prove analytically that the student's common sense understanding of mathematics is correct: when a function has positive derivative, the curve must increase. This is a first instance where the student witnesses that a powerful theorem will have several important corollaries. Showing that a car MUST travel at exactly 45mph at least once on a one hour 45 mile trip is enough to set in the student's mind for 20 years, as evidenced by the opening anecdote. And we can confirm that the student's intuition on working with antiderivatives is correct; there is just one general answer.

Once again, it is helpful to use a graphing device to look at several graphs until the student can themselves identify whether the theorem applies and locate the "c" value where the tangent is parallel to the secant. Studying the Mean Value Theorem after the Intermediate Value Theorem should reveal an increase in student understanding of the logic of a theorem. They should have less difficulty with the "if—then" format, be more comfortable with the idea that the theorem does not always apply, and know what may or may not be concluded if the hypothesis is void. It is interesting to note that the hypothesis of the theorem varies from text to text. Stewart's texts require the function to be differentiable on the interval $[a,b]$, while Larson and Thomas require the weaker condition of continuous on the closed interval and differentiable on the open set. An informal survey of 12 more calculus texts in my office, from Apostol to Ostebee-Zorn, found only one other variation from the traditional weaker hypothesis.

Depending on time constraints in the selection of content, it is interesting to first develop Rolle's Theorem in class and then prove the Mean Value Theorem from it. Rolle's Theorem follows immediately from Fermat's result that "what goes up must come down", so it provides confirmation of one's common sense. It is also nice to show that Rolle's Theorem is a special case of the Mean Value Theorem. The Mean Value Theorem for Definite Integrals is also a direct application of the Mean Value Theorem, and it has a lovely graphical interpretation in addition to its analytical form. Where the Mean Value Theorem invokes parallel line segments as a result on rates, the Mean Value Theorem for Definite Integrals invokes matching areas as a result on accumulation. Completing the proof as a straightforward application of the Mean Value Theorem serves to reinforce the primary theorem and emphasize the power of a theorem may lie in interpolation to similar results. The Mean Value Theorem for Definite Integrals also is highly useful, in averaging functions (such as inventory models) and root mean square measurements.

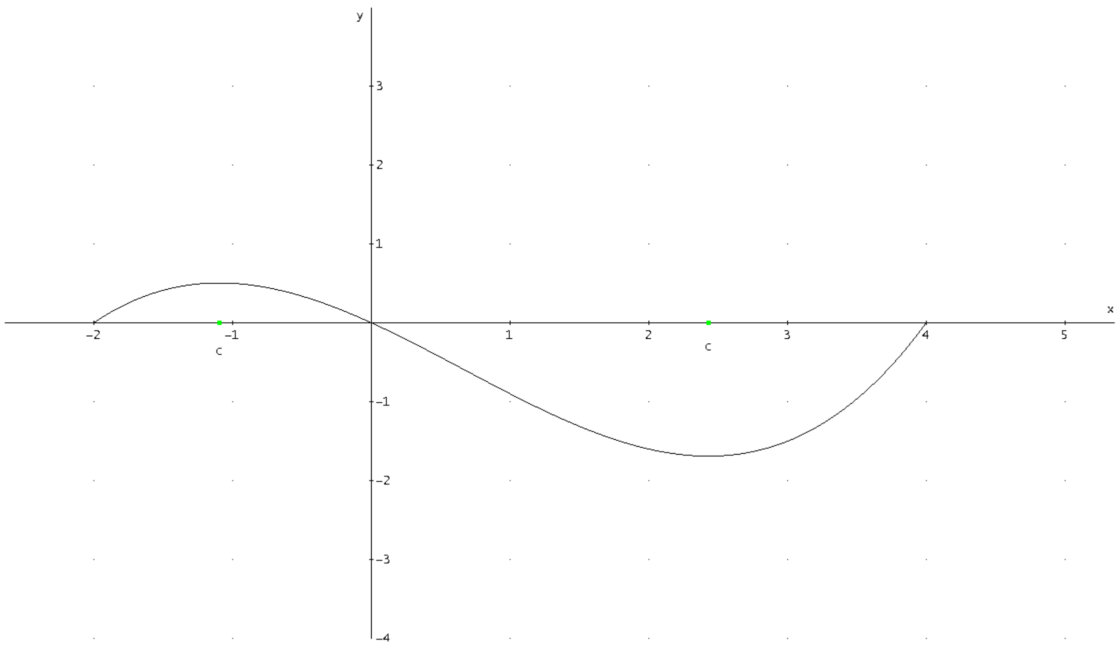

Figure 4: Rolle’s Theorem as “what goes up must come down” (and vice versa)

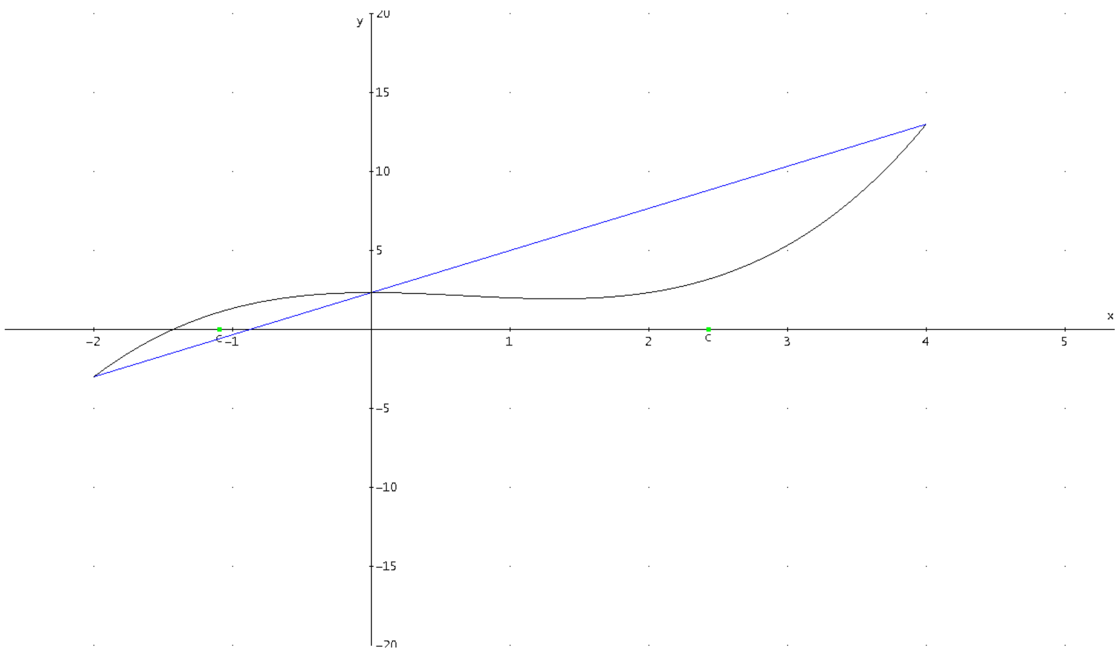

Figure 5: Mean Value Theorem as Rolle’s Theorem with your head atilt.

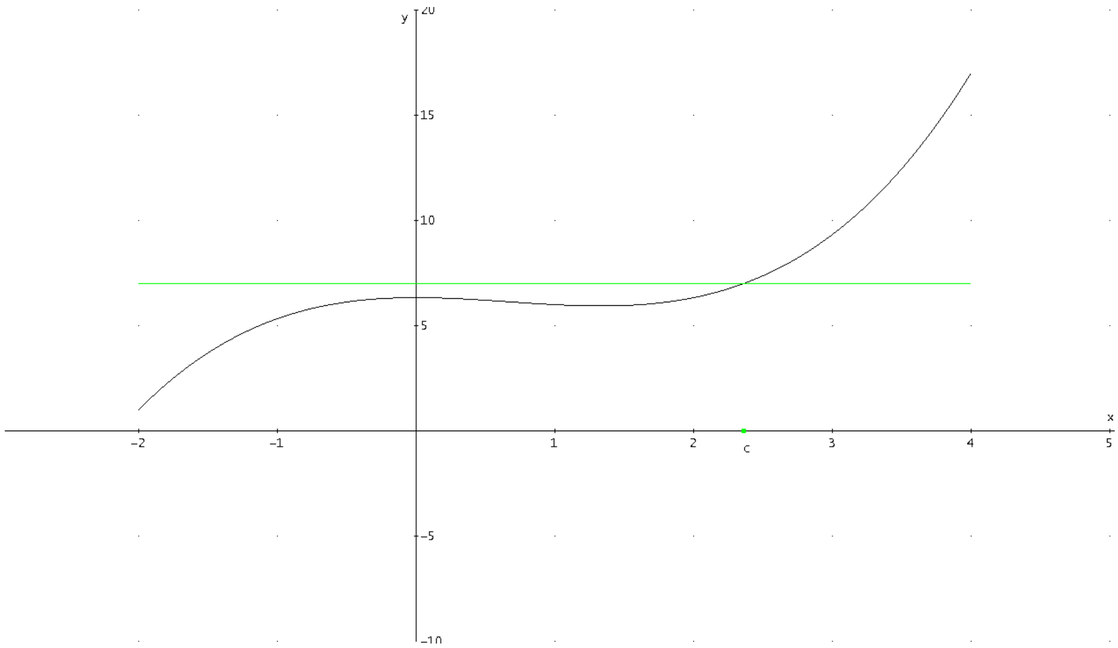

Figure 6: The Mean Value Theorem for Definite Integrals as "cut equals fill on a hill".

As we pursue more advanced calculus topics later in the coursework, we can interface with the content on a deeper level because of the students' heightened understanding of the logical underpinnings of the mathematics. As we return to the formal epsilon-*N* definition of limit for sequences, visualization of the sequence convergence can deepen understanding, in addition to more naïve understanding from computation of sequence values on a CAS or calculator. Then we can actually prove some simple limits using the formal definition, with much deeper student understanding than when they tried similar proofs for function limits at the beginning of calculus. Finally, we can help the students appreciate that limit is the building block of calculus through some of the theorems we prove, such as proving that alternating series converge conditionally.

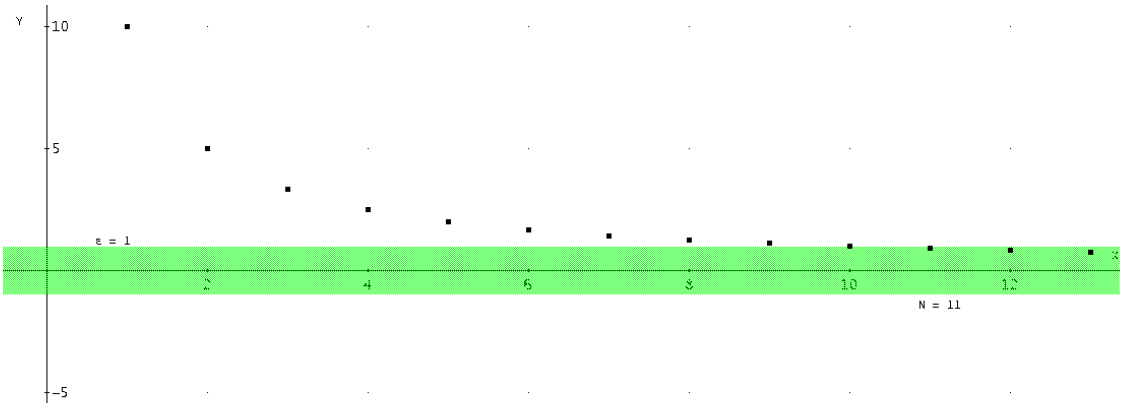

Figure 7: A particular N for a given $\in$.

The study of theorem within more advanced topics, such as improper integrals and infinite series, will develop more advanced reasoning skills as well. Whether investigated as part of improper integrals or infinite series, the Comparison Test Theorem is a powerful tool for developing students' reasoning capabilities. The theorem infers convergence or divergence via an inequality relating the unknown integral (series) to a known one. The idea of using an inequality to infer information about an unknown item should remind students of the Squeeze (Sandwich) Theorem for limits. In addition, the students are reminded of the necessity of validating the hypothesis that the sequence terms are nonnegative first, as they learned continuity was essential in the study of the Intermediate Value Theorem.

What makes the Comparison Test Theorem difficult is that many comparisons are inconclusive. Of four possible inequality situations (e.g., known less than converging, known greater than converging, etc), only two are conclusive. For example in Figure 8 below, when summing the (connected) red sequence "B" of terms of the form $10/(2+\sqrt{n})$, we find it diverges by comparison to the sum of eventually smaller terms $10/n$ (this is the black sequence "C"). However, comparison fails in Figure 9, when you select the more obvious sequence of terms $10/\sqrt{n}$ (Blue "D"). This is a serious disappointment to our students. Many prefer the Limit Comparison Test, which compares the limit of the ratios of the terms (functions), and remedies most inconclusive situations. I try not to introduce this theorem too early, since it reduces the work to process, rather than inference and conclusions from evidence. This is similar to the problem with the evaluation of definite integrals via the Fundamental Theorem, replacing the theory with a process. Similarly, the Ratio and Root Tests for series are useful tools, but don't require such an exercise of reasoned inferences. However, proving the Ratio Test in class via a comparison with a geometric series is very sophisticated. At this point in the course, we can further strengthen appreciation for theorem and proof by showing the students some of these more complicated, delicate proofs. A proof of a theorem such as the Ratio Test involves elegant reasoning and several steps, not just a direct inference from a definition.

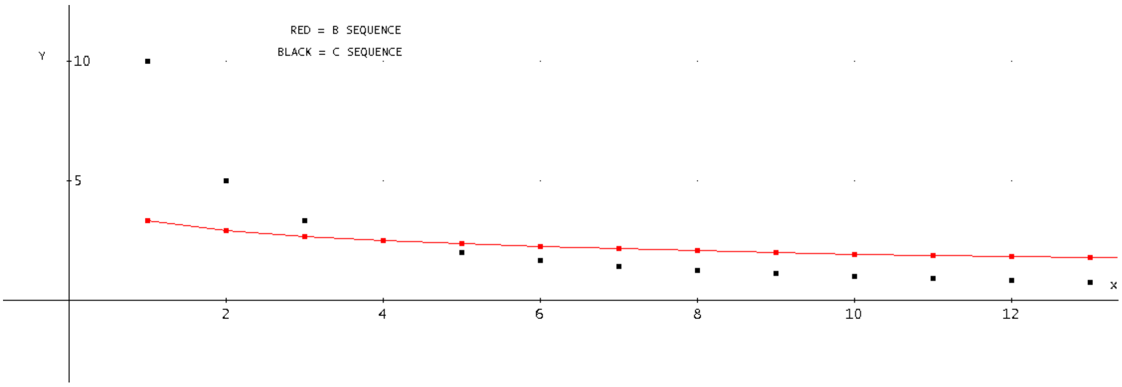

Figure 8: The Comparison Test for series is conclusive.

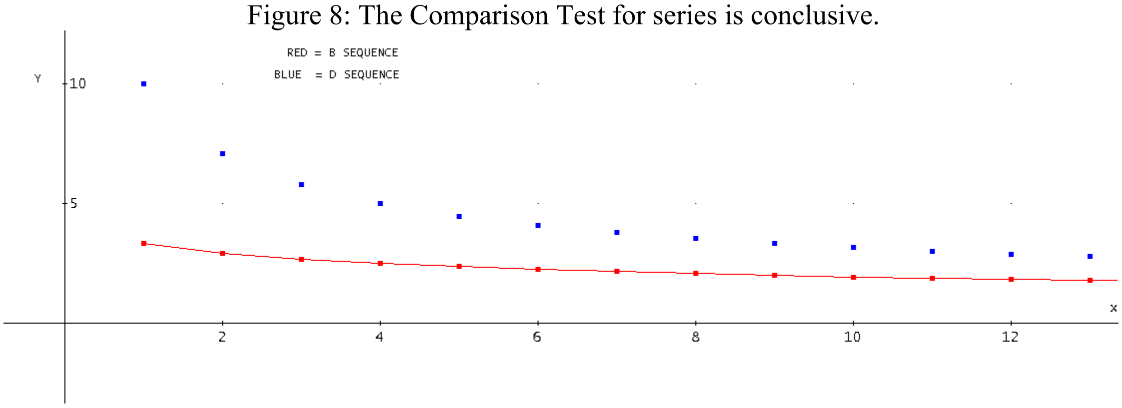

Figure 9: The Comparison Test for series is inconclusive

The theorem on the Test for Divergence of a series (when terms don't approach zero, a series cannot converge) is particularly troublesome for students. Their difficulty is twofold: it is proved (or used, if you like) in contraposition, and the fact that series with small terms may NOT converge is very counterintuitive. The contrapositive demonstrates that sometimes an avenue to proof is not how the theorem is useful later. Students' incorrect intuition is based on faulty reasoning, an expectation that infinite sums obey all laws that apply to finite sums (such as grouping terms two at a time, in the sum $1 - 1 + 1 - 1 + 1 \ldots$). This presents an excellent opportunity to deepen student understanding of the concepts of infinity and indeterminate forms.

To complete an investigation of power series, we arrive at the theorem which states that if a (suitably differentiable) function has a power series centered at a particular a, then it is in fact the Taylor series centered at a. The theorem emphasizes that mathematics procedures require validation, we don't use processes without first proving that the process yields meaningful results. A nice way to assess whether the student has understood this validation process is to tell them a function has a converging power series, say of the form $3 + 4x - x^2/7 + \ldots$ and then ask for some information about the values of the function and its first two derivatives at 0. We can readily assess student understanding of the big picture by asking them to quickly find a series for a given polynomial. Finally, the calculus of the fundamental theorem is reunited in the idea that transcendental and rational functions can be represented by "polynomials" (of infinite degree) and differentiated and integrated accordingly.

Whether the student takes only one calculus course or several, the Fundamental Theorem of Calculus is significant to him/her only after developing the understanding of the logic and mathematics underlying the theorem. Deeper theorems of subsequent courses rely on those underpinnings. Through the study of various theorems in calculus we can achieve the goals of mathematical learning: development of reasoning and communication skills, understanding of rigor, and the ability to implement problem solving strategies. Through theorem we can offer a peek into the world of advanced mathematics, its format and rigor, while achieving those goals.

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Biographical Note

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