CLASSICAL AND COMPUTER METHODS IN ELEMENTARY GEOMETRY[1]

Pavel Pech*, Jaroslav Hora**

*University of South Bohemia, České Budějovice, pech@pf.jcu.cz
**University of West Bohemia, Plzeň, horajar@kmt.zcu.cz, Czech republic

Abstract: We will show that the present stage of development of computer hardware and software enables to solve many elementary and non elementary problems of classical geometry. Whereas classical methods show the beauty of geometry, enable better insight into the situation and better understanding the problem, on the other hand by computer methods we can solve complex elementary and non elementary problems. Computer algebra methods enable automatic proving theorems of elementary geometry, automatic derivation and discovery of geometric formulas, construction of geometric objects which have given properties and which cannot be easily done with a ruler and compass, etc.
On a few examples from geometry of polygons in a plane the strengths and weaknesses of the both methods are demonstrated. It is shown that both classical and computer methods are helpful in teaching mathematics.

Key words: Computer algebra, classical geometry

1 Introduction

During several last semesters one of the authors lead a geometry seminar on using computer methods to solve problems in elementary geometry. The students, who took part in this seminar were participants of the branch of teacher's training in mathematics in their 4^{th} years university study, i.e., they had knowledge of the basic courses in geometry.

In the seminar such methods were stressed which are based on Groebner bases computations. We used the theory of elimination to prove and discover statements from geometry in a plane and a space. We also used this theory to carry out constructions of geometric objects which have given properties and which is not easy to construct by the rule and compass, see [6], [8], [13], [14].

Starting from the statement that the heights of a triangle are concurrent, we investigated both by computer and in a classical way well known problems such as the formula of Heron for the area of a triangle and its generalization – formula of Brahmagupta for the area of an inscribed quadrilateral given by the lengths of its sides, formula of Staudt, Wallace-Simson theorem, Napoleon theorem and further similar topics. The construction of a square with

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the vertices on the four given straight lines in a plane is shown as well.

At the end of the seminar students were engaged in the seminar work, which was aimed on a problem chosen by each student from the recommended offer list from Internet. Here the problems were solved both in a classical and computational way.

2 Automatic theorem proving

Automatic theorem proving concerns with geometry statements of *equality type*, which are of the kind $\mathbf{H} \Rightarrow \mathbf{c}$, where $\mathbf{H}$ is the set hypotheses and $\mathbf{c}$ the conclusion. At first we *algebraize* the geometric problem. This stage is characterized by establishing the set of hypotheses $\mathbf{H}$ whose algebraic form are polynomial equations

$$h_1(x_1, x_2, \dots, x_n) = 0,\ h_2(x_1, x_2, \dots, x_n) = 0, \dots, h_r(x_1, x_2, \dots, x_n) = 0$$

and the conclusion $\mathbf{c}$, which is expressed by the polynomial equation

$$c(x_1, x_2, \dots, x_n) = 0,$$

where the polynomials have coefficients in a base field $\mathbf{K}$. We usually assume that $\mathbf{K} = \mathbf{Q}$, the field of rational numbers. Thus the algebraic form of the statement would be

$$\forall x \quad \{(h_1(x) = 0 \wedge h_2(x) = 0 \wedge\ \dots \wedge h_r(x) = 0) \quad \Rightarrow \quad c(x) = 0.\} \tag{1}$$

The objective of the next step is *verification* of (1), i.e., to decide whether the conclusion follows from the hypotheses or, which is the same, to decide whether the zero set of the conclusion $\mathbf{c}$ contains the zero set of the hypotheses $\mathbf{H}$, i.e., $\mathrm{Zero}(\mathbf{H}) \subset \mathrm{Zero}(\mathbf{c})$. By the famous Hilbert Nullstellensatz the statement (1) is true iff 1 belongs to the ideal $(h_1, \dots, h_r, ct - 1)$ of the hypotheses polynomials and the negated conclusion. However for the most geometry problems it suffices to show that c belongs to the ideal $(h_1, \dots, h_r)$. The simplest way to show the essence of automatic theorem proving is a demonstration on the example. Let us have the following problem.

Prove that the heights of a triangle are concurrent.

The first stage of the automatic proving theorem is to choose an appropriate coordinate system to describe the situation analytically, see Fig. 1. The rule is to choose the coordinate

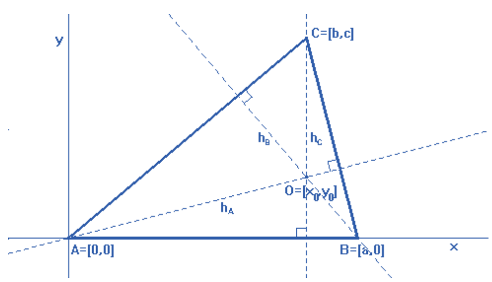

Figure 1:

system in such a way so that we could describe the situation very simply. We place the origin at the vertex A and the axis x into the side AB of a triangle ABC and denote the coordinates of the vertices of the triangle ABC by $A = [0, 0]$, $B = [a, 0]$, $C = [b, c]$. Now we express the equations of the heights h_a, h_b, h_c in this chosen coordinate system

$h_a : (b - a)x + cy = 0,$
$h_b : bx + cy - ab = 0,$
$h_c : x - b = 0.$

Suppose that the heights h_b and h_c intersect at the point $O = [x_0, y_0]$, i.e. the following hypotheses equations are fulfilled

$O \in h_b \Leftrightarrow bx_0 + cy_0 - ab = 0,$
$O \in h_c \Leftrightarrow x_0 - b = 0.$

As the conclusion we want to show that the height h_a contains the point O, i.e., that

$O \in h_a \Leftrightarrow (b - a)x_0 + cy_0 = 0.$

Hence we are to prove the statement of the following form

$$\forall x_0, y_0 \quad bx_0 + cy_0 - ab = 0 \wedge x_0 - b = 0 \quad \Rightarrow \quad (b - a)x_0 + cy_0 = 0. \tag{2}$$

In this very simple case we are able to show that the statement above is valid even by hand. To do this realize that

$$(b - a)x_0 + cy_0 = (bx_0 + cy_0 - ab) - a(x_0 - b).$$

We expressed the polynomial of the conclusion $(b-a)x_0 + cy_0$ as a linear combination of two hypotheses polynomials $bx_0 + cy_0 - ab$ and $x_0 - b$. Thus from the equations $bx_0 + cy_0 - ab = 0,\ x_0 - b = 0$ the equation $(b - a)x_0 + cy_0 = 0$ follows.
We can also equivalently say that the set of all the common solutions (or zeros) of the system of equations $bx_0 + cy_0 - ab = 0, x_0 - b = 0$ is a subset of all solutions of the equation $(b - a)x_0 + cy_0 = 0$.
From the algebraic point of view we are to show that the polynomial $(b - a)x_0 + cy_0$ belongs to the ideal $I = (bx_0 + cy_0 - ab, p - b)$. The decision of whether a polynomial f belongs to the given ideal I or not, is possible by the command `NF(f,I)` which is implemented in most mathematical software. Simply spoken the command `NF(f,I)` or Normal Form of f with respect to the ideal I returns the reminder of a polynomial f if we express it by all the possible algebraic linear combinations of the polynomials from the ideal I. If `NF(f,I)=0` then the remainder equals zero and the polynomial f belongs to the ideal I. The whole process is a generalization of the well known Euclidean algorithm for division of polynomials of one variable and is based on Groebner bases computations. See the nice book [6] where the problem is described into details. In the program CoCoA[2], the software which we will use in this article, we enter

```
Use R::=Q[abcx[0]y[0]];
I:=Ideal(bx[0]+cy[0]-ab,x[0]-b);
NF((b-a)x[0]+cy[0],I);
0
```

[2] Software CoCoA is freely distributed at `cocoa@dima.unige.it`

The answer is 0, i.e. Normal Form of $(b-a)x_0 + cy_0$ equals zero, which means, that the polynomial $(b-a)x_0 + cy_0$ belongs to the ideal $I = (bx_0 + cy_0 - ab, x_0 - b)$ and the statement is valid. The automatic proof is complete. $\square$

Usually the situation is not so easy. Most geometry statements are valid under some *non-degenerate* (also called *subsidiary*) conditions such as points being distinct, three distinct points being not collinear, line segments of nonzero lengths, circles of nonzero radii, etc. The non-degenerate conditions can be algebraically described by the inequalities

$$g_1(x_1, x_2, \ldots, x_n) \neq 0,\ \ g_2(x_1, x_2, \ldots, x_n) \neq 0,\ \ \ldots,\ \ g_s(x_1, x_2, \ldots, x_n) \neq 0.$$

Thus adding these non-degenerate conditions to the hypotheses $(h_1 = 0, h_2 = 0, \ldots, h_r = 0)$ a geometric statement can be translated into the form

$$\forall x \quad \{(h_1(x) = 0 \wedge \ldots \wedge h_r(x) = 0 \wedge g_1(x) \neq 0 \wedge \ldots \wedge g_s(x) \neq 0) \quad \Rightarrow \quad c(x) = 0.\} \tag{3}$$

The statement (3) is true iff 1 belongs to the ideal $(h_1, \ldots, h_r, g_1t_1 - 1, \ldots, g_st_s - 1, ct - 1)$, which consists of the hypotheses polynomials, non-degenerate conditions and the negated conclusion. However for the most geometry problems it suffices to show that c belongs to the ideal $(h_1, \ldots, h_r, g_1t_1 - 1, \ldots, g_st_s - 1)$.
The question now arises: "How to find non-degenerate conditions?" One way is to determine them before the computation, i.e., to rule out the situations such as two points coincide, three points are collinear,... But this method need not be successful. It is not easy in general to determine all possible non-degenerate conditions. The better way consists in the elimination of all *dependent* variables, i.e. those variables which we choose arbitrarily, and a slack variable t in the ideal $(h_1, \ldots, h_r, ct - 1)$ of hypotheses polynomials and the negated conclusion. We get the elimination ideal, which contains polynomials in only *independent* variables. This elimination ideal contains degenerate conditions, say $g_1, g_2, \ldots,\ g_s$ (if there are any). We negate these conditions to obtain the set of polynomials $g_1t_1 - 1, g_2t_2 - 1, \ldots,\ g_st_s - 1$ and add this set of non-degenerate conditions to the ideal $(h_1, h_2, \ldots,\ h_r)$. Instead of the ideal $(h_1, h_2, \ldots,\ h_r)$ we have the new ideal $(h_1, h_2, \ldots, h_r, g_1t_1 - 1, g_2t_2 - 1, \ldots,\ g_st_s - 1)$ and explore whether the conclusion c follows from this new ideal. The whole process now will repeat. Let us look at the next example.

In the example above, suppose that instead of the heights h_b and h_c now the heights h_a and h_b intersect at the point O, i.e. suppose that the equations $(b-a)x_0 + cy_0 = 0$ and $bx_0 + cy_0 - ab = 0$ are fulfilled. Then the conclusion is that the height h_c contains the point C, i.e., $x_0 - b = 0$. We enter

```
Use R::=Q[abcx[0]y[0]];
I:=Ideal((b-a)x[0]+cy[0],bx[0]+cy[0]-ab);
NF(x[0]-b,I);
x[0]-b
```

The answer is $x_0 - b$ which is not zero and the polynomial is not the element of the ideal I. Despite of this the statement can be valid. In such cases non-degenerate conditions should be determined. In order to find non-degenerate conditions we will add to the ideal $I = ((b-a)x_0 + cy_0, bx_0 + cy_0 - ab)$ one more polynomial $(x_0 - b)t - 1$ to obtain the ideal $J = ((b-a)x_0 + cy_0, bx_0 + cy_0 - ab, (x_0 - b)t - 1)$ where t is a slack variable. By the polynomial equation $(x_0 - b)t - 1 = 0$ we can express the fact that the equation $x_0 - b = 0$ doesn't hold for any value x_0. If it does then 1 belongs into the ideal I and related zero set of I would be empty. Hence adding the polynomial $(x_0 - b)t$ to the ideal $((b-a)x_0 + cy_0, bx_0 + cy_0 - ab)$ we suppose that the statement is not valid while all the other conditions are preserved (similarly as by the proof by contradiction). By the elimination of variables x_0, y_0, t from the ideal J which is accessible by the command `Elim(x[0]..t,J)` we will eliminate the *dependent* variables $x[0], y[0], t$ from the ideal J to obtain those polynomials from J which depend only on the remaining *independent* variables a, b, c. We enter

```
Use R::=Q[abcx[0]y[0]t];
J:=Ideal((b-a)x[0]+cy[0],bx[0]+cy[0]-ab,(x[0]-b)t-1);
Elim(x[0]..t,J);
Ideal(-a)
```

and get the only condition $a = 0$. It means that in the triangle ABC the vertices A and B coincide. We will exclude this case by addition of the non-degenerate condition $at - 1 = 0$ to the ideal I to obtain the ideal $K = I \cup \{at - 1\}$.
Now we have

```
Use R::=Q[abcx[0]y[0]t]
K:=Ideal((b-a)x[0]+cy[0],bx[0]+cy[0]-ab,at-1);
NF(x[0]-b,K);
0
```

The proof is now complete also in this case. Why it was necessary to exclude the case $a = 0$ can be seen from the equality $x_0 - b = 1/a((b - a)x_0 + cy_0) - 1/a(bx_0 + cy_0 - ab)$, where the polynomial $x_0 - b$ is expressed as an algebraic linear combination of the polynomials $(b - a)x_0 + cy_0$ and $bx_0 + cy_0 - ab$ but one exception, namely $a = 0$.
We have seen, that two almost the same modifications of the problem can cause unexpected difficulties. See [5], [10], [9], [15], [18] for further study.

3 Solving problems

In this part we will be concerned with solving problems both in a computational and classical way. The purpose is to see two different attitudes to solve problems.
Now we will finish the classical proof of the statement above – the heights of a triangle intersect at one point O, which we have proved in the first part automatically.
For a synthetic proof we can use e.g. the following way. We first draw a triangle ABC with the given heights h_a, h_b, h_c, see Fig. 2. The straight lines through the vertices A, B, C which are parallel to opposite sides BC, AC, AB of the triangle ABC respectively form a new triangle $A'B'C'$. Note that A is a midpoint of $B'C'$, B is the midpoint of $A'C'$ and C is the midpoint of $A'B'$. The heights h_a, h_b, h_c of the original triangle ABC are now perpendicular bisectors of sides of the triangle $A'B'C'$. Thus it suffices to show, that they meet at one point. Suppose that h_a and h_b intersect at the point O, hence it holds $|OB'| = |OC'|$ and $|OC'| = |OA'|$. From which $|OA'| = |OB'|$ follows and we get that O is the point of h_c.
We can also use Ceva theorem or another way to prove our statement. But for all the classical proofs it is necessary to have a *key idea*, which leads to the solution of the problem.
We proved it both in a classical way and in an automatic way. The both methods have their strengths and weaknesses.

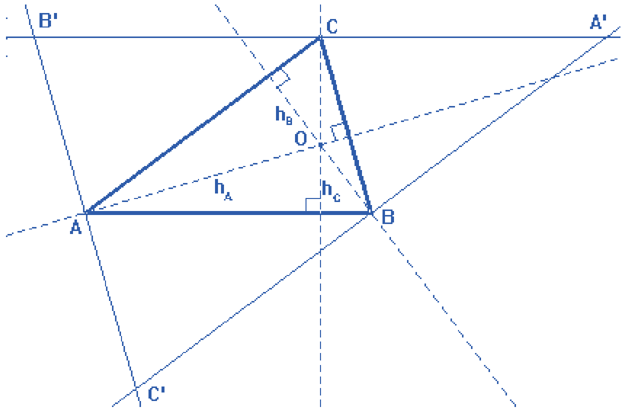

Figure 2: Heights of $\triangle ABC$ are concurrent

The classical way used some knowledge from geometry (we have to determine that the vertices of ABC are the midpoints of the sides of the new triangle $A'B'C'$ then to prove that perpendicular bisectors of the sides of $A'B'C'$ intersect at one point), give us a good overview about the problem, but had one weakness - we had to have the *key idea.* Which is not always easy to find.

On the other hand the automatic proof only needs knowledge of writing the equations of straight lines. The computation was quite automatic. But this method was not so geometric, was not so beautiful and sometimes some unexpected problems, in this case non-degenerate conditions, can occur.

The both method should be combined and used in practice.

To describe *automatic discovery,* we usually start with the formula of Heron. As this is the well-known case will present this theory on the less known next problem.

Let $ABCD$ be a planar quadrangle with sides a, b, c, d and diagonals e, f. Find the formula for the area p of a quadrangle $ABCD$.

First we discover such a formula by computer. As the second step we derive this formula by a classical method.

Choose the coordinate system so that the vertices of a quadrangle $ABCD$ be $A = [0, 0], B = [a, 0], C = [x, y], D = [u, v]$ and $a = |AB|, b = |BC|, c = |CD|, d = |DA|, e = |BD|, f = |AC|$, Fig. 3 We have following relations:

$(x-a)^2 + y^2 = b^2, (u-x)^2 + (v-y)^2 = c^2, u^2 + v^2 = d^2, x^2 + y^2 = e^2, (u-a)^2 + v^2 = f^2, p = 1/2(ay + xu - vy)$.

The elimination of x, y, u, v gives two polynomial equations. The first one

$16p^2 - (-a^4 + 2a^2b^2 - b^4 - 2a^2c^2 + 2b^2c^2 - c^4 + 2a^2d^2 - 2b^2d^2 + 2c^2d^2 - d^4 + 4e^2f^2) = 0$

is the desired relation. After the simplification we obtain

$$16p^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2. \tag{4}$$

This formula by means of which we can express the area of a quadrangle by the all six distances between the four vertices was published by Ch.R. Staudt [17] and we will call it *formula of Staudt.*

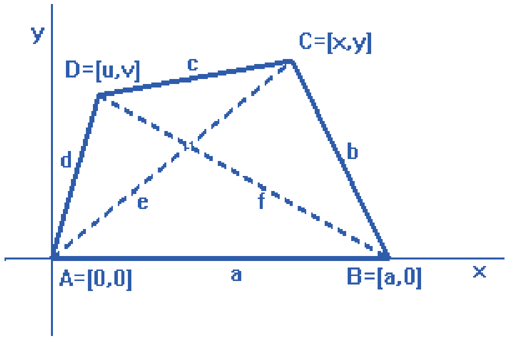

Figure 3: Area of a quadrangle by Staudt

Remark:

1) Note that the formula (4) holds by the given notation for all possible position of vertices A, B, C, D of a quadrilateral. Hence also in a case when $ABCD$ is non convex or even intersects itself. In that case we consider the area of a quadrilateral as the "signed" area.

2) If we set into the formula (4) e.g. $d = 0$ then a quadrilateral becomes a triangle and we get the formula of Heron. Hence (4) is a generalization of the formula of Heron. □

The second polynomial we received by elimination process above is related to the so called *Euler's four points relation,* which expresses the dependence of all six distances a, b, c, d, e, f between four vertices of a quadrangle. It is as follows

$e^4f^2 + e^2(a^2b^2 - a^2c^2 - b^2d^2 + c^2d^2 - a^2f^2 - b^2f^2 - c^2f^2 - d^2f^2 + f^4) - (-a^4c^2 + a^2b^2c^2 - a^2c^4 + a^2b^2d^2 - b^4d^2 + a^2c^2d^2 + b^2c^2d^2 - b^2d^4 + a^2c^2f^2 - b^2c^2f^2 - a^2d^2f^2 + b^2d^2f^2) = 0.$

Euler's four points relation follows from the Cayley - Menger determinant for the volume V of a tetrahedron with edges of lengths a, b, c, d, e, f

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & b^2 & f^2 & a^2 \\ 1 & b^2 & 0 & c^2 & e^2 \\ 1 & f^2 & c^2 & 0 & d^2 \\ 1 & a^2 & e^2 & d^2 & 0 \end{vmatrix} \tag{5}$$

if we put $V = 0$. A comparison of the equation $V = 0$ from (5) with the second equality which we received in elimination process above shows that the both polynomials are the same up to the constant factor 2. □

Now we show a classical approach to establish formula of Staudt (4). From right triangles AED and DEC the equalities $|DE|^2 = d^2 - |AE|^2$, $|DE|^2 = c^2 - |EC|^2$ follow with

$$d^2 - |AE|^2 = c^2 - |EC|^2. \tag{6}$$

Analogously from the right triangles AFB and CFB the equality

$$a^2 - |AF|^2 = b^2 - |FC|^2 \tag{7}$$

follows. Summing up the equalities (6) and (7) gives

$$a^2 - b^2 + c^2 - d^2 = |AF|^2 - |FC|^2 + |EC|^2 - |AE|^2. \tag{8}$$

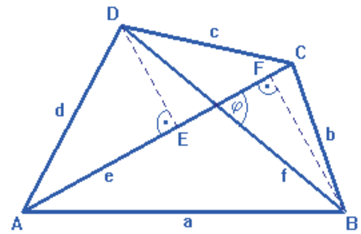

Figure 4: The proof of Staudt's formula

The right hand side in (8) can be written in the form $|AF|^2 - |FC|^2 + |EC|^2 - |AE|^2 = |AF|^2 - |AE|^2 + |EC|^2 - |FC|^2 = (|AF| + |AE|)(|AF| - |AE|) + (|EC| + |FC|)(|EC| - |FC|) = \pm 2e|EF|$, i.e.

$$(a^2 - b^2 + c^2 - d^2)^2 = 4e^2|EF|^2. \tag{9}$$

Further on we see that $|EF| = f\cos\varphi$. A substitution into (9) gives

$$(a^2 - b^2 + c^2 - d^2)^2 = 4e^2 f^2 \cos^2\varphi. \tag{10}$$

Now we will use the well-known formula for the area of a quadrilateral by means of the lengths of diagonals e, f and the angle φ

$$p = \frac{1}{2}\, ef \sin\varphi. \tag{11}$$

Finally the substitution of (11) into (10) with the use of the relation $\sin^2\varphi = 1 - \cos^2\varphi$ gives the formula (4).

In this example we could see, that a discovery by computer can be in some sense "easier" then by classical approach.
The last result we will use to arrive at the well-known formula of Brahmagupta for the area of an inscribed quadrilateral, which is a generalization of the formula of Heron. The problem is as follows.

Given a quadrangle $ABCD$ with the sides $a = |AB|, b = |BC|, c = |CD|, d = |DA|$, which is inscribed into the circle. Find the area of $ABCD$.

To solve the problem by computer we shall take advantage of the Staudt's formula (4) and use the *coordinate free* method. In accordance with the left Fig. 5 by Ptolemy's theorem $ef = ac + bd$ holds. We will eliminate variables e, f from the ideal $(16p^2 - 4e^2f^2 + (a^2 - b^2 + c^2 - d^2)^2, ac + bd - ef)$ and get the only polynomial equation

$$16p^2 = -(a^4 + b^4 + c^4 + d^4) + 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) + 8abcd, \tag{12}$$

or

$$16p^2 = (-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d)$$

or which is the same

$$p = \sqrt{(s - a)(s - b)(s - c)(s - d)}, \tag{13}$$

where $s = 1/2(a + b + c + d)$. This is the well-known formula of *Brahmagupta*, (Brahmagupta, 598 - about 665).

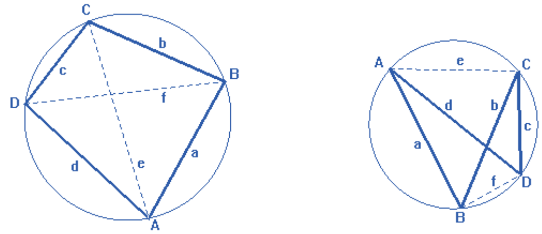

Figure 5: Cyclic quadrilaterals with the sides a, b, c, d – convex and non convex cases

By the figure on the right other position is also possible.

In this case by Ptolemy's relation $ef = ac - bd$ or $ef = bd - ac$. Elimination of e, f in the ideal $(16p^2 - 4e^2f^2 + (a^2 - b^2 + c^2 - d^2)^2, (ac - bd - ef)(ac - bd + ef))$ gives the relation

$$16p'^2 = -(a^4 + b^4 + c^4 + d^4) + 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8abcd \qquad (14)$$

or

$$16p'^2 = (a + b + c + d)(a + b - c - d)(a - b + c - d)(-a + b + c - d).$$

This is the formula for the area of a quadrilateral which intersects itself.

We could also use coordinate method to find the area of an inscribed quadrilateral. The classical proof of the Brahmagupta's formula is omitted and can be found for instance in [1].

4 Seminar works

At the end of the seminar students are engaged in a seminar work, which is aimed on a problem chosen by each student from the recommended offer list on Internet. Seminar works consist of the following parts:

1) Introduction of the problem
2) Description of the problem by a (dynamic) software
3) Solution of the problem in a classical (synthetic) way
4) Automatic proof (discovery) by computer
5) Conclusions (if necessary)
6) Printing of the seminar work.

As you can see from the above structure of a seminar work, it has a wide use of computer by solving problems. First the students searched for an appropriate problem from the Internet offer. Before choosing the problem they mostly consulted it with the teacher if the problem is appropriate to put it into the prescribed form. The second use of a computer is the choice of a (dynamic) geometric software to draw the situation and demonstrate it dynamically if possible. They often used Cabri II. This stage was closely connected with the choice of the text editor in which the final version of the seminar work would be printed. The most commonly two text editors were used – TeX and Word. To do a classical proof of a chosen problem, students used two possibilities - either to find the solution on Internet (at the recommended address) or to find their own solution.
To solve the problem by computer they usually use software CoCoA or Singular which are

for free, or Maple or Mathematica or Derive which are installed in the computer rooms. The most common problems which students encounter by automatic proof or discovery of a statement are as follows:

1) Algebraization of a geometric situation – unsuitable introduction of the system of coordinates, a complex or unsuitable description of geometric situation by algebraic equations.

2) A bad use of a computational method:
- keeping the correct order of all the stages of automatic proving,
- computation of normal form of an ideal,
- elimination of variables, their ordering, which variables to eliminate,
- finding non-degenerate conditions.

3) Understanding a geometric meaning of an algebraic equation.

4) Finding additional conditions to discover or generalize the statement, formulation of the generalized statement.

The next example shows some of the problems mentioned above. It is as follows.

Over sides of triangle ABC construct similar isosceles triangles ABC', BCA', CAB' with an arbitrary angle by the vertices A', B', C'. Then the straight lines AA', BB', CC' intersect at one point S.

Choose the Cartesian coordinate system so that $A = [0,0], B = [a,0], C = [b,c], A' = [k_1,k_2], B' = [l_1,l_2], C' = [m_1,m_2]$. Over the sides of the triangle ABC construct arbitrary isosceles triangles ABC', BCA', CAB' (all outwardly or all inwardly). The problem of this

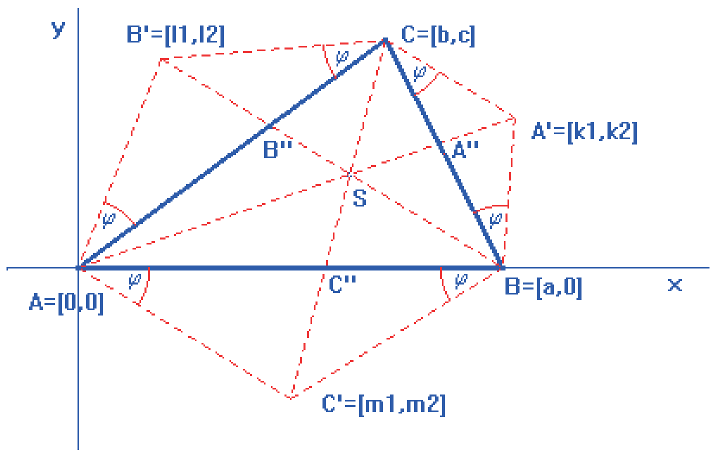

Figure 6: Similar triangles over the sides of ABC

task consists in expression of the notion "outwardly" or "inwardly" only by means of algebraic *equalities.* In this theory it is not possible to use algebraic inequalities, because of working in the field of complex numbers. To describe the point A' as the vertex of an isosceles triangle over the side BC, we can construct two circles with centers B, C with the same radii $|BA'| = CA'|$. Their intersection gives *two* points A' and A'' and we cannot distinguish between outer and inner points A' and A''.
Instead we will use the following method, which makes possible to construct the vertex A'

uniquely and which is due to D. Wang [18].
The vertex A' is the endpoint of a vector, whose initial point is in the center of BC with the length $v|BC|$, where v is a given number, and the same direction as the vector $B-C$ rotated by the angle 90^0 in a positive sense, i.e., it holds $(k_1-(a+b)/2, k_2-c/2) = v(c, a-b)$. Analogously we proceed by B' and C'. The line AA' has the equation $k_1y-k_2x=0$, the line BB': $(l_1-a)y-(x-a)l_2=0$ and the line CC': $(b-m_1)(y-m_2)-(x-m_1)(c-m_2)=0$. Suppose that $S=[s_1,s_2]$ is the common point of the straight lines AA' and BB'. We are to prove that the point S is on the line CC'. We have

```
Use R::=Q[k[1..2]l[1..2]m[1..2]s[1..2]abcv];
I:=Ideal(2k[1]-a-b-2vc,2k[2]-c-2va+2vb,2l[1]-b+2vc,2l[2]-c-2vb,2m[1]-a,m[2]+va,
k[1]s[2]-k[2]s[1],(l[1]-a)s[2]-(s[1]-a)l[2]);
NF((b-m[1])(s[2]-m[2])-(s[1]-m[1])(c-m[2]),I);
0
```

which means that the lines AA', BB', CC' meet at the point S. □

The following classical proof of the above statement which is due to O. Bottema [4], [16] is short end elegant and is worth of reproducing. It is based on the *area method.*

By the Fig. 6

$$|AC''|/|C''B| = \text{ Area } \triangle ACC'/\text{Area } \triangle BCC' = |AC||AC'|\sin(A+\varphi)/|BC||BC'|\sin(B+\varphi)$$

$$= |AC|\sin(A+\varphi)/|BC|\sin(B+\varphi)$$

and similarly

$$|BA''|/|A''C| = |AB|\sin(B+\varphi)/|AC|\sin(C+\varphi)$$

and

$$|CB''|/|B''A| = |BC|\sin(C+\varphi)/|AB|\sin(A+\varphi).$$

We will find that

$$\frac{|AC|}{|AC'|}\frac{|BA''|}{|A''C|}\frac{|CB''|}{|B''A|} = 1$$

and the result now follows from the converse of Ceva's theorem. □

Also this classical proof needs the key idea.

Find the locus of points S by changing the angle φ of similar triangles.

With the same notation we will eliminate variables $k_1, k_2, l_1, l_2, m_1, m_2, v$ in the ideal I. The elimination returns

```
Use R::=Q[k[1..2]l[1..2]m[1..2]vs[1..2]vabc];
I:=Ideal(2k[1]-a-b-2vc,2k[2]-c-2va+2vb,2l[1]-b+2vc,2l[2]-c-2vb,2m[1]-a,m[2]+va,
k[1]s[2]-k[2]s[1],(l[1]-a)s[2]-(s[1]-a)l[2]);
Elim(k[1]..v,I);
Ideal(-s[1]s[2]a^2+s[1]s[2]ab+1/2s[2]a^2b-s[1]s[2]b^2+1/2s[2]ab^2-1/2s[1]^2ac+
1/2s[2]^2ac+1/2s[1]a^2c+s[1]^2bc-s[2]^2bc-s[1]abc+s[1]s[2]c^2-1/2s[2]ac^2);
```

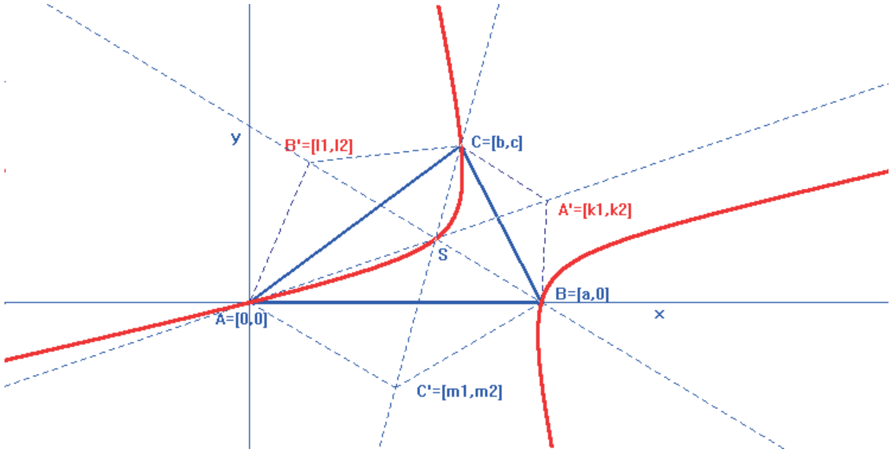

Figure 7: Kiepert's hyperbola

We see that the points $S = [x, y]$ lie (by the standard notation $[x, y]$ instead of $[s_1, s_2]$) on a conic

$$x^2c(a-2b)+2xy(a^2-ab+b^2-c^2)+y^2c(2b-a)+xac(2b-a)+ya(c^2-ab-b^2)=0, \quad (15)$$

which is called *Kiepert's hyperbola.* Kiepert's hyperbola (15) has many interesting properties, for instance it is a rectangular hyperbola, which goes through the vertices of the triangle ABC. It contains also further "remarkable" points of the triangle ABC as the centroid, the orthocenter, outer and inner Fermat's point etc. The Kiepert's hyperbola is closely tied with the Wallace line and Feuerbach's circle [12].
Now we will give the example of non elementary problem. It is as follows.

Four straight lines a, b, c, d are given in a plane. Construct a square $KLMN$ with each vertex on one straight line a, b, c, d.

Choose the coordinate system so that $K = [k_1, k_2]$, $L = [l_1, l_2]$, $M = [m_1, m_2]$, $N = [n_1, n_2]$ and the lines a, b, c, d have equations $a : a_1x + a_2y + a_3 = 0$, $b : b_1x + b_2y + b_3 = 0$, $c : c_1x + c_2y + c_3 = 0$, $d : d_1x + d_2y + d_3 = 0$. Suppose that $K \in a$ and $L \in b$. To ensure that $KLMN$ is a square with the vertices M, N for instance on the lines c, d respectively, we will rotate the vector $L - K$ by 90^0 in the positive sense to obtain the vector $N - K$. Then we rotate the vector $K - N$ by 90^0 in the same sense to obtain the vector $M - N$ an so on, see Fig. 8 We have the following relations:

$K \in a \Leftrightarrow a_1k_1 + a_2k_2 + a_3 = 0,$
$L \in b \Leftrightarrow b_1l_1 + b_2l_2 + b_3 = 0,$
$M \in c \Leftrightarrow c_1m_1 + c_2m_2 + c_3 = 0,$
$N \in d \Leftrightarrow d_1n_1 + d_2n_2 + d_3 = 0,$
$\mathrm{rot}(K - L) = N - K \Leftrightarrow -(l_2 - k_2) = n_1 - k_1,\ l_1 - k_1 = n_2 - k_2,$
$\mathrm{rot}(K - N) = M - N \Leftrightarrow -(k_2 - n_2) = m_1 - n_1,\ k_1 - n_1 = m_2 - n_2,$
$\mathrm{rot}(N - M) = L - M \Leftrightarrow -(n_2 - m_2) = l_1 - m_1,\ n_1 - m_1 = l_2 - m_2,$
$\mathrm{rot}(M - L) = K - L \Leftrightarrow -(m_2 - l_2) = k_1 - l_1,\ m_1 - l_1 = k_2 - l_2.$

We have 12 equations and we are to solve this system with respect to 8 unknowns k_1, k_2, l_1, l_2,

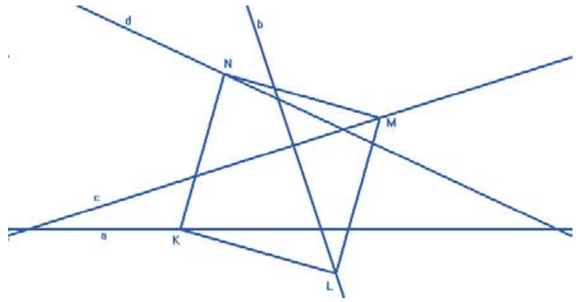

Figure 8: Square $KLMN$ with vertices on four straight lines a, b, c, d

m_1, m_2, n_1, n_2. To easy the computation we put $a_1 = a_3 = 0, a_2 = b_1 = 1, b_3 = c_3 = d_3 = -1$ without any loss of generality. We enter

```
Use R::=Q[k[1..2]l[1..2]m[1..2]n[1..2]a[1..3]b[1..3]c[1..3]d[1..3]];
I:=Ideal(a[1]k[1]+a[2]k[2]+a[3],b[1]l[1]+b[2]l[2]+b[3],c[1]m[1]+c[2]m[2]+c[3],
d[1]n[1]+d[2]n[2]+d[3],-(l[2]-k[2])-(n[1]-k[1]),l[1]-k[1]-(n[2]-k[2]),-(k[2]-
n[2])-(m[1]-n[1]),k[1]-n[1]-(m[2]-n[2]),-(n[2]-m[2])-(l[1]-m[1]),n[1]-m[1]-
(l[2]-m[2]),-(m[2]-l[2])-(k[1]-l[1]),m[1]-l[1]-(k[2]-l[2]),a[2]-1,a[1],a[3],
b[3]+1,c[3]+1,d[3]+1,b[1]-1);
Elim(k[2]..n[2],I);
```

and obtain the solution for k_1

$$k_1 = \frac{b_2c_1 + b_2c_2 + c_1d_1 + c_2d_1 - b_2d_2 - c_1d_2 + c_2d_2 + c_1 - c_2 - d_1}{b_2c_1d_1 + b_2c_2d_1 - b_2c_1d_2 + c_1d_1 - c_1d_2 + c_2d_2}.$$

Similarly we find other unknowns. See the square $KLMN$ in Fig. 8, which was done in Cabri II and is interactive. From the construction we can see that there exist at most four squares with given properties.

We can also proceed in the following way, see [9].

Denote the coordinates of $KLMN$ and the equations of straight lines a, b, c, d as above and consider the square with vertices $A = [1, 0]$, $B = [0, 1]$, $C = [-1, 0]$, $D = [0, -1]$. We are looking for such a similarity φ

$$\varphi : x' = px - qy + r,\ y' = qx + py + s,$$

where p, q, r, s are unknown coefficients, which maps the square $ABCD$ into the square $KLMN$. We have

$\varphi([1, 0]) = [p + r, q + s]$,
$\varphi([0, 1]) = [-q + r, p + s]$,
$\varphi([-1, 0]) = [-p + r, -q + s]$,
$\varphi([0, -1]) = [q + r, -p + s]$,

from which we get the system of equations

$h_1 : a_1(p + r) + a_2(q + s) + a_3 = 0$,
$h_2 : b_1(-q + r) + b_2(p + s) + b_3 = 0$,
$h_3 : c_1(-p + r) + c_2(-q + s) + c_3 = 0$,

$h_4 : d_1(q + r) + d_2(-p + s) + d_3 = 0.$

We are to solve the system of equations $h_1 = 0, h_2 = 0, h_3 = 0, h_4 = 0$ with respect to the unknowns p, q, r, s. We get

```
Use R::=Q[a[1..3]b[1..3]c[1..3]d[1..3]pqrs];
I:=Ideal(a[1](p+r)+a[2](q+s)+a[3],b[1](-q+r)+b[2](p+s)+b[3],c[1](-p+r)+c[2]
(-q+s)+c[3],d[1](q+r)+d[2](-p+s)+d[3],a[2]-1,a[1],a[3],b[3]+1,c[3]+1,d[3]+1,
b[1]-1,p+r-x,p+s-y);
Elim(q..s,I);
```

with the same result as above.

5 Examples of seminar works

Problems which have been solved in seminar works were mostly taken from the Internet address: http://www.cut-the-knot.org/geometry.shtml.
Some of the chosen topics are as follows: Thébault problem, Sum of distances to the sides of an equilateral triangle (Vivianni), Butterfly theorem, Theorem of Menelaus, Ptolemy's theorem, Stewart's theorem, Napoleon's theorem and topics related to Napoleon's theorem, Eyeball problem etc.

On the next pages you can see the example of a seminar work on Zaslavsky problem, which is due to the student of the 4^{th} year university study.

SEMINAR WORK

Ilona Hamanova

University of South Bohemia, Czech Republic

1 Abstract

In this work I am going to prove the Zaslavsky's theorem using the program CoCoA, free available at *http://cocoa.dima.unige.it*. All pictures are made by Cabri geometry II. I found the theorem at *http://www.cut-the-knot.org*, where many other theorems and problems are proved.

2 The theorem

What does the Zaslavsky's theorem say?

Zaslavsky's theorem: Given triangle ABC, point P, and reflection $A'B'C'$ of ABC in P. Let three parallel lines through A', B', and C' intersect BC, AC, and AB in X, Y, Z, respectively. Then X, Y, Z are collinear.

For better understanding see the situation on Fig.1

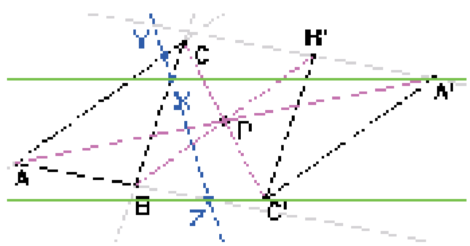

Figure 1: The situation

3 Elementary proof

To prove the theorem we need to add some other points to the picture and to use the Menelaus' theorem.

The points:
Let the line through C parallel to the "parallel triplet" intersect BA' at N, AB' at K and $A'B'$ at Z'. (See Fig.2)

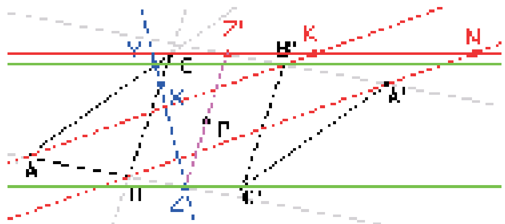

Figure 2: The situation with the points N, K, Z'

Menelaus' theorem:
Let three points F, D, and E, lie respectively on the sides AB, BC, and AC of $\triangle ABC$. Then the points are collinear if $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE} = 1$. (See Fig.3)

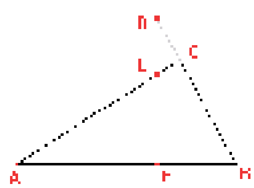

Figure 3: Menelaus' theorem

2

First note that Z' is a reflection of Z in P, so that $B'Z' = BZ$. Also, since the points A, B and Z are collinear, so are their reflections A', B', Z'. In particular, $A'Z' = AZ$. Thus

$$\frac{B'Z'}{A'Z'} = \frac{BZ}{AZ} \tag{1}$$

No we use the Menelaus' theorem. We want to prove that

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1 \tag{2}$$

$$\begin{gathered}
\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = \\
\frac{B'K}{NA'} \cdot \frac{AZ}{BZ} = \\
\frac{B'Z'}{A'Z'} \cdot \frac{AZ}{BZ} = \\
\frac{BZ}{AZ} \cdot \frac{AZ}{BZ} = 1
\end{gathered}$$

We proved the theorem.

4 Automatic proving

To prove the theorem using the program CoCoA we need to locate the situation to the system of coordinates. It is useful to place one point of the triangle ABC to zero point and one side of the triangle on the axe x. For example $A[0,0]$, $B[a,0]$. The whole location of the situation to the system of coordinates is shown on the Fig.4.

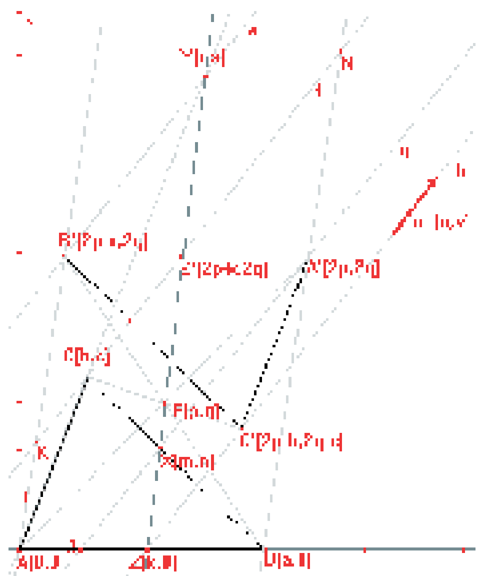

Figure 4: The situation in the system of coordinates

Now we express some facts:

$B' \in e : v(x - 2p + a) - u(y - 2q) = 0$
$Y \in e : v(x - r) - u(y - s) = 0$
$A' \in g : v(x - 2p) - u(y - 2q) = 0$
$X \in g : v(x - m) - u(y - n) = 0$
$C' \in h : v(x - 2p + b) - u(y - 2q + c) = 0$
$Z \in h : v(x - k) - uy = 0$
$X \in BC : c(m - a) - n(b - a) = 0$
$Y \in AC : cr - bs = 0$
$\vec{u} = \overrightarrow{A'X} : v(2p - m) - u(2q - n) = 0$

In CoCoA we enter:

```
Use R::= Q[abcpquvmnkrsxyt];
I:=Ideal(v(x-2p+a)-u(y-2q),v(x-r)-u(y-s),v(x-2p)-u(y-2q),
v(x-m)-u(y-n),v(x-2p+b)-u(y-2q+c),v(x-k)-uy,c(m-a)-(b-a)n,
cr-bs,v(x-b)-u(y-c),v(x-2p+k)-u(y-2q) ,((m-k)s-(r-k)n)t-1);
NF(1,I);
```

The normal form (NF) equals 1 which means that the statement is not true. The following command eliminates the dependent variables m, n, k, r, s, x, y and the slack variable t:

```
Use R::= Q[abcpquvmnkrsxyt];
I:=Ideal(v(x-2p+a)-u(y-2q),v(x-r)-u(y-s),v(x-2p)-u(y-2q),
v(x-m)-u(y-n),v(x-2p+b)-u(y-2q+c),v(x-k)-uy,c(m-a)-(b-a)n,
cr-bs,((m-k)s-(r-k)n)t-1);
Elim(m..t,I);
```

We obtain:`Ideal(-v,u)`. This result gives us the requirements for $-v \neq 0$ and $u \neq 0$. So we enter this two requirements to the ideal and try if the statement is true. We enter:

```
Use R::= Q[abcpquvmnkrsxytwz];
I:=Ideal(v(x-2p+a)-u(y-2q),v(x-r)-u(y-s),v(x-2p)-u(y-2q),
v(x-m)-u(y-n),v(x-2p+b)-u(y-2q+c),v(x-k)-uy,c(m-a)-(b-a)n,
cr-bs,v(2p-m)-u(2q-n),-vw-1,uz-1,((m-k)s-(r-k)n)t-1);
NF(1,I);
```

The result 0 means that the statement is true, so we proved the Zaslavsky's theorem.

CONTS

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