The Correctness, Completeness and Compactness Standards of Computer Algebra Systems and of School Mathematics.

Abstract:

In many cases when solving a school algebra problem (e.g. an equation) using a computer algebra system (e.g. Derive, Maple, Mathematica, MuPAD) we get the answer that is perfectly suitable for both the teacher and the student as well as others. Nevertheless, one may encounter answers having some qualities that are disturbing when used at school, such as the answer being valid on certain conditions only, solving is not brought to an end, the answer containing elements unknown at the specific school level, etc.

The qualities can be represented as deficiencies in relation to correctness, completeness and compactness. Based on the smoothing of disturbing qualities, the answers offered by computer algebra systems may conditionally be divided as follows:
applicable with the help of extra explanations provided to students;
adaptable using the resources of the same computer algebra system;
unsuitable.

The paper also provides examples of smoothing possibilities.

The problems (or rather answers) treated in this paper concern division by zero and extracting the square root of a negative number - from calculating 1/0 to literal equations and inequalities. An analysis of textbooks reveals that the standards vary. There are various conventions, e.g. assume that variables are restricted, check solutions in the end of solving process only, and such like.

In this paper the different standards of computer algebra systems and school mathematics treatment have been compared.

Introduction

This article examines problems obstructing the use of computer algebra systems in teaching and learning mathematics at school. Attempts are made to systematise problems for teachers, software developers and others related to answers offered by computer algebra systems, focusing on two narrower topics (division by zero and square root extraction).

Computer algebra systems have been in use at schools for a fairly long time already. Recently, authors of all the leading systems have claimed mathematics teaching to be an important application of computer algebra. Nevertheless, the implementation of computer algebra software is not always as quick as desired. Comparing the situation of software in algebra with that in dynamic geometry, for instance, the main obstruction seems to be that in many cases teachers are not satisfied with the results put out by computer algebra systems. In many cases when solving a school algebra problem (e.g. an equation) using a computer algebra system (e.g. Derive, Maple, Mathematica, MuPAD) we get the answer that is perfectly suitable for both the teacher and the student as well as others. Nevertheless, one may encounter answers having some qualities that are disturbing when used at school, such as the answer being valid on certain constraints only, the answer containing elements unknown at the specific school level, in equation solving, extraneous roots are also issued, etc. Such cases create difficulties for students and also pose problems for teachers. It may even be said *that many teachers don't understand some of the answers given by a CAS. (Typical questions are: „Why does it give this answer?“, „Why is this antiderivative different from that in the solution book?“, or „Why could it not solve this equation?“) As long as teachers have questions like this, they will not want to use these systems in their teaching – simply because they are afraid that their students might ask similar questions ...* ([Kokol-Voljc & Kutzler 2002]).

The author of this article has studied the capabilities of computer algebra systems in solving school mathematics problems over several years ([Tonisson 2002a], [Tonisson 2002b], [Tonisson 2002c]). The experiments painted a generally positive picture, which, however, included a number of cases where the answers to problems by some or all systems were either slightly different from those expected at the school lesson or given in substantially different forms/terms, or such like. At the same time, it is clear that only in very rare cases can the answers provided by software applications be considered outright erroneous. Why, then, do computer algebra systems give different answers than those expected at school? Do the authors of the systems deem the correct answer to a problem to be different from what it is in school mathematics? B. Kutzler recommended analysing the material gathered with respect to the correctness, completeness and compactness of answers. These concepts can be defined, and, accordingly, related to one other, in a different manner. The section 2.2 provides a more detailed description of the concepts of correctness, completeness in the sense of branches, completeness in the sense of being brought to completion and compactness.

This article provides examples where the answers are significant in the light of the school treatment. The examples are classified in the system of correctness-completeness-compactness. The examples are analysed from the point of view of the school mathematics and discussed with regard to their reasons and possibilities of smoothing. (The work also involved consultations with the developers of computer algebra systems.) In some cases, these features are not so much defects as features offering new possibilities.

It may be stated that the two obvious reasons why the answers obtained by applying the so-called traditional commands and settings differ from those expected at school are:
1) the fact that in computer algebra systems different assumptions are used, for example the default domain is complex numbers whereas at school it is made up of a set of real numbers, or of an even narrower subset;
2) the mathematical and programming inaccuracies (or dissimilar understandings) of the authors of computer algebra systems.

These reasons do not cover all the cases, however. One of the objectives of this article is to find a more exhaustive set of reasons.

It is not a new discovery that the computer algebra systems do not always work as expected, and that there are bugs as well as theoretical and practical limitations. One of the most important articles in the field is [Stoutemyer 1991]. The concluding remarks of that article were relevant back then just as they are relevant today:
The goal here to inspire caution. These systems can be extraordinarily useful if users are aware of underlying assumptions and of their responsibility to verify results.

The current article examines the issues related to division by zero and extracting the square root of a negative number in solving the central algebraic problems – calculation, simplification and solving equations and inequalities. In such problems, the value of expressions may with some variable or subexpression values (for instance, zeros or negative numbers) be indeterminate. The first chapter of the article explores the rules applied at school in dealing with such situations – what to write in the solutions that involve division by zero or extracting the square root of a negative number. There exists probably no uniform school standard that would be internationally applicable; in addition, several differences may occur even within one country. This article is based mainly on Estonian textbooks, but a number of textbooks in English were used as well. Different materials as well as different parts of the same material may provide different possibilities of interpretation. Furthermore, the teacher may adjust some textbook requirements. Consequently, the school treatment given in this article is rather of an illustrative nature and may not be precisely applicable in the various countries and textbooks. Nevertheless, it provides a suitable frame of reference for the examples of computer algebra systems.

The second chapter of the article describes a choice of the answers provided by computer algebra systems Derive 6 (2003), Maple 8 (2002), Mathematica 4.2 (2002) and MuPAD 3.0 (2003). The same chapter explains the basis of classification of the examples and the role of number domains. The third chapter presents the examples in more detail: the answers of different computer algebra systems are compared with each other, with the results expected in the school and with mathematics in general. Attempts are made to explain why one or the other computer algebra system works in this or the other way. A summary of recommendations for smoothing out disturbing features is given in the fourth chapter. As well, a set of problems is presented by which the teacher can determine how the system at their disposal behaves at critical points.

If not specified otherwise, use is made of the corresponding basic commands (for instance, Solve, Simplify) and default settings of the computer algebra systems. The problems have mostly been taken from textbooks, with only an occasional one being specifically composed. In this article quotes borrowed from textbooks, etc. are presented in *italics* and answers recommended in textbooks in `Courier type.`

1 Division by zero and square root extraction at school

1.1 Calculating

In the interests of future discussions this chapter also provides an overview of the school treatment, which could set up certain background for further examples. The following overview is based on a number of school textbooks, handbooks and problem sets written in the English and Estonian languages. Hereinafter they are simply called textbooks. Although the explanations and requirements additionally given by the teacher at the lesson play an important role textbook treatment may obviously be considered as setting the standard. We try to find out what the school mathematics standard is for treating indeterminacy-causing cases in division and square root extraction. What does the student learn in theory and what they are recommended to say and to write in the solutions of problems where expressions may include division by zero or extracting the square root of a negative number?

With respect to division by zero, the school sources studied are unanimous, namely that *division by zero is undefined*. This is explained by means of multiplication. An important argument is the fact that *multiplying by 0 always results in 0*. What, then, must be given as the answer for 1/0? But what about 0/0? In general, the answer provided for the first case in textbooks reads `not defined` (or such like). Some textbooks give the same answer for 0/0 while others read `indeterminate`.

The situation is analogous for square root: *There is no square root for a negative number, for among the numbers as yet known to us there is no such number whose square is negative* ([Form 8]).

How must theoretical knowledge be applied to problem solving? The understanding is cultivated in the student that division by zero and extracting the square root of a negative number are something special and must somehow be avoided. Whether the avoidance should be effected by the compiler or the solver of the problems is in most cases not explicitly stated. Some textbooks do not even contain any "suspicious" calculation problems (e. g. 1/0, $\sqrt{-4}$), thereby precluding student concerns. Others still containing such problems politely state in the directions: *Divide if you can. Explain why you cannot.*

With respect to numerical expressions the teaching received from the theory section is adequate and the solutions can be recorded without problems. Occasionally, of course, expressions like `not defined` (or such like) must be used. Difficulties may arise with problems where a more complicated subexpression, whose value actually equals 0, is reduced without calculation. As a rule, however, such problems are not included in textbooks.

1.2 Simplification

The situation changes when variables appear in problems (including in denominators or under square roots). This results in expressions whose value is determinate at some variable values and indeterminate at others. Simplification (including reduction) of expressions, multiplication of equations by fraction denominators, etc., may result in new expressions whose domain of definition differs from the original one. According to the textbooks, what should be written in the solutions while performing such transformations?

First, the distinguishing of critical situations is taught; for instance, problems are presented:
Find the domain of the expression. What values of variable x have a square root?
When transformation problems emerge, however, the distinguishing of forbidden values is discarded, and the practice is even "legalised".

Even though not always explicitly stated, we always assume that variables are restricted so that division by 0 is excluded ([Barnett & Ziegler 1989]).

The equality is valid only at such variable values where the value of either side of the equality is calculable. For instance, the equality $\frac{x}{x-1}=\frac{x\cdot x}{(x-1)x}$ *is valid only where x≠0 and x≠1. As mentioned above, such restrictions are henceforth not explicitly stated in the equalities* ([Form 9]).

An important difference with respect to irrational expressions is only that it is normally assumed, without adding conditions to expressions, that the variables in the expressions have only such values at which all the radicands are positive ([Form10]).

Simplify. All variables represent positive real numbers ([Barnett & Kearns 1990]).

Unless stated to the contrary all variables are restricted so that all quantities involved are real numbers ([Barnett & Ziegler 1989]).

Such conventions allow the students to remorselessly reduce, expand, isolate variables from the radical, etc. Without a thought to division by zero or extracting the square root of a negative number.

1.3 Equations

In solving expression simplification problems such legalised disregard for forbidden values results in no direct contradictions. The problem resurfaces, however, when solving fractional equations and radical equations. This results in extraneous roots, for such transformations are used in solving that do not ensure the obtainment of an equation equivalent to the preceding one (for instance, the squaring or multiplication by a certain expression of either side). However, the textbooks recommend precisely such transformations. Different variants are used in the textbooks to obtain correct final answers. With respect to fractional equations some textbooks (e. g. [Barnett & Ziegler 1989]) recommend separating such variable values that turn the divider into zero already in the initial phase of the solution process (for instance, before multiplying the fractions by the common multiple). Thus also legitimises multiplication by the common multiple (i.e. guarantees the equivalence of equations). Other textbooks (e. g. [Form 9]) instruct to separate extraneous roots at the end of the solution process by replacement into the original equation. Indeed, in radical equations extraneous roots are separated only at the end of the solution process. Replacement of potential solutions into the original equation is the main technique in solving such equations.

Even before fractional or radical equations the student is introduced to linear and quadratic equations. Although division or square root extraction cannot initially be seen in these equations the student comes to face these techniques during solving. With respect to linear equations, the issue of division by zero emerges in an equation containing 0x on one side and a number other than zero or zero (so-called pseudo-linear equation) on the other. According to the textbook [Form 7] the answer in this case is

`The solution set of this equation is empty set ∅.`

or

`The solution set of this equation is the entire set of numbers known to us, that is, the rational number set` **`Q.`**

respectively.

With respect to quadratic equations the formula requires division by zero where the quadratic term coefficient proves to be zero (the so-called pseudo-quadratic equation); in actual fact, the equation degenerates into a linear equation on such occasions. The problem of square root extraction emerges where the discriminant is negative, in which case the textbooks recommend the following answer:

`The given equation has no real solutions.`

1.4 Literal equations and inequalities

At school literal (parameter-containing) equations are also encountered. These appear both as independent problems and within various formulas. An example might be the solution formula of the equation $ax^2 + bx + c = 0$. In literal (parameter-containing) equations the solution branches out by the values of the parameter(s). Depending on the complexity of the problem such branches may be fairly many in number. The minimal case considered correct in some way would be one where it is assumed by default that the parameter has no "suspicious" values, and then **only the main branch is calculated**. This is often assumed in applied problems (physics, etc.) (A=P+Prt; please express r). The next level is where it is **recorded with the main branch what parameter values** result in the branch. The most thoroughgoing is the level where **all the cases are shown separately**. For instance, the answer to the equation $ax^2 + 2x + 1 = 0$ is ([Form 10]):

If $a=0$, then $x=-0,5$;

if $a \in (-\infty;0) \cup (0;1]$, *then* $x = \dfrac{-1 \pm \sqrt{1-a}}{a}$;

if $a>1$, then no solution.

With respect to the fractional inequality it is noted that the danger of the denominator being zero should be heeded.
Note: On the other hand, P/Q is not defined at the real zeros for Q (division by 0 is not permitted), and the zeros for Q must not be included in the solution set ([Barnett & Ziegler 1989]).

The solution must not contain those values of x that turn the denominator into zero.
(In the example, $x \neq 0$ is carried along.)

With respect to the fractional inequality, the danger of division by zero is rather indicated as a note. With respect to the radical inequality the explanation is more detailed, with the domain of definition being mentioned:

An inequality, including a radical inequality, generally has infinitely many solutions. It is therefore not possible to eliminate extraneous roots from inequalities by means of solution checks. In solving radical inequalities one must be confined to those transformations not resulting in extraneous roots. In this case such a transformation is the squaring of the sides of the inequalities provided both sides have positive numbers. Apart from that, the domain of the inequality must be considered in recording the answer. The domain of the inequality is formed of those variable values at which all the expressions appearing in the inequality are determinate ([Form 10]).

With respect to parameter-containing inequalities branches are normally treated in more detail, since apart from checking an expression's equalling with zero by parameter values also important are the positivity and negativity of the expression.

1.5 Branches

We try to describe everything given in the preceding from the perspective of responding to forbidden branches. We understand a forbidden branch as being a branch containing zero in the divider or a negative number under the square root sign.

Topic	Is responded	Is not responded
Calculating (Degenerate case, always one branch)	not defined, cannot	
Simplification		we assume there are no forbidden branches
Fractional equations	to separate at the beginning	
	extraneous roots to be eliminated at the end	
Radical equations	extraneous roots to be eliminated at the end	
Linear and quadratic equations	the solution set is the entire number set known to us there are no real number solutions; the solution set is the empty set	

Fractional and radical inequalities	to observe the domain	
Literal equations	all the branches separately	sometimes (e.g. in applied problems)
Literal inequalities	all the branches separately	

In the sense of branches, then, forbidden branches are not discussed separately in rather many cases, particularly where simplification is concerned. The elimination of extraneous roots at the end can be categorised in several ways – on the one hand no response is given initially but on the other it is still given ultimately.

It may be assumed that completeness is rejected for the sake of compactness. Apparently, it is more complicated as well as more time- and space-consuming to (repeatedly) record several branches and special cases. Furthermore, repeated recording entails the danger of oversights, etc. As well, it is more difficult to grasp the answer where it contains many special cases and branches, which distract attention from the main line. For the sake of compactness leaving out special cases in simplification is allowed. Their separate recording would shift the focus of the problem: instead of simplification most of the time would be spent on finding the forbidden points. In recording, it might also be pointed out whether it would be sufficient to say something concerning the entire expression contained under the radical sign or in the denominator or the respective variable values should be found separately, which would require particularly much work. As in simplification special values are not recorded for the sake of compactness, similar tendencies can be observed with equations. The technique of replacing extraneous roots into the original equation may be termed as temporary disregard of dangers. The solution to an equation is checked only at the end, it is not added all the time.

In conclusion, it may be said that the standard proves to be thematically dynamic: in theory the matter is clear; however, the scheme given in the presentation of the theory is consistently implemented only in solving numerical problems and with parametric equations. With respect to transformation a compromise is made, which is dangerous in equation solving and is either discarded or compensated for by inserting an extra step into the solution algorithm.

2 Choice of examples and base for classification. Number domain

2.1 Examples

Reviews of answers given by computer algebra systems have already been performed in the past, of course. The most comprehensive of these is presented in an article [Wester 1999] where Michael Wester examines hundreds of problems related to different computer algebra systems. (In reply to my inquiry, Wester stated that no subsequent tables like his have come to his knowledge.) Unfortunately, the table does not contain particularly many problems directly related to school, and the reviews have not been provided from the perspective of school mathematics. Understandably so, because computer algebra systems were primarily developed to support mathematicians in “doing mathematics”. As were such reviews.

In this article we try to evaluate, in relation to the school treatment, the answers given by computer algebra systems in response to the commands of calculation, simplification, equation solution, etc. Here we examine only the answer, since the commands under study solve problems in a single step, as a “black box”. Computer algebra systems have their own interior standards. These are not necessarily well-documented. This article tries to introduce these standards with the help of examples.

Many textbook examples are examined as well as further problems developed from these examples. Only the examples in which the answers are interesting from the point of view of the school are presented in this article. The examples are arranged by their mathematical topics.

Topic	Problem	Problematic answer (System)
Calculating		
	1/0	$\pm\infty$ (Derive)
	$\sqrt{-4}$	2i (All, by default)
	0/sqrt(0)	0 (Maple)
Simplification		
	$\frac{97x}{x}$	97 (All) Is x=0 recorded separately?
Linear equation		
	0a=0	C (MuPAD)
Quadratic equation		
	(4x-1)(x+3)=5x(0.8x+2)	$\{[x = -2.305843009 \cdot 10^{18}], [x = 2.875]\}$ (MuPAD) The correct answer is 3.
Fractional equation		
	$\frac{15}{x^2-4x-5} - \frac{9}{x^2-5x} = \frac{5}{x^2-1}$	-9 and $\pm\infty$ (Derive)
	$\frac{x \cdot x}{x} = 0$	0 (All)
Radical equation		
	$\sqrt{2x+6} + \sqrt{x-3} = 2\sqrt{x}$	-27/7 and 3 (All, by default) At school only 3.
Literal equation		
	ax=1	To what extent are branches presented? Different systems (and sometimes the different commands) work differently.
	ax^2+bx+c=0	To what extent are branches presented? Different systems (and sometimes the different commands) work differently.
	3(a+1)x+3a=2	$\frac{3-3a}{-2a+3(1+a)}$ (Mathematica)
	$\frac{3mx-5}{(m+2)(x^2-9)} = \frac{2m+1}{(m+2)(x-3)} - \frac{5}{x+3}$	The parameter values turning the main solution into one changing the denominator into zero are not given separately. (MuPAD)
Literal inequality		
	ax>1	ax>1 (Derive) $\{-\mathrm{signum}(a)\, x < -\frac{\mathrm{signum}(a)}{a}\}$ (Maple)
	$x - \frac{a}{1-a} < 1 - \frac{x-1}{a-1}$	See next row.
	$(x < 0 \wedge (a < 0 \vee a > 1)) \vee (a < 1 \wedge x > 0 \wedge a > 0)$	

In the next section we will try to classify the above examples. With some inequalities, (e.g. $x^2+2x+1 > \frac{1}{a} - \frac{2}{a^2}$) MuPAD never completes the solving process. We do not discuss such problems as a separate aspect.

2.2 Correctness, Completeness, Compactness

2.2.1 Correctness

The words "correctness", "completeness" and "compactness" can be used to denote various meanings and shades. In this section, we try to clarify the concepts used in this article. In this article we understand "correctness" as correctness of answer in a narrower sense. We leave out the circumstances falling under the subsequent aspects. Thus, for instance, we consider correct those answers that have not been completely simplified ($\frac{3-3a}{-2a+3(1+a)}$). Or where only the main branch has been given. In general, we may consider that textbook answers are correct (provided there are no misprints or other such errors). On rare occasions, faulty answers may be encountered in computer algebra systems. And undoubtedly, incorrect answers can often be found in student works; these, however, are not discussed in this article.

From the above examples, the false answers of equation (4x-1)(x+3)=5x(0.8x+2), the answer 0 of equation $\frac{x \cdot x}{x} = 0$ and conditionally also the question of solutions of radical equation (commented in section 2.3) are suitable for this class.

2.2.2 Completeness in the sense of branches

In problems involving division by zero and square root extraction the value of expressions may with some variable or subexpression values (such as zeros or negative numbers) be indeterminate. This creates the need (particularly in literal equations and inequalities) to add restricting conditions to intermediate results or answer forms, or, in some sense, to divide the solution/answer into branches. This, in turn, creates different possibilities of how to treat branches. (The school treatment is commented in section 1.5.) It is possible to make certain compromises in the treatment of branches (for instance, not to indicate in the solution that x≠0).

From the above examples, the example of simplification $\frac{97x}{x}$ and branch-treatment of literal equation are suitable for this class.

2.2.3 Compactness

The pursuit of compactness provides the motivation to discard completeness in the sense of branches. Compactness is treated in this article only insofar as it is directly related to the topics of branches. Herein, completeness is a kind of "inverted aspect" of completeness in the sense of branches. One of the main criteria of compactness is whether the number of branches is such as to be graspable by the reader. Apart from the number of braches one must also consider how they can be recorded in a graspable manner in order for them not to prove overly complicated to decipher. It depends on the particular student how much and what they can grasp. With respect to textbooks, it is difficult to point out a general separate standard. (For instance, that the answer contain no more than n branches.) Rather, it is reduced to the standard of completeness in the sense of branches. Nevertheless, it may be said that in textbooks as a rule forms are used to distinguish branches,
if CONDITION then EXPRESSION
or
EXPRESSION if CONDITION

The answer presented by the computer algebra system
$(x < 0 \wedge (a < 0 \vee a > 1)) \vee (a < 1 \wedge x > 0 \wedge a > 0)$
may be difficult to read. In actual fact, compactness forms part of a wider set of topics – legibility in a broader sense, notation, etc.

2.2.4 Completeness in the sense of being brought to completion

Completeness can also be considered in the sense of being brought to completion, which shows whether the problem has been solved to the end – whether the answer has been simplified. This aspect generally provides more opportunities for different approaches (for instance, whether the final answer should be presented as a mixed number, improper fraction or decimal fraction and whether the denominator should be rid of irrationality). In this article, we regard as violations of the school treatment such examples in which it is evident that the solution is incomplete and could be continued in plain steps. Again, there are no such examples in textbooks.

From examples in the table, the expression $\frac{3-3a}{-2a+3(1+a)}$ and the unfinished answers of literal inequality are suitable for this class.

2.3 Number Domains

Some examples that seem to be the examples of correctness could be explained through number domains. At school, the main number domains used with these topics are the rational number set **Q** and the real number set **R**. (In reality, expansion to **Q** (or, to be more exact, to $\mathbf{Q}^+$) is done by means of division and to **R** by means of square root). With respect to extracting the square root of a negative number, many of those more experienced in mathematics know that $\sqrt{-1}$ is i, and expanding thus to the domain of complex numbers we can still calculate such expressions. Until then, however, we must say even concerning the expression $(\sqrt{-4})^2$ that it is impossible to find the value from among real numbers. Furthermore, the school curricula in many countries normally do not include complex numbers while in other countries complex numbers are a part of the school curricula.

By the same token, the "boundary" between real and complex numbers is also essential for computer algebra systems. Namely, by default they seem to deal with complex numbers. However, the systems offer resources to restrict the number domain to, say, real numbers. In principle, restrictions can be envisaged at different levels. Whether and how can

- the calculation result;
- the variable value;
- the equation (inequality) solution;
- the entire process be determined in terms of real numbers.

Different systems possess different capabilities. The answer the systems give directly to calculations is a complex number. For instance, the answer to $\sqrt{-4}$ is given as 2i. In all the systems, restrictions can be imposed on variable values; nevertheless, not all the systems behave in the same way. In some systems, for instance, a real number variable can be assigned a complex number value at the "user's own risk". Exclusively real number solutions to an equation can be calculated with Derive and MuPAD. If this is not done, however, the answer to the equation 0*a=0 given by MuPAD, for instance, would be $\mathbb{C}$. Other systems include symbols in the answer (such as {{}}, true, a) to show that the equation is true for all values of the solution variable and that the exact number set cannot be read.

Maple and Mathematica allow the transfer of the entire process to the domain of real numbers (packages RealDomain and RealOnly respectively). Yet with regard to radical equation $\sqrt{2x+6}+\sqrt{x-3}=2\sqrt{x}$ this is completely accomplishable only in Maple. Namely, the solution to this equation obtained at school is 3. By default, all the computer algebra systems give the answer –27/7 and 3. Of the two, however, -27/7 is not appropriate when operating with real numbers only, since a negative number appears under the square root signs. Using the options in Derive (Real Solution) and in MuPAD (assume(x,Type::Real) we still get both answers, which is actually correct, for –27/7 is undoubtedly a real number. The package RealOnly in Mathematica also yields both answers, which is somewhat misleading, however, since only operations involving real numbers, which may also be interpreted as solution checking, should be permitted. Maple's package RealDomain yields 3 alone. (A Real Mode option which eliminates intermediate expressions which may be complex is on the list for a future release of Derive.)

The number domain topics are conditionally represented in Derive if the answer to 1/0 is $\pm\infty$ and, with respect to the fractional equation $\frac{15}{x^2-4x-5}-\frac{9}{x^2-5x}=\frac{5}{x^2-1}$, the other answer apart from -9 is $\pm\infty$.

Likewise, Mathematica implies infinity in answer to 1/0:

```
Power::infy : Infinite expression 1/0 encountered.
```

As well, the issues stated in the lexicon MathWorld ([MathWorld])

There are, however, contexts in which division by zero can be considered as defined. For example, division by zero $z/0$ *for* $z \in \mathbb{C}^* \neq 0$ *in the extended complex plane C-Star is defined to be a quantity known as complex infinity.*

and in the standard IEEE Std 754-1985, for instance, point to infinity. Thus, Derive behaves according to the standard, which, however, differs from the school treatment. Apparently, the problem could be solved by giving explanations to the student; whether they understand the explanations, however, is another matter.

3 Examples in more detail

3.1 Correctness

3.1.1 Automatic simplification as a cause of extraneous solution

Violations of correctness are not frequent in computer algebra systems but still occur occasionally. It must be considered incorrect, for instance, when all the systems offer 0 as the answer to the equation $\frac{x\cdot x}{x}=0$. This answer could be explained by the fact that the original equation is automatically transformed to equation x=0.

We cannot approve such solutions of equations from the point of view of the school. (However, it seems that the textbooks do not include examples like $\frac{x\cdot x}{x}=0$.) It is possible to justify the answers given by the computers algebra systems by using limits (as do the developers of computer algebra systems). Section 3.2.1 discusses some further aspects of automatic simplification.

If we replace 0 into the left side of the original equation and calculate $\frac{0\cdot 0}{0}$, then all the systems behave as in the case of the division 0/0: Derive gives ?, Mathematica

```
Power::infy : Infinite expression 1/0 encountered.
∞::indet :
 Indeterminate expression 0 ComplexInfinity encountered.
```

Maple and MuPAD issue an error message. This shows that computer algebra systems do not check the solution using replacement into the original equation, which is usual practice with students. It would not be quite possible either, since automatic simplification may automatically result in the alteration of the original expression.

3.1.2 Floating point numbers

A vivid example is where MuPAD offers the (pseudo) quadratic equation (4x-1)(x+3)=5x(0.8x+2) instead of the solution 3:

$$\{[x = -2.305843009 \cdot 10^{18}], [x = 2.875]\}$$

The error is due to calculation using decimals: `4-5*0.8;` would result in $4.33680869 \cdot 10^{-19}$ instead of 0. Replacement of 0.8 with the fraction 4/5 would provide the correct answer.

Other systems solve such equations correctly. It would be impossible to accept these answers at school or elsewhere. The inexactitude of operations with floating point numbers seems to be only a weak argument in the case of such simple example.

One must also beware of incorrectness when performing calculations with floating-point numbers using Maple sqrt, for instance. Although 0 is shown as the answer to sqrt(0), Maple would give 0 in answer to 0/sqrt(0). (The other systems behave as with 0/0). sqrt(0)/sqrt(0) would elicit 1 as the answer in Maple. This inaccuracy is disturbing at school and in general. (It is corrected in Maple 9.5.)

3.1.3 Number domain

The next example might be the equation $\sqrt{2x+6} + \sqrt{x-3} = 2\sqrt{x}$, already mentioned under the number domains, where one cannot confine oneself to the domain of real numbers.

3.2 *Completeness in the sense of branches*

3.2.1 Special case in simplification

To simplification of the expression $\frac{97x}{x}$ all the computer algebra systems give the answer 97, without recording the peculiarity of x = 0. As well, the answer to $\frac{x^2 + 4x}{-x}$ is –x-4 or -4-x (Mathematica).

The developers of computer algebra systems justify this simplification by claiming that this imprecision has a set of measure zero and limit of x/x approaches 0 is 1. In addition, the handling of such singularities in the case of large expressions would be unpractical and most of the users expect such simplification.

Actually, the same simplification is done at school. It is another matter whether the computer algebra system (or some other software application) could behave in a more precise manner and separately record special cases. This issue has been examined in [Chuaqui & Suppes 1990], for instance, who has offered the possibility of "carrying along" restrictions in a certain manner (e.g. as sequences).

The trouble is also that the rules and order of automatic simplification is often not documented for the user and some point, which would be important for the learning process, may unexpectedly disappear.

If we determine the variable value, e.g. x=0, and now calculate the value of $\frac{97x}{x}$, then Derive, Mathematica and MuPAD respond as to $\frac{0}{0}$ whereas Maple's answer is 97.

3.2.2 Literal equation

If computer algebra systems solve problems whose answer comprises several branches depending on the parameter values (e.g. the solution formula of the quadratic equation $ax^2+bx+c=0$) then it is possible to record all the branches separately or to focus on the main branch only. For instance, MuPAD records all the branches:

```
•solve(a*x^2+b*x+c=0,x);
```

$$\begin{cases} \mathbb{C} & \text{if } a=0 \wedge b=0 \wedge c=0 \\ \emptyset & \text{if } c \neq 0 \wedge a=0 \wedge b=0 \\ \{-\frac{c}{b}\} & \text{if } b \neq 0 \wedge a=0 \\ \left\{-\frac{b-\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a}, -\frac{b+\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a}\right\} & \text{if } a \neq 0 \end{cases}$$

Derive and Maple, however, do not deal separately with the "zero" parameter values. In Mathematica, it depends on what command (Solve, Reduce or InequalitySolve) is run.

The procedure is analogous for simpler parameter-containing equations. For instance, for ax=1.

a·x = 1 (w.r.t. x)	
Derive, Maple, Mathematica Solve	$\frac{1}{a}$
MuPAD	$\begin{cases} \{\frac{1}{a}\} & \text{if } a \neq 0 \\ \emptyset & \text{if } a = 0 \end{cases}$
Mathematica `InequalitySolve[a*x==1, {a, x}]`	`a < 0 && x == 1/a || a > 0 && x == 1/a`
Mathematica `Reduce[a*x==1, {x}]`	`x == 1/a && a ≠ 0`

The command InequalitySolve finds conditions that must be satisfied by real values of variable(s) in order for the expression to be true. In case of several variables all the variables need to be recorded, for their sequence will determine the form of the answer. The command Reduce simplifies the equations, attempting to solve the variables. Here it is possible and reasonable to record one variable only. When two variables are recorded the segment a≠0 will be left out of the answer.

One of the reasons why special cases are not separately recorded may be the pursuit to keep the answer easy to grasp – compact – and to avoid (superfluous?!) text that distracts attention from the main line. Who cares about one or two special cases as long as the main line is correct!

Emphasising the different branches in the school depends on the topic under discussion. In the case of the problems of physics, nobody would probably require that a=0 should be separately recorded when the aim is to express a from the formula f=ma. In the case of mathematical exercises, however, all branches should be recorded. This would be a correct procedure from the mathematical point of view as well.

3.2.3 Single values that turn the denominator into zero

MuPAD pays no special attention to the parameter values with which the solution value in the main branch becomes such as to change the denominator in the original equation into zero. For instance, in the equation $\frac{3mx-5}{(m+2)(x^2-9)} = \frac{2m+1}{(m+2)(x-3)} - \frac{5}{x+3}$ the answer $\begin{cases} \emptyset & \text{if } m = -2 \vee m = -\frac{3}{2} \\ \left\{\frac{21 \cdot m+38}{6 \cdot m+9}\right\} & \text{if } m \neq -2 \wedge m \neq -\frac{3}{2} \end{cases}$ is offered to the disregard of the fact that neither can a solution be obtained for m=$-3\frac{2}{3}$ or m=$-1\frac{2}{3}$, where x would be 3 and –3 respectively, which would change the denominator in the original equation into zero.

The authors state that 'solve' always assumes that the input is well defined. This means: if 1/(1-m) occurs in the input, 'solve' and the property functions assume that m≠1.

3.3 Completeness in the sense of being brought to completion

Completeness can also be understood to have a second meaning, that of being brought to completion. There are answers in case of which the solution process has been left unfinished. Yet another issue would be the question, what is the simplest form?

3.3.1 Shortly before the end

There may be situations in which little has left until the end:

```
Solve[3*(a+1)*x+3*a==2*a*x+3,x]
```

$$\left\{\left\{x \to \frac{3-3a}{-2a+3(1+a)}\right\}\right\}$$ (Mathematica)

and, using some other command, we can simplify further to a simpler form:

$$\text{Simplify}\left[\frac{3-3a}{-2a+3(1+a)}\right]$$

$$\frac{3-3a}{3+a}.$$

In this problem it is obvious that the answer has not yet been simplified to the end, although it is not a long way off. It is often not unambiguously clear, what is the simplest form.

3.3.2 Clearly discontinued

However, there may also arise a situation where only a little has been done, say, terms containing variables have been brought on one side.

```
> solve(m*x-3*m<x+5,{x});
```

$$\{\mathrm{signum}(m-1)\,x\sim < \frac{\mathrm{signum}(m-1)(3m+5)}{m-1}\}$$

(Maple)

It is possible to remove signums by adding assumptions:

```
> solve({m*x-3*m<x+5},{x}) assuming m::real,m>1;
```

$$\{x < \frac{3m+5}{m-1}\}.$$

Sometimes they have never gotten farther from the beginning

```
SOLVE(a·x > 1, x)
```

$a \cdot x > 1$ (Derive).

It seems that Derive solves those inequalities that after “internal” transformations have the form in which there is a multiplication operation on the one side and 0 on the other or in which the sign of the divisory expression can be determined, e.g. $(a^2+1)x > 1$. Similarly, Maple seems to stop at the point where the solution would branch out.

3.4 Compactness

Examples of compactness, particularly in relation to the number of branches, can be examined in combination with completeness in the sense of branches. The examples given contain a relatively small number of branches. However, even the solution formula of a quadratic equation may pose the danger of “getting lost”. Examples might be given where the answer is several screenfuls long. Particularly when solving parameter-containing equations and inequalities with MuPAD, since MuPAD issues solutions on a branch-by-branch basis but occasionally does it incorrectly by adding branches.

Let us examine here the issue of branch legibility. After solving $x - \frac{a}{1-a} < 1 - \frac{x-1}{a-1}$ relative to x we obtain the solution $(x < 0 \wedge (a < 0 \vee a > 1)) \vee (a < 1 \wedge x > 0 \wedge a > 0)$, which contains the necessary branches yet is difficult to read. In more complex problems the answers are even more difficult to decipher. One of the reasons for the reading difficulties is that the traditional school scheme ‘if CONDITION then EXPRESSION‘ is not followed.

According to the information from the developer of the system, the representation will be improved in future versions.

4 Suggestions and conclusion

4.1 What to suggest to the teacher?

In order to recommend something more tangible to the teacher they must know how the computer algebra system at their disposal functions and what problem-solving devices could be used. The computer algebra system or version at the teacher’s disposal is not necessarily the same as one of those discussed in this article. Following are problems that the teacher could use to test a particular system to find out the system’s behaviour in critical cases. Some of the problems have already been given above while others are different (as a rule, simpler than those given above). Naturally, one can construct a more detailed test, but even this one enables the detection of key problems.

Topic	Problems	What to observe?
Calculating	1/0, 0/0	How is the impossibility of dividing by zero being served?
	$\sqrt{-4}$	Does the answer contain imaginary units, which are alien to school?
	4-5*0.8	Is it 0?
	0/sqrt(0)	Is it presented as 0/0?
	(0*0)/0	Is it presented as 0/0?
Simplification	(97x)/x	Is x=0 recorded separately?
	x:=0 (x*x)/x	Is it presented as 0/0? Or is automatic simplification used?
Linear equation	0*a=0	How is it presented that the solution set is the entire set of numbers? Which one?
	0*a=5	How is the empty solution set presented?
Quadratic equation	(4x-1)(x+3)=5x(0.8x+2)	Are the solutions correct?
Fractional equation	$\frac{1}{x}=\frac{2}{x}+\frac{1}{x-1}$	Is infinity given as the answer?
	(x*x)/x=0	Is it given as the answer that there are no solutions?
Radical equation	$\sqrt{x}=\sqrt{-1}$ $\sqrt{2x}=\sqrt{x-1}$ $\sqrt{x^2-2}=\sqrt{x}$	Are those solutions also given that turn the root base into a negative number?
Literal equation	a*x=1	To what extent are branches presented?
	ax^2+bx+c=0	Is it graspable if there were more branches?
	$\frac{3mx-5}{(m+2)(x^2-9)}=\frac{2m+1}{(m+2)(x-3)}-\frac{5}{x+3}$ $\frac{1}{x-2}=\frac{m}{1-m}$	Are the parameter values turning the main solution into one changing the denominator into zero given separately?
Literal inequality	ax>1	Is the problem solved to the end?
	ax>0	Is the problem solved to the end?
	(a^2)x>1	Is the problem solved to the end?
	2*(a+1)*x=1	Is the problem simplified to the end?

In testing it is also necessary to observe whether appropriate premises, domain, etc. are established. As well, different commands might be tested apart from the so-called main commands.

If the teacher has identified the behaviour of a particular system and its shortcomings they have to decide how to surmount the undesired effects. Based on the smoothing of disturbing qualities, the answers offered by computer algebra systems may conditionally be divided as follows:
adaptable using the resources of the same computer algebra system;
applicable with the help of extra explanations provided to students;
unsuitable.

Interestingly, the reasons for and surmounting of many problematic effects can be explained in terms understandable at school:

- check the solution by placing it into the original equation (however, it might not always help);
- not simplified until the end;
- inexact calculation.

There are many ways **to use a computer algebra system** for rendering the solution more suitable "on the spot". We categorise them as follows:

- Changing the domain of definition or other premises.

- (Additionally) Selecting another command.
    - Performs better (e.g. In Mathematica, InequalitySolve works more accurately than Solve).
    - Continues (e.g. Simplify).
- Changing the expression or equation (e.g. using 4/5 instead of 0.8).
- Using "human" activities. (e.g. replacement of the solution into the original equation).
- Writing of short programs by which the above-mentioned devices can be implemented in a more automated manner.

Although it may be possible to invent other possibilities the list can in a sense be considered exhaustive. Undoubtedly, better results can be obtained if the makers of computer algebra systems themselves develop the systems in this direction.

Indeed, **explaining the answer** is almost always possible in principle. For the explanation to be adequate and convincing, however, the teachers themselves should know as precisely as possible why the computer algebra system gives just the answer it gives. If there is a clear and rational computer algebra system standard it can be explained in a simpler manner. It would be particularly good if the terminology understood at school could be used. Although in several cases it could be done, in other cases the explanation may prove to be too complicated for the respective age. As well, it must be observed whether the explanations are linked with the syllabus. Reasonable linking allows approaches that are developing and appropriate.

Apart from explanations it must be decided whether the answer is satisfactory or should be complemented or improved manually, for instance.

4.2 Conclusion

In conclusion, it may be said that to investigate the answers given in textbooks and by computer algebra systems such a system-method can be used where correctness and completeness in the sense of branches and in the sense of being brought to completion are treated separately. Virtually all the problematic cases within the topics under study can be discussed under these aspects in the course of the treatment of the number domain. In problems related to these topics, particularly noteworthy is completeness in the sense of parts, for which different textbooks present different conventions (apparently for the sake of compactness). Other components come into greater focus when analysing the answers given by computer algebra systems (and apparently also those given by students, which are not discussed in this study).

On some occasions, the school treatment and the different computer algebra system standards coincide (e.g. no recording singularities in simplification) while on others they do not. One system (particularly Mathematica) may feature different commands bearing more or less affinity with school mathematics.

The reasons for the deviations from the school treatment are various. They can be exhaustively classified as follows:

- The number domain or other premises differ from those used at school;
- Inaccuracies in designing, programming, etc.;
- For the sake of the compactness of the output or programming or resources some answers are not treated separately or to the end.

Some of these (d)ef(f)ects are easier to explain and surmountable "on the spot", particularly those that can be explained (and surmounted) in terms understandable at school. Unfortunately, not all the possible explanations are necessarily fit for the age of the students. Some of the effects are of a more complicated nature and would apparently require the changing of the computer algebra system. Or the existence of an intelligent intermediate program between the student and the computer algebra system ([Prank & Tonisson 2001]). Interestingly, some problems can be surmounted using fairly simple, so-called schoolchildish devices, such as checking the solution of an equation by replacing it into the original equation. In the future it is planned to study programming possibilities of computer algebra systems in order to make more school-friendly procedures.

There also arises the question whether the computer algebra system should put out more than the student should write on paper. For instance, to show the special cases, even though this is not done at school (e.g. in simplification). If there is the danger that this results in the output losing its compactness the showing or hiding of special cases may be made optional for the user.

Virtually every section of the study can be investigated in more detail (in part, I have already done that) and written about in an article. Newer versions (particularly those of Maple and Mathematica) and other systems should be explored. Undoubtedly, such reviews can and must be done on other topics as well. The first priority would be problems related to cube root and other radicals, as well as those related to logarithm-taking. The treatment of square root can also be developed into that of absolute value. For instance, the textbook presents the convention *"Accordingly, if not specifically required, we write* $a\sqrt{y}$ *instead of* $\sqrt{a^2 y} = |a|\sqrt{y}$ ([Form 10]).

This topic also overlaps with the issues of equivalence of expressions (or equations), which, among other things, poses greater mathematical challenges.

Although the article may seem to mainly concern negative aspects, the author is convinced that computer algebra systems have an important role to play in improving mathematics teaching, and the rectification of small defects would help to better perform the task.

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