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**TIME-2004**

## **Technology and its Integration into Mathematics Education**

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6<sup>th</sup> Int'l Derive & TI-CAS Conference**

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### **5 years of teaching mathematics to students with mandatory symbolic calculators : the good, the bad and the ugly!**

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### **ABSTRACT**

Since the fall of 1999, every students in our classrooms have the same symbolic calculators (TI-89, TI-92 Plus or Voyage 200). We have witnessed just about every possible negative reaction to having this technology available for all students. From making fun of it, to seeking its weaknesses, to asking students not to use it, up to using it for everything even to solve a simple equation like  $x + 2 = 5$ . After almost 5 years, the dust has settled down.

First, on the good side, we will show examples, in various topics, of what students can now do in classrooms and how these calculators have changed the way we teach part of the math curriculum. On the bad side, and this is true in general, the more we use technology, the more we tend to depend on it. Students now rely more on these calculators to do basic mathematics. They are less skilful in manual calculations. We will show examples of calculations which are best left to the calculators, examples our students would have some problems resolving manually; this is caused mainly by a lack of practice. We had to make room for the intelligent use of technology and cutting back on some long manual calculations was an easy way to do it. Finally, the ugly side of this on-going experience. Good students tend to be even better but bad students tend to be even worse. At the low end of the scale, some students cannot do mathematics without a symbolic calculator and usually don't understand much of what the technology is doing for them.

We have seen, in the last few years, an evolution in course topics, in exam questions linked to the use of symbolic calculators. The major part of the curriculum remains the same but with an intelligent use of the technology, we can now change the way we view and explore, with the students, some aspects of mathematics.

## 1. Introduction, a bit of history

This presentation should give you an overview of our experience, of the ups and down of dealing with technology in math teaching on a campus-wide level. We will use some material from past conferences and give new examples and new insights on this ongoing experiment.

Almost 5 years now since the big decision: math instructors, as a group and with the approval of our university, have decided to make it a requirement, as of the fall of 1999, that all new students entering an undergraduate engineering degree program obtain a TI-92 Plus (now Voyage 200) or TI-89. They learn how to use them efficiently in their first math course (Calculus). Why this decision? In the 1996-1999 period, the TI-92 (Plus) and the TI-89 made their way into the classrooms and brought about many problems. In that period, these symbolic hand-held calculators from Texas Instruments were the only ones available with general algebraic computation capabilities. Using symbolic computations, these calculators could perform derivatives, integrals, Taylor series, etc. which was essentially the content of our introductory Calculus course. The following screen illustrates such calculations.

The screenshot shows a TI-89 calculator interface with the following content:

- Top menu bar: F1 (2nd), F2 (Algebra), F3 (Calc), F4 (Other), F5 (PrgmIO), F6 (Clean Up)
- Integration result:  $\int (e^{-2t} \cdot \sin(5t)) dt = \frac{-5 \cdot e^{-2t} \cdot \cos(5t)}{29} - \frac{2 \cdot e^{-2t} \cdot \sin(5t)}{29} + C$
- Taylor series expansion:  $\text{taylor}(t \cdot e^{-2t}, t, 6)$ 

$$\frac{-4 \cdot t^6}{15} + \frac{2 \cdot t^5}{3} - \frac{4 \cdot t^4}{3} + 2 \cdot t^3 - 2 \cdot t^2 + t$$
- Input line: **taylor(t\*e^(-2t),t,6)**
- Bottom status bar: MAIN, RAD AUTO, FUNC 5/30

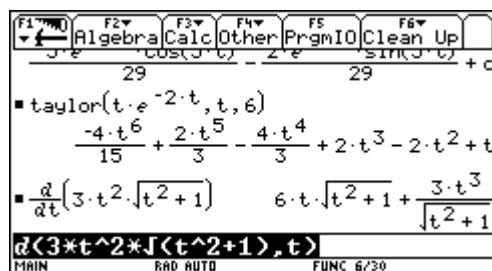
Screen 1

Of course, as more and more students brought this type of calculator in the classroom, it challenged how we taught mathematics. For example, some students couldn't understand why we would take 15 minutes to do a problem by hand, using the classical approach, when the calculator gave the same answer in 15 seconds! It was also becoming more difficult to design tests that would correctly assess student learning. To some extent, and in an effort to minimize inequities, this led us to essentially supply answers to questions. For example:

Show that the derivative of  $f(t) = 3t^2 \sqrt{t^2 + 1}$  is

$$f'(t) = \frac{3t(3t^2 + 2)}{\sqrt{t^2 + 1}}.$$

It is interesting to notice that the calculator doesn't give the answer shown above, see screen 2. The student still has to perform some algebraic simplifications to get the required answer.



Screen 2

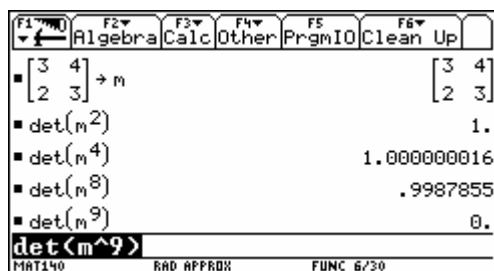
Fortunately, at our university, a majority of math instructors had already begun using Computer Algebra Systems (CAS) and started having students work on more sophisticated or complex problems using DERIVE or MAPLE software. We should mention that this was facilitated by the availability of computers and projectors or LCD panels in classrooms. So why not extend this to have more active students using Symbolic calculators in more dynamic classrooms?

Of course, we had to make room for the introduction of this technology. We decided to put less emphasis on certain aspects, for example limits, calculating derivatives with the algebraic definition, some techniques of integration. This left us more room for exploring the use of the calculator and exploring concept with the aid of technology.

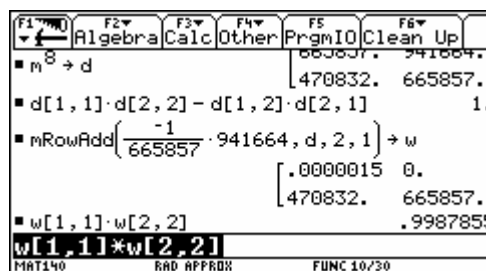
Implementing this experience wasn't an easy task. We did see some resistance from a small number of colleagues, some of them making fun of the use of these machines. The fear of being replaced by a machine..., if the calculator can do all these math calculations, what should I be still teaching? Of course, the next step is to find examples where the technology is in error or where it can't find an answer, then concluding that we should discard the technology since it is not reliable and can give wrong answers. People thinking like this forget that these calculators should be viewed only as tools and that we need to know what the technology is doing, we need to know the concepts behind it, and how they work. Exploring why the symbolic calculator gives a certain answer may involve doing more maths than one can suspect.

Consider this classic example from basic matrix algebra: using  $m = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ , we easily compute 1

as the determinant of this matrix, no need for technology here! Having shown students basic properties of determinants, they know that the determinant of any positive integer power of  $m$  will still be 1. Have them verify this result with their calculators, **in approx mode**. Have them compute  $\det(m^2)$ ,  $\det(m^4)$ ,  $\det(m^8)$  ... (see screen 3 below). What is going wrong?



Screen3



Screen 4

Of course, since we are in approx mode, we could answer that this is due to round off errors, but that is not enough. In fact, it is also due to the fact that the calculator will use a Gaussian elimination to compute the determinant, even for a 2 by 2 matrix. We can verify with students that this is in fact the case (screen 4). This simple example illustrate what we have often said about technology, if well used, you will end up doing more math then without it.

We do have to mention that all students of a same course (no matter how many different groups there are) will be doing the same final exam (with the first Calculus course, this can mean over 20 groups of students). This means that you have to be careful when planning the use of technology in your math classes. When we were only having students work on computers with Derive or Maple, this wasn't a problem since this aspect was only tested on team assignments with more challenging problems, demanding the use of CAS. Each group or instructor could have (or not) its own set of problems for exploring technology. When all students have a CAS system on their desk, you have to decide to what extent they can use it and for what. Of course, we could decide that calculators are not allowed for exams and still use the same old exams, testing the same topics as if technology didn't exist. But students would then ask why we are making the symbolic calculator mandatory. For an experience like this one to be successful, students as well as math instructors have to see an advantage in using the technology. It demands a careful dosage between classical approach and exploring with the aid of technology. The large part of our teaching remains the same as before, using classical approach and a good blackboard. The symbolic calculators are more used for exploration, motivation and going beyond the usual material seen in a traditional approach.

Considering the last 5 years spent trying to integrate this technology into our classrooms, we can offer some important remarks:

- give instructors a symbolic calculator! Give them access to the technology. Take the time necessary to ensure that instructors are at ease with it, show them how they could benefit from it. Be prepare to have a lot of discussions;

- don't try to keep all the classical topics and manual calculations; you have to make room for new technology;
- encourage students to use the technology intelligently, do calculations with them in the classroom, use the View Screen to show them what you are doing;
- show students common mistakes that are made with the calculator, illustrate its limitations;
- insist on the comprehension and the importance of mathematical syntax. Clarify what they should be able to do manually and what can be done with the calculator.

One must realize that this technology hasn't changed our math courses that much. More emphasis is now put on concepts instead of manual skills. Plus, we can explore more complex problems, which are probably closer to what students will encounter in their engineering classes. As a final note, we are now teaching to what may be called "the Nintendo generation". They appreciate working with these machines and realize that they need to really understand the mathematics behind them to make efficient use of the technology. They ask more questions, do more exploration than they used to and are happy doing it.

Finally, we have noticed in the past few years that students arriving at our university seem to be lacking some basic math and science topics or abilities. We can't expect them to be as skilful in manual calculations as students were a decade (or generation) ago. Some would argue that technology is responsible for this and that these calculators are there to compensate for these weaknesses in basic math. While this is partly true, we do consider these symbolic calculators to be a fantastic tool for reviewing and exploring these lacking topics. You can remind a student of the definition of a function but when you defined one with the calculator, you can explore and illustrate this concept, the composition, substitution...

## 2. The good...

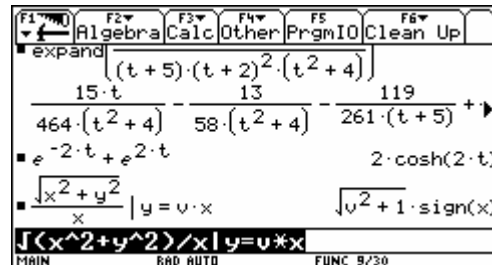
Let's now look at examples where the use of the symbolic calculators has, in our opinion, benefited or changed the way we teach.

**Example 1:** Using the TI can even be an excuse to introduce new subject matter. For example, we ask students to enter on their calculators, the next expression, a sum of conjugate exponentials:

$$e^{-2t} + e^{2t}$$

The resulting simplification done by the TI, see screen 5, gives us the opportunity to introduce hyperbolic functions. When teaching first order differential equations, one technique demands substituting  $v \cdot x$  for  $y$ , resulting in an expression depending only on  $v$ . Students usually have no

problem obtaining  $\sqrt{1+v^2}$  manually from the expression  $\frac{\sqrt{x^2+y^2}}{x}$  with this substitution. Using the calculator (screen 5) to verify this simplification, we have to explain the meaning of this new (for them) function,  $\text{sign}(x)$ . It is a good opportunity to let them know that the result they obtain manually was right only if we consider the variable  $x$  to be positive!

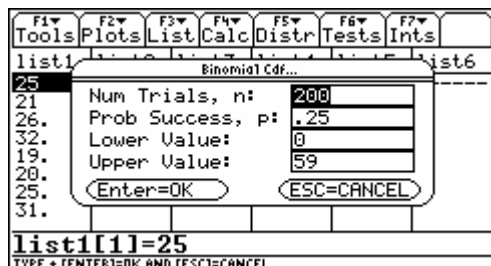


Screen 5

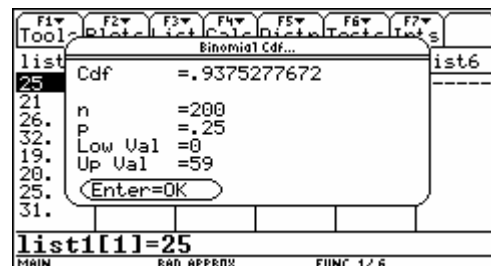
As an added bonus, since all the students have on their desks this powerful calculator, they become more active during lectures. The use of the calculator, as we have seen from the examples above, is a source, in itself, of new math.

**Example 2:** our engineering students have to follow an introductory course on Statistics. A classic approach of this subject demands a lot manual arithmetic calculations, many approximations and extensive use of tables (Normal, Student, etc.) for calculating probabilities and for interval estimation as well as hypothesis testing. A good part of these is now left to the calculator, with the Flash Application Stats/List Editor, or to the computer; time saved is now put on seeing more topics and on better interpreting the results calculated. In 13 weeks, we can go from basic statistics and graphs, to classical distributions, hypothesis testing up to linear and multiple linear regression.

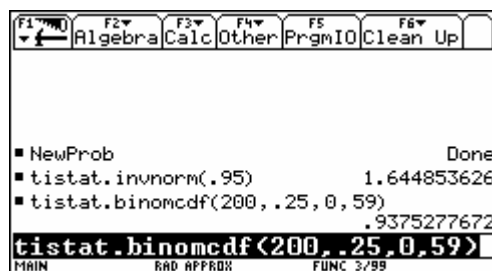
Here is an example where approximations are no longer needed: evaluate  $P(X < 60)$  if  $X \sim B(n=200, p=0.25)$ . Usually this is done by approximating the binomial distribution with a normal approximation. With the TI calculator, this can be done in the Stats/List editor (see screen 6 a) and b)) or directly in the home screen using the exact distribution (screen 7).



Screen 6 a)



Screen 6 b)



Screen 7

Although we can still teach these classical approximations, explaining, for example, how to obtain the usual formula for a confidence interval for a proportion, they are not a necessity since with the calculator we could get an interval using the exact binomial distribution. As an added bonus we can then have confidence intervals on smaller samples.

Here is another example illustrating how the symbolic calculators can modify the way we teach.

The breaking strength of yarn, which we will consider being normally distributed, must have a mean value of at least 100 psi. A random sample of 40 specimens gives an average breaking strength of  $\bar{x} = 98.8$  psi with a standard deviation of  $s = 2.4$  psi. Should the fiber be judged acceptable with a significance level of 5%?

Since the sample size is over 30, this problem is traditionally done using a normal distribution even if the exact distribution of the statistic  $t = \frac{\bar{x} - u_0}{s/\sqrt{n}}$  would be a Student distribution with 39

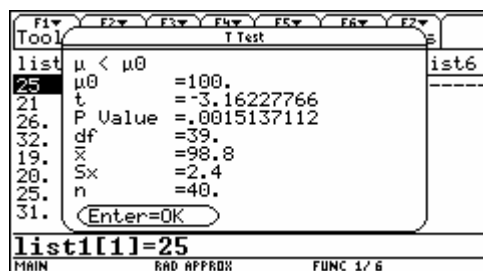
D.F. We usually use a normal distribution since we would need a more detailed table for the Student distribution. With the TI calculators these limitations are eliminated. With the Stats/List Editor you can have the calculator do the complete test (see screen 8 a) and b)) and then have students put more time on interpretation of the results. You can also see that the calculator gives an answer with a p-value, which is what more complex software on computer have been doing for years. In this case, the student should still be able to write manually how this p-value was calculated:

$$P\left(t = \frac{\bar{x} - u_0}{s/\sqrt{n}} < \frac{98.8 - 100}{2.4/\sqrt{40}}\right) = P(t < -3.1623) \text{ using 39 degrees of freedom, see screen 9.}$$

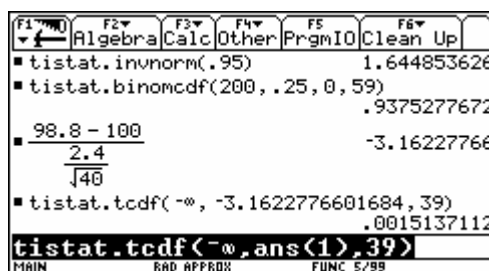
Since the p-value, 0.0015137, is less than 5% we can conclude that the fiber is not acceptable (regarding the mean breaking strength).



Screen 8 a)

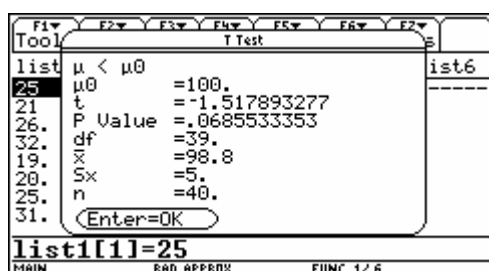


Screen 8 b)



Screen 9

Technology makes it easy to explore this example: what happens if the sample standard deviation is 5 psi instead of the above value? Screen 10 shows that this would mean that we won't reject the null hypothesis, concluding that it is possible that the yarns is acceptable and that the difference observed in the sample (a sample mean less than 100 psi) may be attributed to sampling variations.



Screen 10

This Flash Application has had a great positive impact on our introductory course on statistics. Don't believe this means life is simpler for students! Many would prefer being graded more on long calculations which are learned easily with some practice and have less points given to good interpretation of results.

Example 3: part of an exam can be given without the use of the calculator, for verifying basic derivative abilities for example. On the other hand, another part of the exam should be given taking into account the fact that students have a powerful tool at their disposal. We can now ask problems which would be too demanding for manual calculations and we should be verifying if students know how to use well these machines. In a course where students were asked to program on their TI calculators some basic numerical techniques, this question was asked in the final exams: the following program takes a square matrix as input. What will it do?

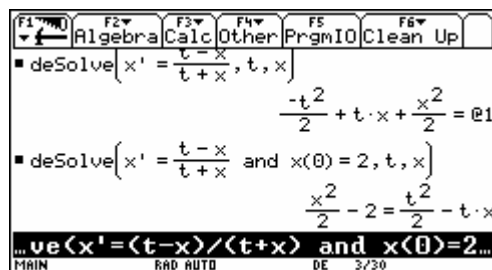




Screen 11

This small question will verify if each student did the work they were supposed to do (this is not a program they had seen before). One could argue that a student could just type in the code and see for himself what the program does. We respond, good for him. This means he did do some work with the calculator, knowing how to write a program and where to get the results. Furthermore, he has to be precise in writing a small paragraph to describe what the program does. Someone else could note that the student doesn't have to use the calculator to give a right answer. Again, we respond good for him, we were talking about programming numerical techniques after all.

**Example 4:** Another familiar math subject for engineering students is Differential Equations. Once again, the TI can be very useful. The calculator can solve, algebraically, some basic first and second order equations. Screen 12 gives an example of the calculator solving a first order differential equation: a general solution and a particular one given the initial condition.

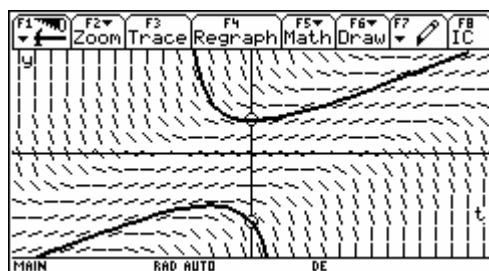


Screen 12

Algebraic solutions to general first order differential equations can be a tricky business. The TI isn't able yet to solve, for example, the equation below.

$$\frac{dx}{dt} = 1 - (t - x)^2$$

Students learn that this equation is easily solved, manually, by using the substitution  $v = t - x$ , which transforms it into a separable form. Although the calculator can't find an exact solution, it can solve it numerically. On the following screen, you will find a direction field for the above equation. Two integral curves are also shown for initial conditions  $x(0)=1$  and  $x(0)=-2$ ; they were determined using a Runge-Kutta method.

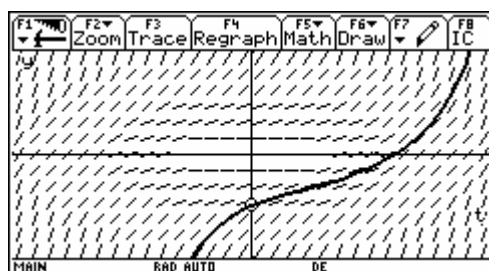


Screen 13

We can now easily investigate differential equations that have no solution in terms of elementary functions. A classical example is given by the following equation.

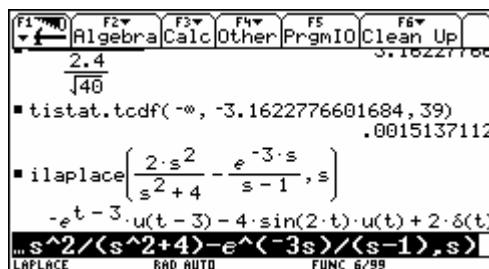
$$\frac{dx}{dt} = t^2 + x^2 \text{ with } x(0) = -1$$

On screen 14, below, you will find the direction field for this equation, illustrating the behavior of the integral curves, as well as a solution curve for the initial condition mentioned above.



Screen 14

In the same course, when computing inverse Laplace transforms, we will let student use the Expand command on the symbolic calculator for partial fraction decomposition. But we will have them use a basic Laplace transform table and the usual techniques seen in class to complete the computation of the inverse. This is inline with what we have always put forward: students should be able to do basic math manually, but the calculator can pick up some more tedious calculations. Furthermore, the technology should also be used to verify results. Students could use (see screen 15) the excellent program by Lars Frederiksen, to compute the inverse Laplace transform and verify a result done manually.



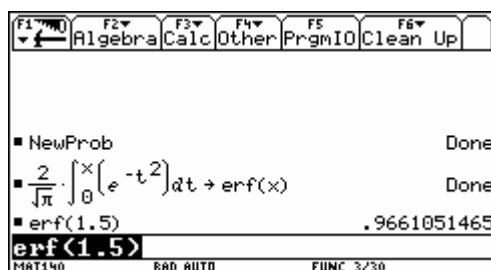
Screen 15

### 3. The bad and the ugly...

Let's now look at some negative aspects of this experience. One annoying effect is that you will have students (few) only interested in the right sequence of button to push to get the good answer and not willing to listen to any explanation on what and how to solve a problem. We must admit this is not new, before symbolic calculators we would have some students only interested in the right formula to get the good answer.

Another bad aspect we have witnessed over the past years is having math instructors discrediting this technology instead of learning its use and its limitations and seeing it as a tool to do and explore more mathematics. We have seen colleagues perhaps not wanting to change the way they've been teaching math; more surprising was seeing others, using CAS on computers and having a similar reaction to CAS calculators. One explanation could be that some instructors having spent a lot of time and energy mastering one CAS software and having prepared a lot of examples for students may not be willing to start over with another CAS system, especially if that new system challenges the way they teach and interact with students in classrooms. This negative side effect is subsiding since the CAS calculators are more and more widely accepted in the curriculum of our engineering programs.

Not knowing... What can these CAS calculators do? We have put a lot of energy to try to answer this question for the benefit of all faculty members and instructors. But they have to see an interest in the use of these CAS. Even then, communication is often a problem. For example, one civil engineering faculty, favorable to the use of symbolic calculators, was thinking of asking his students to use MatLab software on computers. He just wanted his students to be able to calculate the value of the error function  $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ . We showed him that the calculator could do the calculations he wanted just by entering this formula, see screen 16.



Screen 16

He was quite satisfied, considering the fact that he could have student do it in classrooms instead of computer labs. But we were lucky to meet him at just the right time.

Most students will be less skillful in manual calculations with the use of this technology. But this seems to be true even when they start at the university level. We would argue that this is in fact caused by a lack of practice; we are having them do less manual calculations to give us room to explore and do more mathematics with the aid of CAS calculators. When we had students 15 years ago with calculators not having a function to calculate factorials or a number of combinations, every student would easily compute  $\binom{50}{2}$  doing  $\frac{50 \times 49}{2}$ . They didn't have a choice. Since all scientific calculators now have a factorial function available, students will compute instead  $\frac{50!}{2! \times 48!}$ , which correspond to the definition we gave them. But many calculators now have a function for calculating directly the number of combinations (screen 17). Students using this function will tend to forget its definition in term of factorials since they don't need it for computation.

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
<div> <div>■ NewProb</div> <div>■ <math>\frac{50 \cdot 49}{2}</math></div> <div>■ <math>\frac{50!}{2! \cdot 48!}</math></div> <div>■ nCr(50, 2)</div> <div><b>nCr(50, 2)</b></div> </div> <div>Done</div> <div>1225</div> <div>1225</div> <div>1225</div> <div>1225</div>					
<div>MAT140</div> <div>RAD AUTO</div> <div>FUNC 4/30</div>					

Screen 17

We have seen the good students in our classrooms become even better with the use of technology, but at the lower end, really bad students tend to become worse. On the other hand, since these bad students, as it has always been the case, will be leaving the system anyway, we should perhaps not pay too much attention to them. Let us be clear, we do think that students having some difficulties can benefit greatly using technology if its use is well monitored. Some instructors tend to remember too much these bad students. We should be focusing more on the average student, keeping in mind that with technology it is easy to get good students to do extra work. Even the average ones will want to do more exploration with the CAS calculators. But manual calculations are more tedious for them today; for the average student, these two examples (a partial fraction expansion and an indefinite integral) are not easy problems even in their basic calculus course:

$$\frac{2x^3 + 7}{(x^2 - 4)(x^2 + 4x + 13)(x + 5)} \quad \text{and} \quad \int e^{-2t} \cos(3t) dt$$

A simpler fraction expansion or a simpler integral would be easier for them, these two examples demanding quite a few manual algebraic computations. But if students have to do this integral in our differential equations course, we do tell them not to use the classical manual way of doing this problem (they will of course use their calculators to get the correct answer). And this is true not only since the CAS calculators are on every student's desk. We were also telling students the same thing 10 years ago, student then had to buy a math handbook to get these integrals done rapidly.

## **4. Conclusion**

In our experience, students will acquire and use technology if they see an interest for them in it. The same is true for math instructors and faculty members. We tend to forget that students learn math for a reason. In our engineering curriculum, they will come across a lot of math in engineering and science courses. Ten years ago, all students starting an electrical engineering degree knew they had to buy an HP-48SX calculator. This machine eased a lot of math problems although the majority of math instructors didn't really know what it could do. Students were told that this calculator could easily solve systems of linear equations, do matrix calculations, numerical integrals, etc. Over the years, a lot of programs were written for this calculator giving students an even more powerful tool. Faculty members in that department were used to work with this calculator. Then came more powerful calculators, the symbolic calculators.

In our university, part of the key to the success of implementing mandatory use of TI symbolic calculators rest in the hands of faculty members teaching engineering and science courses. If they expect or demand that students do manually some calculations, then we have to ensure that our math courses satisfy this demand. But when they realize that they can study more complex problems, when they realize that they can do more science or engineering with the CAS calculator easing off some math calculations, then they see an interest in technology. All this can be done in a classical classroom, without any special equipment and with students being active and following our calculations showed to them with a Viewscreen. But we do have to get technology available to faculty members. In our experience, the process of learning what technology is available and how to adapt course curriculum is a slow process, it demands a lot of time, resources and efforts to ensure the success of implementing the use of CAS in mathematics education.

After 5 years of mandatory use of these symbolic calculators at ETS, we do think we are well on the path of success. We will continue to show how, when used well, a CAS calculator can have students explore more math, deal more challenging problems and, something very important, have students have fun doing mathematics.

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