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`RESIDUES.MTH`: Solving Problems of integration using the residue theorem

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Abstract

In this paper we present the file RESIDUE.MTH, created to be used in subjects that deal with complex variable, aimed at Engineering students. Such file contains a serie of macros which permit to solve integration problems using the residue theorem.

The macros contained in the file can be grouped within the following blocks:

- Compute of residues (of a given singularity, of the singularities of rational functions).
- Complex integrals using the residue theorem.
- Improper integrals of a rational function.
- Improper integrals of the form $f(x)\sin(kx)$ or $f(x)\cos(kx)$ where $f(x)$ is a rational function.

We also show in the paper some examples of applications that have been carried out with our students of Telecommunication Engineering. The macros have been developed in order to be used as didactical tools with explications of what the macros do step by step (using DERIVE as a PeCAS or as a white-box CAS).

Finally, we include the conclusions obtained after using this file with our students and also some future work on this subject.

1 Introduction

The use of computer as a tool to support teaching has been extended in University, especially in practical subjects. The choice of the program to be used is one of the most important matters. For the practical exercises with computer we develop in Engineering, among the great amount of mathematical software available, we have chosen Derive mainly due to its easiness of use in front of other mathematical software.

The elaboration of application files is very useful when solving specific problems because we can solve these problems easier than using the standard functions of Derive. In this paper we present the file `RESIDUE.MTH`, created for being used in subjects that deal with complex variable, aimed at Engineering students. Such file contains a serie of macros which permit to solve integration problems using the residue theorem.

2 Residue theorem

Let $f(z)$ be a complex function with a finite numbers of singularities inside a Jordan curve $\mathcal{C}$. The following theorem, known as *residue theorem,* establishes the value of the line integral of $f(z)$ over $\mathcal{C}$.

Residue Theorem

Let $f(z)$ be a complex function which is analytic inside and on a Jordan contour $\mathcal{C}$ except for a finite number of singularities z_1 , z_2 , $\ldots$, z_n inside $\mathcal{C}$, then

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z \;=\; 2\pi\,i\left(\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z)\right) \;=\; 2\pi\,i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z)$$

where $\operatorname*{Res}_{z=z_k} f(z)$ is the residue of $f(z)$ in the singularity z_k.

2.1 Finding the residue

If $z=a$ is a pole of order k of $f(z)$ its residue can be calculated as:

$$\operatorname*{Res}_{z=a} f(z) \;=\; \lim_{z\to a}\left[\frac{1}{(k-1)!}\,\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\Big((z-a)^k f(z)\Big)\right]$$

In the particular case of $z=a$ being a simple pole, the residue is:

$$\operatorname*{Res}_{z=a} f(z) \;=\; \lim_{z\to a}\Big[(z-a)f(z)\Big]$$

2.2 Compute of the residue with Derive

The following command has been developed in `RESIDUE.MTH` in order to calculate the residue of the pole a of order k of a function $f(z)$.

```
poleresidue(f, a, k) := LIM(1/(k - 1)!·dif((z - a)^k·f, z, k - 1), z, a)
```

Example:

```
#1 poleresidue(z^2/(z^2-1)^2,1,2)
```

#2 $\dfrac{1}{4}$

On the other hand, the following program has been developed in order to obtain the singularities and their residues of a rational expression $\dfrac{p}{q}$.

```
residues(p,q,sing,singqs,evit:=[],pqs,fsimp,v,n:=1,filas,res:=[],a,orden) :=
  Prog(
    sing := polinomy_factors(q),
    pqs := simplify_polinomies(p, q),
    fsimp := pqs sub 1/pqs sub 2,
    singqs := polinomy_factors(pqs sub 2),
    filas := DIM(sing),
    IF(filas = 0,
      RETURN res),
    LOOP(
      IF(n = filas,
        Prog(
```

```
        a := sing sub n,
        IF(not MEMBER?(a, singqs) or complex_limit(fsimp, z, a) < inf,
          Prog(
            res := APPEND(res, [[" ", a, " is ", " ", " removable ", " "]]),
            evit := APPEND(evit, [a])
          )
        ),
        n:+ 1
        ),
      exit
      )
    ),
  v := (matrix_numerator_denominator(p, q)) sub 2,
  filas := DIM(v),
  n := 1,
  LOOP(
    IF(n > filas,
      RETURN res,
      Prog(
        IF(div(v sub n sub 1, z) = 1,
          Prog(
            a := - (v sub n sub 1 - z),
            orden := v sub n sub 2,
            IF(not MEMBER?(a, evit),
              res := APPEND(res, [["pole ", a, " order ", orden, " residue",
                     poleresidue(fsimp, a, orden)]])
            )
          )
        ),
        n :+ 1
        )
      )
    )
  )
```

Example:

`#1 residues(z^2,(z^5-z)^2)`

`#2`

$$\begin{bmatrix} & 0 & \text{is} & & \text{removable} & \\ \text{pole} & i & \text{order} & 2 & \text{residue} & -\frac{3i}{16} \\ \text{pole} & -i & \text{order} & 2 & \text{residue} & \frac{3i}{16} \\ \text{pole} & 1 & \text{order} & 2 & \text{residue} & -\frac{3}{16} \\ \text{pole} & -1 & \text{order} & 2 & \text{residue} & \frac{3}{16} \end{bmatrix}$$

3 Applying Residue Theorem for computing Complex Integral

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In order to calculate $\displaystyle\oint_{\mathcal{C}} \frac{p(z)}{q(z)}\,\mathrm{d}z$ using the residue theorem, the following program has been developed in DERIVE. Notice that partial results are also given in order to use the result with pedagogical aims.

```
ComplexIntegral(p, q, izqecu, derecu, r_, s_ := 0, d_ := [], i_ := 1, val_) :=
  Prog(
    izqecu := SUBST(izqecu, z, x + #i·y),
    r_ := residues(p, q),
    LOOP(
      IF(i_ > DIM(r_),
        IF(d_ = [],
          Prog(
            vector_display(["The singularities and their residues are", r_,
                            "As all poles are outside the integration curve,
                            Cauchy's integral theorem affirms that the value
                            of the integral is nule:"]),
            RETURN "I = 0"
            ),
          Prog(
            vector_display(["The singularities and their residues are", r_,
                            "The poles inside the integration curve are",
                            d_, "Thus, the value of the integral is 2pi·i
                            multiplied by the sum of the corresponding residues,
                            that is:"]),
            RETURN APPEND("I = ", number_to_string(2·pi·#i·s_))
            )
          ),
          Prog(
            val_ := SUBST(izqecu,[x,y],[RE(r_sub i_ sub 2),IM(r_sub i_ sub 2)]),
            IF(val_ = derecu,
              Prog(
                string_display("In this case the residue theorem can not be
                                applied since there is at least a pole on the
                                integration curve. Thus,"),
                RETURN "The value of the integral, using this method,
                        is UNKNOWN")
              ),
            IF(FIRST(r_ sub i_ sub 1) = "p" and val_ < derecu,
              Prog(
                s_ := s_ + r_ sub i_ sub 6,
                d_ := APPEND(d_, [r_ sub i_ sub 2])
                )
              ),
            i_ := i_ + 1
            )
```

```
        )
      )
    )
```

Example:

```
#1 ComplexIntegral(z^2+4,z^3+2z^2+2z,|z-#i|,5/4)

    The singularities and their residues are
    pole  -1+i  order  1  residue  -1/2+3i/2
    pole  -1-i  order  1  residue  -1/2-3i/2
    pole  0  order  1  residue  2
    The poles inside the integration curve are
    -1+i   0
    Thus, the value of the integral is 2pi·i multiplied by the sum of the
    corresponding residues, that is:

#2                                       "I = (-(3pi))+(3pi·i)"
```

4 Resolution of improper integrals using Residue theorem

In order to calculate improper integrals the following items must be considered:

- The improper integral of a continuous function $f(x)$ in $[0, \infty)$ is defined as

$$\int_0^{\infty} f(x)\, \mathrm{d}x = \lim_{R \to \infty} \int_0^R f(x)\, \mathrm{d}x$$

  When the limit on the right exists, the improper integral is said to be *convergent* and its value is the value of such limit. In other case, the improper integral is said to be *divergent.*

- If $f(x)$ is a continuous function in $\mathbb{R}$, the improper integral $\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x$ is defined as

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\, \mathrm{d}x \; + \lim_{R_2 \to \infty} \int_{0}^{R_2} f(x)\, \mathrm{d}x$$

  When both limits exist the improper integral is said to be *convergent,* and its value is the sum of the value of both limits. In other case, the improper integral is said to be *divergent.*

  The *Cauchy's Principal Value* (CPV) associated to $\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x$ is defined as

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)\, \mathrm{d}x$$

  and, **if the improper integral is convergent**,

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, \mathrm{d}x$$

  Nevertheless, it can occurs that CPV exists although the integral $\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x$ is divergent.

- Let $f(x)$ be an even function, that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$. In this case, if CPV of the integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists such integral is convergent. Thus, if $f(x)$ is an even function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \quad \text{convergent} \iff \int_{0}^{\infty} f(x)\,\mathrm{d}x \quad \text{convergent} \iff \int_{-\infty}^{0} f(x)\,\mathrm{d}x \quad \text{convergent}$$

and

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x = \int_{0}^{\infty} f(x)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

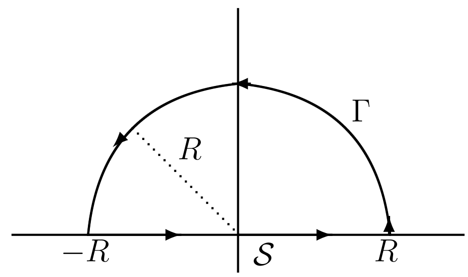
- It is important to point out that if the function is not even, the above result does not held. Nevertheless, if the improper integral is convergent, its value is the CPV.

Let now present the theoretical aspects needed to compute improper integrals using Residue Theorem.

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a real rational function and $q(x) \neq 0$ for all real x.

In order to calculate an integral such as $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ the integral

$\displaystyle\oint_{\mathcal{C}} f(z)\,\mathrm{d}z$ must be considered where $\mathcal{C} \equiv \mathcal{S} \bigcup \Gamma$ is the contour of the curve formed by the segment $\mathcal{S}$ on OX axe from $-R$ to R and the semi-circunference Γ on OX axe which diameter is the segment $\mathcal{S}$, as shown in the picture on the right. $R \to \infty$ will be considered later.



As shown above, if $f(x)$ is an even function, this fact can be used to determinate the value of $\displaystyle\int_{0}^{\infty} f(x)\,\mathrm{d}x$.

In order to calculate integrals such as

$$\int_{-\infty}^{\infty} f(x)\,\cos(ax)\,\mathrm{d}x \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\,\sin(ax)\,\mathrm{d}x$$

the integral $\displaystyle\oint_{\mathcal{C}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z$ will be considered, where $\mathcal{C}$ is the same curve than the previous case. Remind that $\mathrm{e}^{i\,ax} = \cos(ax) + i\,\sin(ax)$ and, thus, the real part of the result will be considered to calculate $\displaystyle\int_{-\infty}^{\infty} f(x)\,\cos(ax)\mathrm{d}x$, and the imaginary part of the result to calculate $\displaystyle\int_{-\infty}^{\infty} f(x)\,\sin(ax)\mathrm{d}x$.

The following result is very important in order to calculate integrals such as the ones shown above:

> Let Γ be the semi-circunference of the previous picture and $|f(z)| \leq \dfrac{M}{r^k}$ for $z = r\mathrm{e}^{i\,\theta}$, where k and M are constants. In this case:
>
> 1. If $\quad k > 1 \implies \displaystyle\lim_{r\to\infty}\left(\int_{\Gamma} f(z)\,\mathrm{d}z\right) = 0.$
> 2. If $\quad k > 0 \implies \displaystyle\lim_{r\to\infty}\left(\int_{\Gamma} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z\right) = 0.$

Notice that in the conditions of the previous result and being $\mathcal{C}$ the curve showed before, the following equations hold:

$$\lim_{R\to\infty} \oint_{\mathcal{C}} f(z)\,\mathrm{d}z \;=\; \lim_{R\to\infty} \int_{\mathcal{S}} f(z)\,\mathrm{d}z + \underbrace{\lim_{R\to\infty} \int_{\Gamma} f(z)\,\mathrm{d}z}_{0} \;=\; \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

Let $\mathcal{P}$ be the upper semi-plane which is the limit of $\mathcal{C}$ when $R \to \infty$. In this case, the CPV of the real integral can be calculated as:

$$\mathrm{CPV}\left(\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\right) = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x \;=\; \lim_{R\to\infty} \oint_{\mathcal{C}} f(z)\,\mathrm{d}z \;=\; \oint_{\mathcal{P}} f(z)\,\mathrm{d}z$$

Even more:

$$\lim_{R\to\infty} \oint_{\mathcal{C}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{e}^{i\,ax}\,\mathrm{d}x + \underbrace{\lim_{R\to\infty} \int_{\Gamma} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z}_{0}$$

Thus :

$$\begin{cases} \mathrm{CPV}\left(\displaystyle\int_{-\infty}^{\infty} f(x)\,\cos(ax)\,\mathrm{d}x\right) = \mathcal{R}\mathrm{e}\left(\displaystyle\oint_{\mathcal{P}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z\right) \\[2ex] \mathrm{CPV}\left(\displaystyle\int_{-\infty}^{\infty} f(x)\,\sin(ax)\,\mathrm{d}x\right) = \mathcal{I}\mathrm{m}\left(\displaystyle\oint_{\mathcal{P}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z\right) \end{cases}$$

The following result will allow to use the previous result in a more operative way:

> Let $f(z)$ be an analytic function in the upper semi-plane $\mathcal{P} \equiv \mathcal{I}\mathrm{m}(z) \geq 0$ except in a finite set of singularities $S_{\mathcal{P}}$ with none of them on the real axe. In this case:
>
> 1. If $\displaystyle\lim_{z\to\infty} z\,f(z) = 0 \Longrightarrow |f(z)| \leq \frac{M}{r^k}$ for $z = \mathrm{e}^{i\,\theta}$, where k and M are constants with $k > 1$.
> 2. If $\displaystyle\lim_{z\to\infty} f(z) = 0 \Longrightarrow |f(z)| \leq \frac{M}{r^k}$ for $z = \mathrm{e}^{i\,\theta}$, where k and M are constants with $k > 0$.

Thus, joining these two results:

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Let $f(z)$ be an analytic function in the upper semi-plane $\mathcal{P} \equiv \mathcal{I}\mathrm{m}(z) \geq 0$ except in a finite set of singularities $S_{\mathcal{P}}$ with none of them on the real axe and let $\mathrm{CPV}(I)$ be Cauchy's Principal Value of the integral I. In this case:

1. If $\lim\limits_{z\to\infty} z\, f(z) = 0 \quad \Longrightarrow \quad \mathrm{CPV}\left(\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\right) = \oint_{\mathcal{P}} f(z)\,\mathrm{d}z = 2\pi\, i \sum\limits_{z_k\in S_{\mathcal{P}}} \operatorname*{Res}\limits_{z=z_k} f(z).$

2. If $\lim\limits_{z\to\infty} f(z) = 0 \quad \Longrightarrow$

   (a)

$$\begin{aligned}
\mathrm{CPV}\left(\int_{-\infty}^{\infty} f(x)\cos(ax)\,\mathrm{d}x\right) &= \mathcal{R}\mathrm{e}\left(\oint_{\mathcal{P}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z\right) \\
&= \mathcal{R}\mathrm{e}\left(2\pi\, i \sum_{z_k\in S_{\mathcal{P}}} \operatorname*{Res}_{z=z_k} f(z)\,\mathrm{e}^{i\,az}\right) \qquad a>0
\end{aligned}$$

   (b)

$$\begin{aligned}
\mathrm{CPV}\left(\int_{-\infty}^{\infty} f(x)\sin(ax)\,\mathrm{d}x\right) &= \mathcal{I}\mathrm{m}\left(\oint_{\mathcal{P}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z\right) \\
&= \mathcal{I}\mathrm{m}\left(2\pi\, i \sum_{z_k\in S_{\mathcal{P}}} \operatorname*{Res}_{z=z_k} f(z)\,\mathrm{e}^{i\,az}\right) \qquad a>0
\end{aligned}$$

4.1 Rational improper integrals with Derive

The following program has been developed in order to calculate rational improper integrals with Derive using the Residue Theorem. Notice that partial results are also given in order to use the result with pedagogical aims.

```
RationalImproperIntegral(p, q, pz, qz, r_, s_, d_, i, n, par) :=
  Prog(
  IF(LIM(x·p/q, x, inf) /= 0,
    RETURN string_display("The residue theorem can not be applied since
                           the limit of zf(z) is not zero when z tents to
                           inf")
    ),
  IF(p/q = SUBST(p, x, -x)/SUBST(q, x, -x),
      par := "As the integration function is an even function, the existence
              of Cauchy's Principal Value leads to the convergence of the
              integral. Moreover, the value of the integral extended to
              [0,inf) or (-inf,0] is half of the result. Thus, the
              value of the integral extended to (-inf,inf) is:",
      par := "As the integration function is not an even function, the
              convergence of the integral extended to (-inf,inf) can
              not be assumed and nothing can be said about the convergence of
              the integrals extended to (-inf,0] and [0,inf). On the
              other hand, if the integral extended to (-inf,inf) is
```

```
                convergent, its value would be:",
        par := "As the integration function is not an even function, the
                convergence of the integral extended to (-inf,inf) can
                not be assumed and nothing can be said about the convergence of
                the integrals extended to (-inf,0] and [0,inf). On the
                other hand, if the integral extended to (-inf,inf) is
                convergent, its value would be:"
      ),
    pz := SUBST(p, x, z),
    qz := SUBST(q, x, z),
    r_ := residues(pz, qz),
    s_ := 0,
    d_ := [],
    i := 1,
    n := DIM(r_),
    LOOP(
      IF(i > n,
        Prog(
          vector_display(["lim [zf(z)] = 0 when z tents to inf and there are
          no real roots. Thus, Cauchy's Principal Value can be obtained by the
          residue theorem. The singularities and their residues are", r_, "The
          poles inside the upper semi-plane are", d_, "Thus, Cauchy's Principal
          Value is 2pi·i multiplied by the sum of the corresponding residues. ",
          par]),
          RETURN APPEND("I = ", number_to_string(2·pi·i·s_))
          ),
        Prog(
          IF(IM(r_ sub i sub 2) = 0,
            Prog(
              string_display("In this case the residue theorem can not be applied
                              since there is at least a real pole. Thus,"),
              RETURN "The value of the integral, using this method, is UNKNOWN"
              )
            ),
          IF(IM(r_ sub i sub 2) > 0,
            Prog(
              s_ := s_ + r_ sub i sub 6,
              d_ := APPEND(d_, [r_ sub i sub 2])
              )
            ),
          i := i + 1
          )
        )
      )
    )
```

Example:

```
#1 RationalImproperIntegral(1,x^6+1)
```

```
lim [zf(z)] = 0 when z tents to inf and there are no real roots. Thus, Cauchy's
Principal Value can be obtained by the residue theorem. The singularities and thei
residues are
pole  ((sqrt(3))/(2))+((i)/(2))  order  1  residue
(-((sqrt(3))/(12)))-((i)/(12))
pole  ((sqrt(3))/(2))-((i)/(2))  order  1  residue
(-((sqrt(3))/(12)))+((i)/(12))
pole  (-((sqrt(3))/(2)))+((i)/(2))  order  1  residue
((sqrt(3))/(12))-((i)/(12))
pole  (-((sqrt(3))/(2)))-((i)/(2))  order  1  residue
((sqrt(3))/(12))+((i)/(12))
pole  i  order  1  residue  -((i)/(6))
pole  -(i)  order  1  residue  (i)/(6)
The poles inside the upper semi-plane are
((sqrt(3))/(2))+((i)/(2))   (-((sqrt(3))/(2)))+((i)/(2))   i
Thus, Cauchy's Principal Value is 2·pi·i multiplied by the sum of the
corresponding residues.  As the integration function is an even function, the
existence of Cauchy's Principal Value leads to the convergence of the integral.
Moreover, the value of the integral extended to [0,inf) or (-inf,0] is half
of the result. Thus, the value of the integral extended to (-inf,inf) is:
```

#2 I = (2pi)/(3)

4.2 Rational improper integrals multiplied by a cosine function with Derive

The following program has been developed in order to calculate this kind of integrals with Derive using the Residue Theorem. Notice that partial results are also given in order to use the result with pedagogical aims.

```
CosineRationalImproperIntegral(c, p, q, pz, qz, m, r_, s_, d_, i, n, par) :=
  Prog(
    IF(LIM(p/q, x, inf) /= 0,
      RETURN string_display("The residue theorem can not be applied since the
      limit of f(z) is not zero when z tents to inf")
      ),
    IF(c·p/q = SUBST(c·p, x, -x)/SUBST(q, x,-x),
      par := "As the integration function is an even function, the existence
              of Cauchy's Principal Value leads to the convergence of the
              integral. Moreover, the value of the integral extended to [0,inf)
              or (-inf,0] is half of the result. Thus, considering the real
              part of the result (for dealing with a cosine function), the value
              of the integral extended to (-inf,inf) is the real part of:",
      par := "As the integration function is not an even function, the
              convergence of the integral extended to (-inf,inf) can not be
              assumed and nothing can be said about the convergence of the
              integrals extended to (-inf,0] and [0,inf). On the other hand,
              considering the real part of the result (because is a cosine
```

```
          function) and if the integral extended to (-inf,inf) is
          convergent, its value would be the real part of:",
   par := "As the integration function is not an even function, the
          convergence of the integral extended to (-inf,inf) can not be
          assumed and nothing can be said about the convergence of the
          integrals extended to (-inf,0] and [0,inf). On the other hand,
          considering the real part of the result (because is a cosine
          function) and if the integral extended to (-inf,inf) is
          convergent, its value would be the real part of:"
   ),
  pz := SUBST(p, x, z),
  qz := SUBST(q, x, z),
  m := v(- dif(c, x, 2)/c),
  r_ := residues(#e^(#i·m·z)·pz, qz),
  s_ := 0,
  d_ := [],
  i := 1,
  n := DIM(r_),
  LOOP(
    IF(i > n,
      Prog(
        vector_display(["lim f(z) = 0 when z tents to inf and there are not
                        real roots. Cauchy's Principal Value can be obtained
                        using residue theorem. The singularities and their
                        residues are", r_, "The poles inside the upper
                        semi-plane are", d_, "Thus, Cauchy's Principal Value
                        is 2pi·i multiplied by the sum of the corresponding
                        residues. ", par, 2·pi·i·s_, "that is:"]),
        RETURN RE(2·pi·#i·s_)
        ),
      Prog(
        IF(IM(r_ sub i sub 2) = 0,
          RETURN string_display("In this case the residue theorem can not
                                be applied since there is at least a real
                                pole.")
          ),
        IF(IM(r_ sub i sub 2) > 0,
          Prog(
            s_ := s_ + r_ sub i sub 6,
            d_ := APPEND(d_, [r_ sub i sub2])
            )
          ),
        i := i + 1
        )
      )
    )
  )
```

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Example:

```
#1 CosineRationalImproperIntegral(COS(x), 1, x^2 + x + 1)
```

```
lim f(z) = 0 when z tents to inf and there are not real roots. Cauchy's Principal
Value can be obtained using residue theorem. The singularities and their
residues are
pole  (-1/2)+((sqrt(3)i)/(2))  order  1  residue
(sqrt(3)(e)^((-((sqrt(3))/(2)))-((i(pi)+(1))/(2))))/(3)
pole  (-1/2)-((sqrt(3)i)/(2))  order  1  residue
-((sqrt(3)(e)^(((sqrt(3))/(2))-((i(pi)+(1))/(2))))/(3))
The poles inside the upper semi-plane are
(-1/2)+((sqrt(3)i)/(2))
Thus, Cauchy's Principal Value is 2pi i multiplied by the sum of the
corresponding residues.  As the integration function is not an even function,
the convergence of the integral extended to (-inf,inf) can not be assumed
and nothing can be said about the convergence of the integrals extended to
(-inf,0] and [0,inf). On the other hand, considering the real part of the
result (because is a cosine function) and if the integral extended to
(-inf,inf) is convergent, its value would be the real part of:
(2sqrt(3)pi(e)^((-((sqrt(3))/(2)))-((i)/(2))))/(3) that is:
```

```
#2                    2·sqrt3·pi·#e^(- sqrt3/2)·COS(1/2)/3
```

4.3 Rational improper integrals multiplied by a sine function with Derive

The following program has been developed in order to calculate this kind of integrals with Derive using the Residue Theorem. Notice that partial results are also given in order to use the result with pedagogical aims.

```
SineRationalImproperIntegral(s, p, q, pz, qz, m, r_, s_, d_, i, n, par) :=
  Prog(
    IF(LIM(p/q, x, inf) /= 0,
      RETURN string_display("The residue theorem can not be applied since the
      limit of f(z) is not zero when z tents to inf")
      ),
    IF(s·p/q = SUBST(s·p, x, -x)/SUBST(q, x, -x),
      par := "As the integration function is an even function, the existence
              of Cauchy's Principal Value leads to the convergence of the
              integral. Moreover, the value of the integral extended to [0,inf)
              or (-inf,0] is half of the result. Thus, considering the imaginary
              part of the result (for being a sine function), the value of the
              integral extended to (-inf,inf) is the imaginary part of:",
      par := "As the integration function is not an even function, the
              convergence of the integral extended to (-inf,inf) can not be
              assumed and nothing can be said about the convergence of the
              integrals extended to (-inf,0] and [0,inf). On the other hand,
              considering the imaginary part of the result (because is a sine
              function) and if the integral extended to (-inf,inf) is
              convergent, its value would be the imaginary part of:",
```

```
      par := "As the integration function is not an even function, the
              convergence of the integral extended to (-8,8) can not be
              assumed and nothing can be said about the convergence of the
              integrals extended to (-inf,0] and [0,inf). On the other hand,
              considering the imaginary part of the result (because is a sine
              function) and if the integral extended to (-inf,inf) is
              convergent, its value would be the imaginary part of:"
      ),
    pz := SUBST(p, x, z),
    qz := SUBST(q, x, z),
    m := v(- dif(s, x, 2)/s),
    r_ := residues(#e^(#i·m·z)·pz, qz),
    s_ := 0,
    d_ := [],
    i := 1,
    n := DIM(r_),
    LOOP(
      IF(i > n,
        Prog(
          vector_display(["lim f(z) = 0 when z tents to inf and there are not
          real roots. Cauchy's Principal Value can be obtained using residue
          theorem. The singularities and their residues are", r_, "The poles
          inside the upper semi-plane are", d_, "Thus, Cauchy's Principal Value
          is 2pi·i multiplied by the sum of the corresponding residues. ", par,
          2·pi·#i·s_, "that is:"]),
          RETURN IM(2·pi·#i·s_)
          ),
        Prog(
          IF(IM(r_ sub i sub 2) = 0,
            RETURN string_display("In this case the residue theorem can not
            be applied since there is at least a real pole.")
            ),
          IF(IM(r_ sub i sub 2) > 0,
            Prog(
              s_ := s_ + r_ sub i sub 6,
              d_ := APPEND(d_, [r_ sub i sub 2])
              )
            ),
          i := i + 1
          )
        )
      )
    )
```

Example:

```
#1 SineRationalImproperIntegral(SIN(x), x, x^2 - 1)
```

```
  In this case the residue theorem can not be applied since there is at least a
  real pole.
```

4.4 Improper integrals with DERIVE

Finally, in order to deal with all considered cases of improper integrals, the following program in DERIVE has also be developed in the package RESIDUE.MTH:

```
ImproperIntegral(f, p, q := 1) :=
 IF(VARIABLES(q) = [],
   RationalImproperIntegral(f, p),
   IF((STRING(f)) sub 1 = "COS",
     CosineRationalImproperIntegral(f, p, q),
     IF((STRING(f)) sub 1 = "SIN",
       SineRationalImproperIntegral(f, p, q),
       "Error in the parameters")
     )
   )
```

Example:

```
#1 ImproperIntegral(1,x^6-1)
```

```
  In this case the residue theorem can not be applied since there is at least a
  real pole. Thus,
```

```
#2                 The value of the integral, using this method, is UNKNOWN
```

```
#3 ImproperIntegral(SIN(x), x^2, x^2 + 1)
```

```
  The residue theorem can not be applied since the limit of f(z) is not zero when
  z tents to inf
```