Programming line and multiple integrals with Derive

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The main objective of this material is to be used as a guide for the workshop.

The contents are divided in two blocks: the first block presents the Derive's functions to be used along the workshop. The second block describes the tasks to be developed in the workshop.

The didactical method to be used in the workshop can be summarized in the following two points:

1. During the first part of the workshop, the lecturer will show the participants how to create different macros (commands) to solve typical problems of line and multiple integrals, including some graphical utilities to help in the process of resolution.
2. The second part will be dedicated to help the participants in building additional macros related with this subject.

The description of the tasks to be developed in the workshop is given by means of the following four items:

- **Formulation of the problem**, where the generic problem to be solved is described.
- **Theoretical remarks**. In this item, a very short review of some theoretical aspects are considered.
- **Derive's command**. In this third item, the Derive's command which solves the corresponding problem is described.
- **Auto-check exercise**. Finally, a specific exercise must be solved using the generated command. The execution and final result are also presented in order to check the correctness of the command involved.

Block I: DERIVE's functions

- Use of the conditional function (`if`)

```
if ( TEST ,
    ACTION IF TEST IS TRUE ,
    ACTION IF TEST IS FALSE ,
    ACTION IF TEST IS UNDETERMINED
   )
```

- Term's substitution in an expression (`subst`)

```
subst ( EXPRESSION WHERE TO SUBSTITUTE ,
        VARIABLES TO BE SUBSTITUTED ,
        NEW VALUES
      )
```

- Derivative of a function (`dif`)

`dif(f, x, n)` to calculate $f^{(n)}(x)$

`dif(f, x)` to calculate $f'(x)$

- Integral of a function (`int`)

`int(f, x, a, b)` to calculate $\displaystyle\int_a^b f(x)\,\mathrm{d}x$

`int(f, x)` to calculate $\displaystyle\int f(x)\,\mathrm{d}x$

CONTS

Block II: Tasks to develop in the workshop

Multiple Integrals

Triple Integral using cartesian coordinates

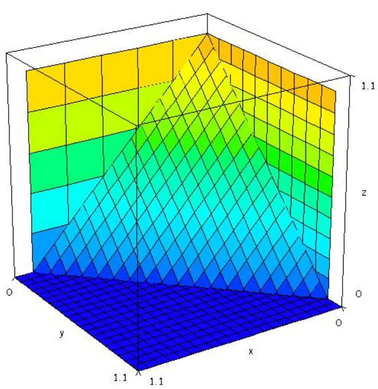
- **Theoretical remarks**

  The calculation of a triple integral requires three defined integrals. The integration order must be taken into account: if the integration limits of some variables are not constants, the first variable of integration must be one whose integration limits are not constants while the last integral must be always done with respect to a variable whose integration limits are constants.

- **DERIVE's command**

```
triple(f,u,u1,u2,v,v1,v2,w,w1,w2) :=
   int( int( int( f, u, u1, u2 ), v, v1, v2 ), w, w1, w2 )
```

- **Auto-check exercise**



Calculate $\displaystyle\iiint_{\mathcal{V}} (x + yz)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ where $\mathcal{V}$ is the region bounded by planes: $x+y+z=1$, $x=0$, $y=0$ and $z=0$.

Execution: `triple(x+yz,z,0,1-x-y,y,0,1-x,x,0,1)`

Solution: $\dfrac{1}{20}$

Triple Integral using cylindrical coordinates

CONTS

- **Theoretical remarks**

Let us consider the change to cylindrical coordinates given by: $\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \\ z = z \end{cases}$. The jacobian of this transformation is ρ. Thus:

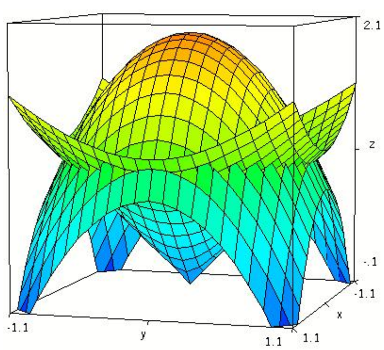
$$\iiint_{\mathcal{V}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\mathcal{D}} \rho\, f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, z)\,\mathrm{d}z\,\mathrm{d}\rho\,\mathrm{d}\theta$$

where $\mathcal{V}$ is the region in cartesian coordinates and $\mathcal{D}$ is the new region in cylindrical coordinates.

- **Derive's command**

```
triplecylindrical(f,u,u1,u2,v,v1,v2,w,w1,w2) :=
  triple(ρ subst(f,[x,y,z],[ρ cos θ,ρ sin θ,z]),u,u1,u2,v,v1,v2,w,w1,w2)
```

- **Auto-check exercise**



The volume of a region $\mathcal{V}$ can be determined by: $\mathrm{vol}(\mathcal{V}) = \iiint_{\mathcal{V}} 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Find the volume of the solid bounded below by cone $z = +\sqrt{x^2+y^2}$ and above by paraboloid $z = 2 - x^2 - y^2$ using cylindrical coordinates.

Execution: `triplecylindrical(1,z,rho,2-rho^2,rho,0,1,theta,0,2pi)`

Solution: $\frac{5\pi}{6}$

CONTS

Double Integral using cartesian coordinates

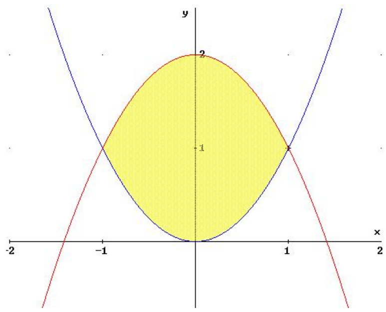
- **Theoretical remarks**

  The calculation of a double integral requires two defined integrals. The integration order must be taken into account: if the integration limits of one of the variables are not constants, this must be the first variable of integration while the last integral must be always done with respect to the variable whose integration limits are constants.

- **DERIVE's command**

```
double(f,u,u1,u2,v,v1,v2) := int( int( f, u, u1, u2 ), v, v1, v2 )
```

- **Auto-check exercise**



Calculate $\iint_{\mathcal{R}} (x^2 + y^3)\, dx\, dy$ where $\mathcal{R}$ is the region bounded by $y = x^2$ and $y = 2 - x^2$.

Execution: `double(x^2+y^3,y,x^2,2-x^2,x,-1,1)`

Solution: $\frac{176}{35}$

CONTS

Double integral using polar coordinates

- **Theoretical remarks**

Let us consider the change to polar coordinates given by: $\left\{\begin{array}{l} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{array}\right.$. The jacobian of this transformation is ρ. Thus:

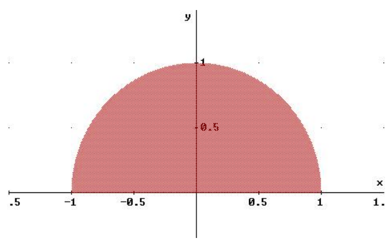
$$\iint_{\mathcal{V}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{D}} \rho\, f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta$$

where $\mathcal{V}$ is the region in cartesian coordinates and $\mathcal{D}$ is the new region in polar coordinates.

- **Derive's command**

```
doublepolar(f,u,u1,u2,v,v1,v2) :=
    doble(ρ subst(f,[x,y],[ρ cos θ,ρ sin θ]),u,u1,u2,v,v1,v2)
```

- **Auto-check exercise**



Calculate $\displaystyle\iint_{\mathcal{R}} \frac{x^2y^2}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$ where $\mathcal{R}$ is the region bounded by $\left\{\begin{array}{c} x^2+y^2=1 \\ y \geq 0 \end{array}\right.$.

Execution: `doublepolar(x^2y^2/(x^2+y^2),rho,0,1,theta,0,pi)`

Solution: $\dfrac{\pi}{32}$

Triple Integral using spherical coordinates

- **Theoretical remarks**

Let us consider the change to cylindrical coordinates given by: $\begin{cases} x = \rho \cos\phi \cos\theta \\ y = \rho \cos\phi \,\mathrm{sen}\,\theta \\ z = \rho \,\mathrm{sen}\,\phi \end{cases}$.

The jacobian of this transformation is $\rho^2 \cos\phi$. Thus:

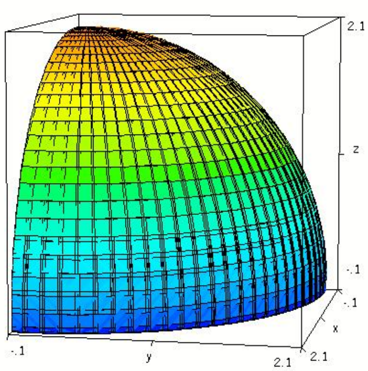
$$\iiint_{\mathcal{V}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\mathcal{D}} \rho^2 \cos\phi\, f(\rho\cos\phi\cos\theta, \rho\cos\phi\,\mathrm{sen}\,\theta, \rho\,\mathrm{sen}\,\phi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

where $\mathcal{V}$ is the region in cartesian coordinates and $\mathcal{D}$ is the new region in spherical coordinates.

- **DERIVE's command**

```
triplespherical(f,u,u1,u2,v,v1,v2,w,w1,w2) :=
    triple(ρ² cos φ subst(f,[x,y,z],[ρ cos φ cos θ,ρ cos φ sin θ,ρ sin φ]) ,
           u,u1,u2,v,v1,v2,w,w1,w2)
```

- **Auto-check exercise**



Calculate $\displaystyle\iiint_{\mathcal{V}} \frac{xyz}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ where $\mathcal{V}$ is the solid $x^2+y^2+z^2 \le 4$ with $x \ge 0$, $y \ge 0$ and $z \ge 0$.

Execution: `triplespherical(xyz/sqrt(x^2+y^2+z^2),rho,0,2,theta,0,pi/2,phi,0,pi/2)`

Solution: $\frac{4}{5}$

Line Integrals

Exact differential in $\mathbb{R}^3$

Build a command named EXACTDIFFERENTIAL3 to check if a differential in $\mathbb{R}^3$ is an exact one.

- **Theoretical remarks**

  $P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z$ is an exact differential in $\mathbb{R}^3$ if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$ y $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$.

- **Derive's command**

```
exactdifferential3(p,q,r) :=
if ( dif(p,y)=dif(q,x) and dif(p,z)=dif(r,x) and dif(q,z)=dif(r,y) ,
     "This is an exact differential" ,
     "This is not an exact differential" ,
     "This is not an exact differential"
   )
```

- **Auto-check exercise**

  Check if the following differentials are exact or not:

  1. $(yz+y+z)\ \mathrm{d}x + (xz+x+z)\ \mathrm{d}y + (xy+x+y+2z)\ \mathrm{d}z$

     Execution: `exactdifferential3(yz+y+z,xz+x+z,xy+x+y+2z)`

     Solution: This is an exact differential

  2. $(\mathrm{e}^x+1)\ \mathrm{d}x + (x+z)\ \mathrm{d}y + (xy+x+y+2\mathrm{e}^z)\ \mathrm{d}z$

     Execution: `exactdifferential3(ê^x+1,x+z,xy+x+y+2ê^z)`

     Solution: This is not an exact differential

Exact differential in $\mathbb{R}^2$

Build a command named EXACTDIFFERENTIAL2 to check if a differential in $\mathbb{R}^2$ is an exact one.

- **Theoretical remarks**

  $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ is an exact differential in $\mathbb{R}^2$ if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

- **DERIVE's command**

```
exactdifferential2(p,q) :=
if ( dif(p,y)=dif(q,x) ,
     "This is an exact differential" ,
     "This is not an exact differential" ,
     "This is not an exact differential"
   )
```

- **Auto-check exercise**

  Check if the following differentials are exact or not:

  1. $(xy^2 + x + 1)\,\mathrm{d}x + (x^2y - 2)\,\mathrm{d}y$

     Execution: `exactdifferential2(xy^2+x+1,x^2y-2)`

     Solution: This is an exact differential

  2. $xy\,\mathrm{d}x + 2x\,\mathrm{d}y$

     Execution: `exactdifferential2(xy,2x)`

     Solution: This is not an exact differential

CONTS

Potential in $\mathbb{R}^2$

Build a command named POTENTIAL2 to calculate the potential function of an exact differential in $\mathbb{R}^2$ (the command must previously check the condition in which such potential function exists).

- **Theoretical remarks**

  If the differential $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ in $\mathbb{R}^2$ is an exact one, its potential function $U(x,y)$ can be built by:

$$U(x,y) = \int_0^x P(t,0)\,\mathrm{d}t + \int_0^y Q(x,t)\,\mathrm{d}t$$

- **DERIVE's command**

```
potential2(p,q) :=
if ( dif(p,y) = dif(q,x) ,
     int(subst(p,[x,y],[t,0]),t,0,x) + int(subst(q,[x,y],[x,t]),t,0,y) ,
     "The potential function does not exist" ,
     "The potential function does not exist"
   )
```

- **Auto-check exercise**

  Build the potential function, if there exists, of the following differentials:

  1. $(xy^2 + x + 1)\ \mathrm{d}x + (x^2y - 2)\ \mathrm{d}y$

     Execution: `potential2(xy^2+x+1,x^2y-2)`

     Solution: $x^2\left(\frac{y^2}{2} + \frac{1}{2}\right) + x - 2y$

  2. $xy\,\mathrm{d}x + 2x\,\mathrm{d}y$

     Execution: `potential2(xy,2x)`

     Solution: The potential function does not exist

CONTS

Potential in $\mathbb{R}^3$

Build a command named POTENTIAL3 to calculate the potential function of an exact differential in $\mathbb{R}^3$ (the command must previously check the condition in which such potential function exists).

- **Theoretical remarks**

  If the differential $P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z$ in $\mathbb{R}^3$ is an exact one, its potential function $U(x,y,z)$ can be built by:

$$U(x,y,z) = \int_0^x P(t,0,0)\,\mathrm{d}t + \int_0^y Q(x,t,0)\,\mathrm{d}t + \int_0^z R(x,y,t)\,\mathrm{d}t$$

- **DERIVE's command**

```
potential3(p,q,r) :=
if ( dif(p,y) = dif(q,x) and dif(p,z)=dif(r,x) and dif(q,z)=dif(r,y) ,
     int(subst(p,[x,y,z],[t,0,0]),t,0,x) +
     int(subst(q,[x,y,z],[x,t,0]),t,0,y) +
     int(subst(r,[x,y,z],[x,y,t]),t,0,z) ,
     "The potential function does not exist" ,
     "The potential function does not exist"
    )
```

- **Auto-check exercise**

  Build the potential function, if there exists, of the following differentials:

  1. $(yz + y + z)\ \mathrm{d}x + (xz + x + z)\ \mathrm{d}y + (xy + x + y + 2z)\ \mathrm{d}z$

     Execution: `potential3(yz+y+z,xz+x+z,xy+x+y+2z)`

     Solution: $x\big(y(z+1)+z\big) + z(y+z)$

  2. $(\mathrm{e}^x + 1)\ \mathrm{d}x + (x + z)\ \mathrm{d}y + (xy + x + y + 2\mathrm{e}^z)\ \mathrm{d}z$

     Execution: `potential3(ê^x+1,x+z,xy+x+y+2ê^z)`

     Solution: The potential function does not exist

Line Integral of a non-exact differential in $\mathbb{R}^2$

CONTS

Build a command named LINEINTEGRAL2 to calculate the line integral of a non-exact differential in $\mathbb{R}^2$ along a given curve.

- **Theoretical remarks**

  Let $\mathcal{C}$ be the curve parametriced by:

$$\begin{aligned} \vec{\alpha} : [a,b] &\longrightarrow \mathbb{R}^2 \\ t &\longrightarrow \vec{\alpha}(t) = \big(c_1(t), c_2(t)\big) \end{aligned}$$

  The line integral of $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ along C, is defined as:

$$\int_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \int_a^b \Big[P\big(c_1(t),c_2(t)\big)c_1'(t) + Q\big(c_1(t),c_2(t)\big)c_2'(t)\Big]\,\mathrm{d}t$$

- **Derive's command**

```
lineintegral2(p,q,c1,c2,a,b) :=
   int(subst(p,[x,y],[c1,c2]) dif(c1,t) +
       subst(q,[x,y],[c1,c2]) dif(c2,t) , t, a, b)
```

- **Auto-check exercise**

  Calculate $\quad I = \oint_C xy\,\mathrm{d}x + 2x\,\mathrm{d}y \quad$ along the circunference $x^2 + y^2 = 2^2$.

Execution: `lineintegral2(xy,2x,2cost,2sint,0,2pi)`

Solution: 8π

CONTS

Line Integral of a non-exact differential in $\mathbb{R}^3$

Build a command named LINEINTEGRAL3 to calculate the line integral of a non-exact differential in $\mathbb{R}^3$ along a given curve.

- **Theoretical remarks**

  Let $\mathcal{C}$ be the curve parametriced by:

$$\begin{array}{rcl} \vec{\alpha} : [a,b] & \longrightarrow & \mathbb{R}^3 \\ t & \longrightarrow & \vec{\alpha}(t) = \big(c_1(t), c_2(t), c_3(t)\big) \end{array}$$

  The line integral of $P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z$ along C, is defined as:

$$\begin{aligned} I &= \int_{\mathcal{C}} P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z \\ &= \int_a^b \Big[P\big(c_1(t),c_2(t),c_3(t)\big)c_1'(t) + Q\big(c_1(t),c_2(t),c_3(t)\big)c_2'(t) + R\big(c_1(t),c_2(t),c_3(t)\big)c_3'(t)\Big]\,\mathrm{d}t \end{aligned}$$

- **Derive's command**

```
lineintegral3(p,q,r,c1,c2,c3,a,b) :=
  int(subst(p,[x,y,z],[c1,c2,c3]) dif(c1,t) +
      subst(q,[x,y,z],[c1,c2,c3]) dif(c2,t) +
      subst(r,[x,y,z],[c1,c2,c3]) dif(c3,t) , t, a, b)
```

- **Auto-check exercise**

  Calculate $I = \int_C (\mathrm{e}^x + 1)\,\mathrm{d}x + (x+z)\,\mathrm{d}y + (xy + x + y + 2\mathrm{e}^z)\,\mathrm{d}z$ along the segment which joins $(0,1,2)$ with $(2,-1,6)$.

Execution: `lineintegral3(ê^x+1,x+z,xy+x+y+2ê^z,2t,1-2t,2+4t,0,1)`

Solution: $2\mathrm{e}^6 - \mathrm{e}^2 - \dfrac{19}{3}$