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The Constant Feature: Spanning K–12 Mathematics

When I reflect on my many years of teaching mathematics, one of the most enjoyable and fruitful ideas for calculator use came from an article that appeared more than twenty-five years ago (Judd 1976). It gave suggestions for using the constant feature of a simple calculator to enhance mathematics learning. I have used Judd's idea successfully in many classrooms at grades levels ranging from kindergarten through twelfth grade. This article revisits the constant feature for four-function calculators and extends it for use with graphing calculators and computer algebra systems (CASs) for a broad range of topics.

In this article, I provide examples of applying the power of the constant feature to topics that include counting and operations, consumer mathematics, algebra, probability, and iterative processes, with implications for both conceptual understanding and problem solving. The examples are based on the unifying concept of recursion, which starts with an initial value and repeatedly applies a rule. With the increasing availability and power of technology, recursion is an important idea for the mathematics classroom.

When racing to a million, students are amazed at how few multiplications are required

Readers may wish to follow the entire development to see how the constant-feature idea connects a variety of results, or they can simply look for the ideas that are suited to the grade levels that they teach. The *constant feature* is the feature of a calculator that allows users to add (or subtract, multiply, or divide) using the same number more than once without entering it each time. For example, to add test scores of 89, 95, 76, 76, and 76 on some calculators, users enter

89 + 95 + 76 = = =,

where each extra equal sign adds another 76.

THE CONSTANT FEATURE

Depending on the type of calculator, the constant feature can be used in one of several ways:

On many simple calculators, the keystrokes

10 + 1 = = = = = . . .

generate 10, 11, 12, 13, 14, 15, and so forth. Some machines require the keystrokes

10 + + 1 = = = = = . . .

to achieve the same result. Others may require using a constant key or operation key, such as

+1 OP
10 OP OP OP OP . . .

On many graphing calculators, the keystrokes are

10 ENTER
+ 1 ENTER ENTER ENTER ENTER
ENTER . . .

Some variation among calculators also exists with respect to the number that is the constant, especially for multiplication. Thus, depending on the calculator,

10 × 2 = = = = = . . .

may generate either 10, 20, 40, 80, 160, 320 . . . (where 2 is the constant multiplier) or 10, 20, 200, 2000, 20000, 200000, . . . (where 10 is the constant multiplier). Identifying the method that works is easy, and one of the methods usually works for any calculator; however, a classroom set of calculators should have the same constant feature. For simplicity, the notation for only one version of the constant feature is given in this article, but most of the examples can be done on almost any calculator, with only minor adjustments similar to the ones that have previously been described. Many of the ideas in this article can also be adapted for use with spreadsheets.

USING A FOUR-FUNCTION CALCULATOR TO BUILD CONCEPTS

Concepts as elementary as counting and number sense can be illustrated in an informative and

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entertaining way by using the constant feature. To count forward by 1's, the user enters

1 + 1 = = = = . . .

Students in kindergarten can use these keystrokes to "race" by 1's, stopping without going past a given number, say 20, to reinforce their understanding of counting. With the added rule that going past the target number (in this situation, 20) results in "crashing," students must think and use number sense—not merely go as fast as possible.

To count backward from 20 to 0, students enter

20 – = = = = . . .

Students who "crash" by going past 0 are introduced to negative numbers. A graphing-calculator display has advantages even in the early grades, since it shows several entries at a time, not just the most recent one.

The keystrokes

7 + 7 = = = = . . .

generate 7, 14, 21, 28, 35, . . . , which illustrates multiplication as repeated addition; whereas

50 – 8 = = = = = =

shows that 8 can be taken from 50 six times with a remainder of 2, thereby illustrating division (with a remainder) as repeated subtraction. These examples help build concepts, both in seeing multiplication as repeated addition and in seeing division as repeated subtraction, as well as in seeing the connections between the operations, as shown in **figure 1.**

Similarly, exponentiation can be shown as repeated multiplication, with the keystrokes

2 * 2 = = = = . . .

When starting with 1 and racing to a target of 1,000,000 by doubling, students are invariably amazed at how few multiplications are required, and they learn a valuable lesson about exponential functions. Experience shows that most students "crash" because they do not stop at 524,288, the optimal number. Students have an opportunity to see that 2^{20} already exceeds one million.

$$\begin{tikzcd}[row sep=huge, column sep=huge] + \arrow[r, "\text{inverse}"] \arrow[d, "\text{repeated}"] & - \arrow[d, "\text{repeated}"] \\ \times \arrow[r, "\text{inverse}"'] & \div \end{tikzcd}$$

Fig. 1

Readers may wish to use the electronic calculator and accompanying hundreds board, which is available on the "Learning about Number Relationships and Properties of Numbers Using Calculators and Hundred Boards" link on the NCTM Illuminations Web site in the pre-K–2 grade band. There the constant feature can be used to show patterns on a hundreds board. For example, **figure 2** shows how it might be used to investigate the least common multiple of 6 and 8 by entering 6 + 6 = = = = . . . and then 8 + 8 = = = = . . .

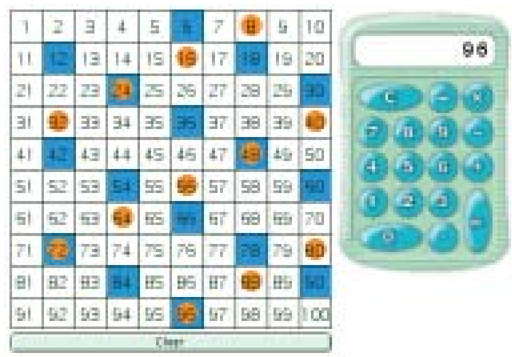


Fig. 2

Additional problems A

1. What keystrokes are needed on your calculator to do each of the following?
   a) Count by 5s
   b) Count by 10s starting with 100
   c) Count backward by 2s from 50
   d) Triple the previous number, starting with 1
2. What name can be given to the set of numbers generated by each of these keystrokes?
   a) 1 + 2 = = = = . . .
   b) 1 * 2 = = = = . . .
3. 60 – 8 = = = = = = = suggests a repeated subtraction model for what division computation?
4. What is the first integral power of 2 that exceeds one billion?
5. What is the inverse operation for exponentiation?

USING A FOUR-FUNCTION CALCULATOR TO SOLVE PROBLEMS

In addition to modeling operations, the constant feature can be used to solve problems. For example, consider the following problem:

> How many school shirts can the class buy for $100 if the cost is $12 per shirt plus a $5 handling charge for the entire order? ➔

It involves a linear relationship, which students can understand by noticing that the cost includes a fixed charge of $5 and increases by $12 for each shirt purchased. This situation suggests the calculator keystrokes

5 + 12 = = = = . . .

Note that this use of the constant feature precedes using a table feature, in that students are not required to invent a formula to create the output. However, this technique can assist in finding a general formula, such as

$$C = 5 + 12S,$$

since the cost of S shirts arises from beginning with 5 and adding 12 S times.

Similarly, the question "How long will a deposit of $100 in a savings account at 4 percent compounded yearly need to reach $150?" involves an exponential relationship. The solution can be obtained by realizing that the initial amount is $100 and that each new amount is the previous amount increased by 4 percent; in other words, it is 104 percent times the previous amount. The keystrokes are

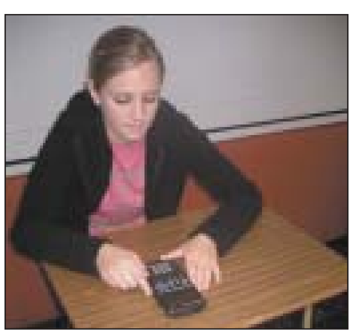
100 * 1.04 = = = = . . . ,

or on some calculators,

1.04 * 100 = = = = . . .

The teacher must carefully develop the idea that the multiplier in this example is 1.04 (and not simply 0.04), since the new amount is the old amount plus a 4 percent gain.

Problems similar to the previous one are important for all students; but in a traditional program, they are typically done (if done at all) by using logarithms in an advanced algebra course, which is taken by perhaps half the students in eleventh grade. With technology, students in the middle grades can solve these problems by using a four-function calculator to obtain a solution to the nearest integer. This example of "mathematical power for all in a technological society" (NCTM 1989, p. 255) also shows how "the computational capacity of technological tools extends the range of problems accessible to students" (NCTM 2000, p. 24).



Photograph by James E. Schultz;

Additional problems B

Find the answers, and give the keystrokes.

1. If you are 440 miles from home on the second day of a trip and if you continue driving at an average speed of 55 MPH, how many more hours will it take to reach a destination that is 710 miles from home?
2. If canoe rental costs $8 for one hour and $6 for each additional hour, for how many hours can you rent a canoe if you have $40? (Notice that this problem is slightly different from the example.)
3. If you pay a deposit of $10 to use a computer in an Internet café and the cost per hour (or part of an hour) is $1.35, for how many hours can you use the computer if you want to leave with at least $4.50 of your deposit?
4. If the population of Kenya is 30.3 million and if it is increasing at a rate of 1.5 percent per year, in how many years will it reach 35 million?
5. If you start with a penny on the first day and double the previous day's amount, after how many days will you reach $1 million? (For example, on the third day you will have 4 cents.)

USING A GRAPHING CALCULATOR TO SOLVE PROBLEMS

Computing the repayment of a credit-card account or other loan involves doing more than one operation. For example, if you begin with a credit-card balance of $1000 and have a monthly interest charge of 1.5 percent and a minimum payment of $25, how long will you need to pay the balance, and what is the total amount of interest? The new monthly balance with the interest added in is 1.015 times the former balance minus the payment. This problem can be solved on many graphing calculators as follows:

1000 ENTER
Ans * 1.015 – 25 ENTER ENTER ENTER ENTER . . . ,

which shows the decreasing balance. Students can learn an important lesson by seeing that more than 61 payments of $25 are needed. Thus, more than five years are needed to pay the balance, and interest costs are more than $500. This exercise, possibly one of the most valuable practical experiences that a student can obtain in a mathematics class, could be done by all students in the middle grades.

Just as constants can be numbers, they can be matrices, which have many applications to problem solving. For example, consider this problem, which is modified from one in *Discrete Algorithmic Mathematics* (Maurer and Ralston 1991, p. 372).

A car rental agency has offices in New York and Los Angeles. Each month, half of the cars in New York go to Los Angeles and one-third of the cars in Los Angeles go to New York. Starting with 600 cars in each office, how many cars will be in each office three months later? What happens in the months that follow? Why?

The number of cars in New York after the first month is

$$\frac{1}{2} \times 600 + \frac{1}{3} \times 600 = 500,$$

and the number of cars in Los Angeles is

$$\frac{1}{2} \times 600 + \frac{2}{3} \times 600 = 700,$$

This problem can be represented by using matrices. To begin, use the 1×2 matrix

$$[600 \quad 600] = A,$$

which is called an *initial stage matrix,* to represent the number of cars in New York and Los Angeles, respectively, and use the following 2×2 matrix, called a *transition matrix*, to represent the fraction of cars moving each month:

$$\begin{array}{ccc} & & \text{To} \\ & & \begin{array}{cc} \text{NY} & \text{LA} \end{array} \\ \text{From} & \begin{array}{c} \text{NY} \\ \\ \text{LA} \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = B \end{array}$$

Then the computation can be represented in matrix form as follows:

$$[600 \quad 600]\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = [500 \quad 700]$$

The computations for the next two months in matrix form (rounding to the nearest integer) are

$$[500 \quad 700]\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = [483 \quad 717]$$

$$[483 \quad 717]\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = [481 \quad 719]$$

As with numbers, this process amounts to entering an initial value (matrix A) and repeatedly multiplying by a constant (matrix B):

A ENTER
× B ENTER ENTER ENTER ENTER . . .

The display will indicate the monthly inventories of cars when the user continues to press ENTER. After three months, 481 cars are in New York and 719 are in Los Angeles. After just one more month, a steady state of

$$[480 \quad 720]$$

is achieved (to the nearest integer) because each time the matrix

$$[480 \quad 720]$$

is multiplied by B, the answer is again

$$[480 \quad 720].$$

This feature can be used to solve a variety of problems involving matrix multiplication that are included in high school mathematics curricula. A thorough discussion of this idea, along with student activities, can be found in "Probability, Matrices, and Bugs in Trees" (Young 1998).

The constant feature can demonstrate many iterative processes, such as the divide-and-average method of approximating square roots. It might be presented to students as an example of an algorithm, since it is not needed when a square-root key is available. For example, in finding $\sqrt{20}$, let 4 be a first approximation. Since 4 is less than $\sqrt{20}$ and $20/4 = 5$ is greater than $\sqrt{20}$, use 4.5 as a next approximation by averaging 4 and 20/4. Continue the process by using the divide-and-average method. The calculator keystrokes

4 ENTER
(Ans + 20 / Ans) / 2 ENTER ENTER
ENTER ENTER ENTER . . .

converge to 4.472135955 in just four iterations.

As another example, to approximate a zero of $x^3 - x^2 + 5x + 3$, solve $x^3 - x^2 + 5x + 3 = 0$ for x by using the $5x$ term to obtain

$$x = \frac{-x^3 + x^2 - 3}{5},$$

which suggests the iterative equation

$$x_{n+1} = \frac{-x_n^{\,3} + x_n^{\,2} - 3}{5}.$$

Then using $x_0 = -1$ gives

$$x_1 = \frac{-(-1)^3 + (-1)^2 - 3}{5} = -0.2$$

and

$$x_2 = \frac{-(-0.2)^3 + (-0.2)^2 - 3}{5} = -0.5904.$$

The sequence converges to –0.518 within ten iterations. The calculator keystrokes

The constant feature can be used to solve a variety of problems involving matrix multiplication that are included in high school mathematics curricula

```
-1 ENTER
(-Ans ^ 3 + Ans ^ 2 - 3) / 5 ENTER ENTER
                          ENTER ENTER . . .
```

give –1, –0.2, –0.5904, –0.4891, –0.5287, . . . Two early calculator articles (Waits and Schultz 1979; Schultz and Waits 1979) discuss this method and give many examples of formidable equations that can be solved by using an iterative process, although these articles do not advocate using the constant feature.

Additional problems C

1. How long will it take to pay off a car loan of $2000 with a monthly interest charge of 1 percent and a payment of $50?
2. What monthly payment (to the nearest dollar) will pay off the car loan in problem 1 in twelve months?
3. Consider the car-rental problem, starting with 500 cars in New York and 500 cars in Los Angeles and with 30 percent of the cars in New York going to Los Angeles and 20 percent of the cars in Los Angeles going to New York. How many cars are in New York after two months? What happens in the months that follow?
4. Make up and solve a similar car-rental problem that involves cars in three cities.
5. Compute the matrix product [480 720] B in the original car problem.
6. Let c be the number of cars in New York, and let $1200 - c$ be the number of cars in Los Angeles. Use the given transition matrix B to solve for c in the matrix equation

$$[c \quad 1200 - c]\, B = [c \quad 1200 - c].$$

   In other words, find the value of c for which the initial stage matrix does not change when it is multiplied by the transition matrix.
7. Use the divide-and-average method to find $\sqrt{60}$ to the nearest hundredth.
8. Find the zeros of the function $f(x) = x^3 - 8x + 1$ to the nearest hundredth by using the method of iteration. Hint: Solve for x in the $-8x$ term to get one solution. Solve for x in the x^3 term to get two additional solutions.
9. Find a solution to $5 - x^2 = 2^x$ to the nearest hundredth by using the method of iteration. Hint: Solve for x in the x^2 term, and use both + and – the radical.

A CAS can be used to generalize the graphing calculator problems to give insight into the corresponding formulas

USING A CAS TO BUILD CONCEPTS

A CAS can be used to generalize the graphing-calculator problems to give insight into the corresponding formulas. The graphing-calculator keystrokes for the shirt problem,

```
5 ENTER
+ 12 ENTER ENTER ENTER ENTER . . .
```

can be generalized using a CAS to find the cost c for n shirts costing s dollars each and a handling charge of h. Students can see from the corresponding calculator keystrokes that

```
h ENTER
+ s ENTER ENTER ENTER ENTER . . .
```

gives h, $s + h$, $2s + h$, $3s + h$, $4s + h$, which suggests the formula $c = ns + h$.

In a similar way, insights can be given for such algebraic expressions as $5a$ and a^5, much like the keystrokes on the four-function calculator using

```
a ENTER
+ a ENTER ENTER ENTER ENTER
```

and

```
a ENTER
× a ENTER ENTER ENTER ENTER.
```

Arithmetic and geometric progressions can be obtained in much the same way:

```
a ENTER
+ d ENTER ENTER ENTER ENTER
```

yields $a + 4d$, suggesting the formula

$$t_n = a + (n - 1)d$$

for the nth term of an arithmetic progression with first term a and difference d. Similarly,

```
a ENTER
* r ENTER ENTER ENTER ENTER
```

yields ar^4, suggesting the formula $t_n = ar^{n-1}$ for the nth term of a geometric progression with first term a and ratio r.

Illustrating how the numbers from Pascal's triangle arise in the binomial theorem can be accomplished by entering the keystrokes

```
x + y ENTER
EXPAND (Ans * (x + y)) ENTER ENTER
                          ENTER ENTER . . .
```

Note that the EXPAND command may be required to show the desired form.

Additional problems D

(Note: Problems 3–5 can be done on some graphing calculators by using a fraction feature or an i key.)

1. Find the population p of a city after ten years if the population starts at a and increases by a fixed amount d each year.
2. Find the population p of a city after ten years if the population starts at a and increases by rate r over the previous year.
3. Describe the fractions that result when you enter the following in exact mode:

```
1 ENTER
```

1 + 1 / Ans ENTER ENTER ENTER ENTER . . .

4. Evaluate the same keystrokes in approximate mode. Describe what happens.
5. Using the representation for the complex number *i,* enter

i ENTER
∗ *i* ENTER ENTER ENTER ENTER . . .

What pattern do you see?

EXTENSIONS

Creating a counter

Some calculators have a built-in counter for the constant function. On others, sequences can be used to create a counter when using the problem functions. Examples:

- In the shirt problem,

  {0, 5} ENTER
  + {1, 12} ENTER ENTER ENTER ENTER, . . .

  displays {1, 17}, {2, 29}, {3, 41}, {4, 53}, . . . , which counts the number of shirts purchased for the given amount.
- In the compound-interest problem (a nice challenge for students),

  {0, 100} ENTER
  ∗ {1, 1.04} + {1, 0} ENTER ENTER ENTER ENTER, . . .

  displays {1, 104}, {2, 108.16}, {3, 112.49}, {4, 116.99}, . . . , which counts the number of years corresponding to the given balance. Note the role of the identity for multiplication in the multiplication part and the identity for addition in the addition part of the linear combination of ∗ {1, 1.04} + {1, 0}.

Logistic functions

In *Chaos: Making a New Science,* James Gleick (1987) discusses logistic functions of the form

$$x_{n+1} = cx_n(1 - x_n),$$

which model growth, such as fish populations that vary with the number of predators. For example, if $x_0 = 0.5$ and $c = 1.2$, the sequence 0.5, 0.3, 0.252, 0.226195, . . . can be displayed using

.5 ENTER
1.2 ∗ Ans ∗ (1 – Ans) ENTER ENTER ENTER ENTER, . . .

This sequence converges to 1/6. To see this convergence, solve $x = 1.2x(1 - x)$ for x. Next, use the constant feature to explore what happens in each of the following cases:

a) $x_0 = 0.5$ and $c = 1.5$

b) $x_0 = 0.5$ and $c = 3.2$

c) $x_0 = 0.5$ and $c = 3.5$

Example (*b*) suggests the idea of bifurcation, in which the population eventually oscillates between two values. The sequence in example (*c*) bifurcates again to approach four values.

Markov chains

The following problem can be solved by using matrices and the constant function:

> If a fast-food restaurant offers one of three equally likely randomly chosen different prizes with each visit, find the probability of obtaining all three different prizes in five visits.

Begin with an initial matrix $A = [1\ 0\ 0]$, representing the probability of obtaining one, two, or three different prizes, respectively, in one visit. The transition matrix B shown below indicates the probability of having a given number of prizes after making one more visit.

$$\text{Prizes after previous visit}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \text{Prizes after next visit} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

For example, the entries in the first row indicate that if you have one prize and make another visit, the probability that you will then have one, two, or three different prizes, respectively, are 1/3 that you receive the same prize, 2/3 that you receive one of the two remaining different prizes, and 0 that you go from having one prize to three prizes in a single visit.

If you use the constant feature as follows, you will find that the probability of having all three prizes in five visits is about .617. Use the keystrokes

A ENTER
∗ B ENTER ENTER ENTER ENTER

CONCLUSION

This article has presented examples that show the application of the idea of recursion to topics beginning with simple counting and leading to powerful iteration methods, all with the support of readily available technology. Each example has a place in the K–12 mathematics curriculum and helps students enhance their conceptual understanding and tackle problems long before they study advanced topics. Together, the examples enable students to see a powerful connection among important ideas, including computing credit-card interest, solving complicated equations, solving problems using matrices, exploring the relationship between the binomial theorem and Pascal's triangle, and operating with complex numbers. With this beginning, you and your students can explore and discover vast amounts of interesting and useful mathematics.

SOLUTIONS

Minor modifications may be necessary for various calculators.

Additional problems A

1. *a*) 5 + 5 = = = = . . .
   b) 100 + 10 = = = = . . .
   c) 50 – 2 = = = = . . .
   d) 1 * 3 = = = = . . .
2. *a*) odd (whole) numbers
   b) powers of 2
3. 60 ÷ 8 is 7 with remainder 4.
4. 30
5. logarithms

Additional problems B

1. about 5 hours; 440 + 55 = = = = =
2. 6, with $2 left over; 8 + 6 = = = = =
3. 4; 10 – 1.35 = = = =
4. 10; 30.3 * 1.015 = = = = . . .
5. 28; .01 * 2 = = = = . . .

Additional problems C

1. 52 months
2. $178
3. 425 cars. The number of cars eventually stabilizes at 400 cars in New York and 600 cars in Los Angeles.
4. Answers will vary. Use a 1×3 initial stage matrix and a 3×3 transition matrix.
5. [480   720]
6. Use matrix multiplication, and solve
$$c \cdot 1/2 + (1200 - c) \cdot 1/3 = c$$
to obtain $c = 480$.
7. 7.75
8. –2.89 or .13 or 2.76
9. 1.48 or –2.19

Additional problems D

1. $a + 10d$
2. $a(1 + r)^{10}$
3. 1/1, 2/1, 3/2, 5/3, 8/5, . . . , that is, fractions using consecutive Fibonacci numbers
4. The decimals approach 1.618 . . . , that is, the golden ratio, or $(1 + \sqrt{5})/2$.
5. i, –1, –i, 1, i, –1, – i, 1, . . . The pattern repeats in groups of 4, since $i^4 = 1$.

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