CONTS

TI-Based Learning Environment: Developing Conceptual Understandings of Function-Related Concepts

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Abstract: The foundation for this paper consists of (a) a study carried out in three graphing calculators based Precalculus classes and (b) authors' teaching experiences in technology-oriented environments. The study considered two intertwined aspects. The first aspect dealt with enabling students to build a stronger conceptual understanding of functions and related concepts and the second was to study what conditions provide for a successful learning environment utilizing graphing calculators. The key factors identified and applied were: long-term exposure to ill-structured problems; writing about the concepts; the teacher answering questions with appropriate questions/prompts to provide for scaffolding; cooperative learning; and the teacher's proficiency with graphing calculator. The students developed a deeper understanding of the concepts and they were more willing to attempt complex problems. Their communication skills improved. The study indicates that problem-based learning in a technology oriented environment provides appropriate conditions for developing critical thinking and communication skills. Authors are using this study as a springboard to elaborate further on the TI-based learning environments and emerging calculator-related issues; preconceptions and misconceptions; and indispensable and dispensable mathematical abilities and skills related to the concepts studied.

Introduction

Authors' deliberations in this paper are based on the study that was conducted by the first author during her graduate work and under supervision of the second author. The aim of the study was to investigate two intertwined learning goals. The first one was enabling students to build a strong conceptual understanding of functions and related concepts and the second was to study what conditions provide for a successful learning environment utilizing graphing calculators. Some results of this study have already been reported in Thiel & Alagic (2004).

The goal of this paper is to elaborate on the aspects of the technology used and conditions that had to be changed to accommodate the learning environment for appropriate use of graphing calculators. Furthermore, authors are looking at students' preconceptions and misconceptions and how to address them, as well as searching for dispensable and indispensable mathematical concepts in the environment centered around graphing calculators. For that purpose this paper is "unbalanced" in the following way: *The Study* section describes the research only as much as it is necessary to introduce the reader to the context of authors' source of information and reflections. It comprises defining the problem, literature review and design of the study. *Findings* part of the paper is elaborating on Students' achievement, Students' understanding, Teacher's understanding, Preconceptions and misconceptions, and Indispensable and dispensable mathematical concepts. The appendix includes four labs used by students during this study.

The Study

Defining the Problem

In the current NCTM 2000 Principles and Standards for School Mathematics (NCTM, 2000), the learning principle states: "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (p. 20). Understanding is the key. The basic goal of education is to prepare students to be life long learners and to function effectively in the world. In order for this to occur, students must understand the mathematics that they are taught. Based on research of how people learn (Cobb, 1994; Bransford, Brown & Cocking, 2000), a learner acquires knowledge by construction of new knowledge from old, rather than knowledge transmission and the recording of information conveyed by others. Students must "author their own knowledge, by revising and creating new understandings out of existing ones" (Applefield, Huber, & Moallem, 2001, p. 3).

Year after year, there are still students who enter the mathematics classroom not remembering the basic concepts that were "taught" to them previous year. It seems that for some students the mathematics classroom is a day after day endless sequence of facts and procedures that make little sense. It is no wonder that the next year, students do not remember what they have learned, they have made no connections. So in an effort to "cover the material" that is in the syllabus we drill and practice and then the next year, do the same thing. Not only have many students forgotten most of the facts and procedures, but also they seem unable to apply many of the learned procedures to new problem situations. The transfer of knowledge does not occur. This is directly related to their lack of conceptual understanding of the mathematical ideas (Bransford et al., 2000).

The authors experiences confirm what research has found. Students could "do" lots of mathematics, but many of them did not seem able to make the transfer of knowledge from one situation to another. It was the introduction of the graphing calculator, in the early 1990's, and it's capacity to provide students with multiple representations of the problem that caused the first author to seek opportunities to learn about this tool and how it could impact her teaching of mathematics.

Students develop new knowledge through an active process, not through passive reception of information. For learners to construct meaning they must view each new experience in light of what they already know about the topic. Students' current knowledge base provides the initial context for creating new understanding. They must have opportunities to engage in activities that allow them to interpret new information and create new understandings based on their previous knowledge base. No matter the sophistication of the learner, their pre-existing knowledge base will have a powerful influence on what is learned and how previous concepts change (Applefield et al., 2001).

Three guiding questions formed the basis of this study. How will students develop conceptual understanding of functions and function-related concepts in a problem-based learning environment that requires significant utilization of graphing? In what way might this approach result in a student-centered approach to problem solving? And lastly what decisions will the teacher need to make during the course of the research to accomplish the teacher's year-long learning goals?

Literature Review

The purpose of this study is to focus on problem-based learning a) facilitated by cooperative group activities and b) utilizing multiple representations including technology based ones, and how these issues impact students' attitude and achievement. Therefore topics such as problem-based learning, technology, cooperative learning and naturalistic research paradigm were the basis of this literature review. Two studies, on problem-based learning, revealed positive outcomes when students were engaged in solving ill-structured problems. In a study on cooperative learning opportunities, improvement in student understanding of concepts was also documented. Studies involving the use of software and the graphing calculator to create learning environments that promote student understanding had positive results.

Problem-Based Learning. Research by Mergendoller, Maxwell and Bellisimo (2000), in a pilot study, compared Problem-Based Learning (PBL) to traditional instruction in an economics classroom. Three teachers who had attended a week-long training workshop taught one hundred eighty-six students in nine classes. Two of the nine classes were comparison classes and were taught using traditional lecture/discussion approach. The treatment classes were given ill-structured problems and were allowed to explore, research, and cooperatively discuss solutions to the problem. As students worked on the problem, they discovered that understanding economics concepts was essential to framing and solving the problem. The data suggested weak empirical confirmation that PBL classes spark student interest in the subject studied and teach students to learn from their mistakes. This was a pilot study and the researchers admit that a larger population and more rigorous testing methods need to be used.

Gallagher, Stepien and Rosenthan (1992) studied the effects of problem-based learning on problem solving with students in a 3-year state-supported residential school for students talented in mathematics and science. Eighty-seven junior and senior students were in the experimental group and forty-four students were in the control group. Intervention consisted of three process-oriented goals: (a) to lead students to discover the interdisciplinary character of most "real world" problems, (b) to require students to engage in the process of solving an ill-structured problem, and (c) to improve students' problem-solving skills. Analysis of the data on pre and post-test ill-structured problems indicated that students who had been given the intervention treatment improved in their problem solving abilities. Students in the control group did not have the same level of improvement even though all students in this school had been introduced informally to problem-solving strategies. Engagement in solving ill-structured real-world problems seemed to provide the difference in student learning.

Cooperative Learning. Whicker, Bol and Nunnery (1997) investigated the effects of cooperative learning on student achievement and attitudes in two precalculus classrooms. The 31 participants attended a rural, lower-middle class school in the mid-South. Before the beginning of the treatment phase, students in the cooperative group class were instructed in the rules for small-groups. They were instructed to explain their answers to each other and to direct questions to their teammates rather than the teacher. Students were told that they were not finished with their task until they were certain that all members of the group could score 100% on the test. The classroom procedure for the treatment group was teaching the lessons, group study, testing, and team recognition. The teacher presented the material to both the treatment group and the comparison group in 5 to 8 days, depending on the length of the chapter. The students in the treatment group then spent the next two class periods studying the teacher-developed review sheets that included the correct answers, while students in the comparison group used their two days to individually study the review sheets. After the two-day study sessions, all students took individual tests on the chapter material. Students in the experimental group scored higher on the tests than students in the comparison groups. The differences in the scores increased throughout the study, leading researchers to conclude that it takes some time for the benefits of cooperative learning to become apparent. There were decreases in test scores for both groups, but the decline in scores for the comparison group was greater than the decline in scores for the experimental group. The researchers attribute these declines to the fact that the material studied was advanced in nature and was increasingly difficult throughout the 6 weeks. Students in the experimental group favorably evaluated the cooperative learning procedure. They liked receiving help from others and working with other students. The only negative side to their responses was the sameness of the group as they were not reassigned during the 6-week study.

Technology. Confrey, Piliero, Rizzuti, and Smith (1990), supported by grants from Apple Classrooms of Tomorrow, conducted a study based on constructivist framework. The goal of the project was to create a learning environment that promoted student instruction of mathematical concepts through repeated cycles of developing a problem, working to solve the problem and reflecting on the solution to the problem. A class of 22 students in their fourth year of mathematics participated in this study. The teacher of the class had sixteen years of experience and described her teaching as traditional. Teacher development during this project included a week of intensive preparation during the summer and continued support during the school year. Research findings indicated that students of average and above abilities were able to use the available software to construct strong conceptualizations of linear and exponential functions. The teacher's role in this developmental process was shown to be as strong as the role of the software. In this project, the teacher was also a learner. Not only did her view of mathematics change during the course of the research, but her willingness to devote more instruction time to problem solving activities and student-initiated activities increased. By allowing students to acknowledge alternative ways of approaching and solving problems and sharing those insights, the teacher felt that her students had developed great understanding of the mathematics and had taken on greater responsibilities for their own learning. Conclusions drawn from this study included the following: (a) sustainable changes in classrooms requires a systematic, but incremental approach; (b) the teacher is a critical participant in the process; (c) institutional changes such as longer classroom periods and partnership among teachers are necessary; (d) students need support throughout the change process; and (e) methods for assisting students in working effectively in groups must be developed and supported with forms of assessment that promote student's own evaluation of their work.

Smith and Shotsberger (1997) considered the effect of integrating the graphing calculator into a semester-long college algebra course. The project focused on the extent to which student achievement, attitude, and problem-solving methods were effected by the use of the graphing calculator. A total of 114 students in four classes were involved in the study. Two teachers participated, each teaching a calculator and a non-calculator based class. All four classes used the same non-calculator based textbook. The teachers were provided with training in calculator usage. The results of the study indicated that students utilizing the calculator scored consistently higher on the achievement tests (62.9%) than did the non-calculator sections (59.6%). Females had significantly higher scores in both the calculator and non-calculator classes. Pre and post measures of student attitudes were collected using a researcher-developed questionnaire related to feelings about learning mathematics, doing mathematics and valuing mathematics. Using these measures the post measure of attitude of calculator students was slightly higher than that of non-calculator students. In an additional survey, students asked questions concerning out-of-class usage responded that they considered the calculator to be integral to their course work. Use of the graphing calculator had a positive impact on the achievement and attitude of these students.

Naturalistic Paradigm. Three research principles guide a naturalistic research study. The first principle recognizes that multiple viewpoints of an event are essential in order to understand the learner's existing base of knowledge. Connecting theory verification to theory generation is the second principle of naturalistic research. The third principle pays special attention to studying cognitive activity in context. One way to address this principle is to study the learning process in the natural setting in which it occurs without intervention (Moschkovich & Brenner, 2000).

Moschkovich (1996, 1998) integrated a naturalistic paradigm into her study to explore students' conceptions of linear functions and to examine how these conceptions change. Two ninth-grade algebra classes were studied. Students were observed working in groups or with a teacher during two curricular units. Classroom observations as well as videotaped student conversations, and written assessments were utilized to determine the students understanding of linear functions. The study reflected a difference both in design and analysis from quantitative studies. The cycle of data collection began by observing students at the beginning of the unit in a natural setting, the classroom, making conjectures based on these observations, and then addressing these conjectures directly in the design of the learning process. Conjectures based on observations were corroborated through the analysis of written and videotaped data. A central objective of the study was to consider multiple points of view, especially the points of view of the students. The goal was to describe how the students approached the connection between lines and equations. As the study progressed, two perspectives were maintained; the researcher's perspective, with formal mathematics training, and the students' perspective, as they reflected and conversed with each other.

Design of the Study (Thiel & Alagic, 2004)

Three precalculus classes in an independent college-preparatory urban mid-western school were involved in this study. A total of 49 students participated in the semester-long project. Students in this school are required to take 4 units of mathematics consisting of Algebra I, Algebra II, Geometry, and one other course to meet graduation requirements. The fourth course could be Precalculus or Advanced Placement Statistics. Approximately 50% of the student body complete more mathematics than is required and take some level of calculus before graduation. Consequently about 50% of any senior class is enrolled in Precalculus or Statistics. The study was done in three Precalculus classrooms, grades 10, 11 and 12.

The teacher/researcher in this study had participated in extensive training incorporating the graphing calculator into the classroom. Three summer sessions at a Precalculus-Calculus Lead-Teacher Training at the North Carolina School of Science and Mathematics provided the teacher with experience in problem-based learning utilizing the graphing calculator. The teacher has been integrating the graphing calculator into the classroom as a tool for discovery and to answer the “what if” questions since her first professional development with graphing calculators. The teacher believes that when students experience a “learning diet” rich in explorations and discoveries, followed by making connections with the mathematical concepts behind it, students are better able to develop strong conceptual understandings. Allowing the students to discover a mathematical concept, in an appropriate context, rather than just telling them about the concept, enables students to make better connections with their prior knowledge, and therefore, between new and existing concepts.

The use of the graphing calculator was an integral part of every activity and every student had his/her own TI-83 graphing calculator. The teacher also had an overhead calculator and a viewscreen for projecting onto the board. Students were already experienced with the use of the calculator as it is a required tool in the Algebra I and Algebra II classroom. They were knowledgeable in using lists, tables, function evaluation, and the graphing features of the calculator.

The naturalistic approach to research as well as rules of good teaching required that the teacher first have a grasp of what the students did understand. In order to assess this, two pre-test situations were employed. Students were asked to describe four key mathematical concepts in as rich a language as possible. Those concepts were function, domain, range and inverse. These descriptions provided base-line data on which of the multiple representations students used as well as which of the concepts students understood. The second pretest was an ill-structured problem involving concepts that will be used at later time. Naturalistic research necessitates the consideration of a wide variety of viewpoints of any one particular event or concept understanding. Toward that goal a variety of assessment tools were used. At the end of the first quarter and at the end of the semester, students wrote about the concepts that they have been studying. Pairs of students also volunteered to be recorded as they conversed about the concepts. Rich documentation for each student, including tests, labs, group problems and written work, was kept in the form of a portfolio.

During the semester two ill-structured problems were presented to the students The solution to a problem was due one week following the test on the material in the corresponding chapter. As the material in a unit was taught, students gained more insight into possible solutions of the problem, but it was never “taught” in the classroom. A rubric for grading the ill-structured problems was given each group when the problem was assigned. For each problem, students could request additional information, but only if they could justify to the teacher the relevance of the information to a line of inquiry. For the second problem, students were given twenty-minute time segments about twice each week to work on their problem solutions. This was a better working arrangement for the students. During each in-class work session, one member of the group was to record, in a spiral notebook, what the group did that day, what worked, what didn’t work and where they were headed. Approximately once every two weeks, the teacher read the student notes and made comments or suggestions.

During the time spent on either chapter, a variety of methods were utilized to help students “construct” their own knowledge. Data gathering labs, exploration labs and less complicated ill-structured problems were done in cooperative groups, where both group work and individual work were required of the students.

As another means of assessing student progress toward conceptual understanding, four pairs of students were taped as they held discussions. These taping sessions took place after the first quarter and at the end of the semester. Students volunteered to have their conversations recorded. It was assumed that the students who volunteered for the first quarter recording session felt fairly confident in themselves.

One of the critical issues in the constructivism paradigm is that students construct their new understandings of a concept from their prior understanding of that concept. Pre-assessment would let the teacher know which concepts were clearly understood by the students and which concepts were unclear or not understood at all. For each of the four major concepts: function, domain, range, and inverse, students were assessed on a scale of zero to five for each method of representation, zero being the lowest score. Each entry represents the average total score (out of 20) that the students earned in each concept. Total scores are a summation of the four methods of representation (verbal, analytical, numerical and graphical).

	Female 10	Male 10	Female 11	Male 11	Female 12	Male 12
Function	1.0	2.2	1.0	1.7	0.8	2.8
Domain	0.5	1.0	0.7	1.5	0.8	0.8
Range	0.5	1.4	0.9	1.1	0.9	1.0
Inverse	0.5	1.2	0.9	1.4	1.0	1.0

Table 1: Pre-Assessment Averages (Thiel & Alagic, 2004)

The pre-assessment situations provided insight and confirmed the researcher's feelings that even though these students had had three years of high school mathematics, few if any of them could give a meaningful definition of a function, describe the concepts of domain and range, or had any idea what the inverse of a function meant. Though it was suggested that students use multiple representations, many had no idea what that meant. Student response to this pre-assessment was less than positive, as many of them left definitions blank.

At the end of the quarter and semester, students again wrote definitions of the four key concepts in as rich a means as possible, using multiple representations. The results of the end-of-semester assessment are below. As before, for each of the four major concepts, function, domain, range, and inverse, students were assessed on a scale of zero to five for each method of representation, zero being the lowest score. Each entry represents the average total score (out of 20) that the students earned in each concept. Total scores are a summation of the four methods of representation (verbal, analytical, numerical and graphical).

	Female 10	Male 10	Female 11	Male 11	Female 12	Male 12
Function	20.0	18.6	17.8	17.5	16.6	16.5
Domain	16.0	17.6	15.8	16.4	14.0	14.3
Range	16.0	16.6	14.9	15.4	14.2	13.8
Inverse	17.5	17.6	17.3	16.7	15.2	13.5

Table 2: End-of-Semester Averages (Thiel & Alagic, 2004)

Taped conversations between pairs of students again occurred at the end of the semester. The teacher wrote down equations and sketched any graphs that either student used. In this way the transcripts of the conversations could more accurately reflect what actually occurred. These conversations were a vast improvement from those at the quarter. The pair of students discussing functions was able to include in their discussions all the following types of representations: verbal, analytical, graphical and tabular. The discussion of domain and range was also richer than previously recorded. This pair of students used analytical, graphical and table methods as well as verbal information to describe these concepts. The most improvement could be seen in the discussion of inverses. These students seemed to have a reasonably clear understanding of the graphical look of an inverse as well as the rotation about the y = x line. There was some confusion in how to write inverse notation, but even that was corrected during the discussion.

Findings

The ongoing discovery of increasingly efficient technology-based tools and their increasing use in schools lends support to the view that a paradigm shift in teaching and learning mathematics is taking place. For many teachers, understandings of these ideas are grounded in the ways they learned them during their formal education. They are aware of these changes and many of them are involved in the processes of these changes in their schools. The research on use of technology reveals the challenges that the blending of technology and teaching/learning mathematics pose. And, it is ultimately the mathematics teachers, not the technological tools that continue to be the key to the success of the mathematical learning environment (e.g., Garofalo, Drier, Harper, Timmerman, & Shockey, 2000; Kaput, 1992; NCTM, 2000, Alagic, 2003).

In this context, the purpose of the research reported here was to study two related aspects of the learning process; the broadening of students' conceptual understanding of certain concepts, and the teacher's insight into the development of those conceptual understandings. Both aspects were studied while utilizing problem-based learning, cooperative group structures and graphing calculators. The key factors identified as relevant and applied during this study were: long-term exposure to ill-structured problems, writing about the concepts, the teacher answering questions with appropriate questions/prompts to provide for scaffolding, cooperative learning; and the teacher's proficiency with graphing calculator. The students developed a deeper understanding of the concepts and they were more willing to attempt complex problems. Their communication skills improved. The study indicated that problem-based learning in a technology-oriented environment provides appropriate conditions for developing critical thinking and communication skills (Thiel & Alagic, 2004).

Students

Students' Achievement. At the middle of the semester the teacher/researcher was disappointed in the overall student averages. Student understanding of functions was beginning to take shape, but many misconceptions were still apparent. The concepts of domain and range were less developed, though students had some understanding of the issues at hand. The concept of inverses was completely misunderstood. The students had made no real connections with the concept of inverse in their previous course. What they did connect with was the elementary idea of the inverse/reciprocal of a number and so assumed that the inverse of a function would be the same as the inverse of a number, one divided by the function. They made connections, just not the correct ones. Although significant improvement had occurred at this time, the teacher was very concerned at this point in time because the class was approximately two full weeks behind where the class had been in previous years (Thiel & Alagic, 2004).

The taped conversations at the end of the semester indicated that some of the previous misconceptions about the four key concepts had been cleared up. It is important to remember that the students volunteering for this task probably had a great deal of confidence in their understanding and they knew what kinds of questions were going to be asked of them. In the conversation about functions, the pair of students utilized verbal, analytical, graphical and tables as a means of describing the concept. They also pointed out examples graphically and using tables which were not functions. The conversation about domain and range did include all four methods of description, but a misconception about the domain and range of a table of values continued to surface as it had at mid-semester. The conversation about inverses also contained all four methods of description. Here a misconception about how to write an inverse was cleared up in the discussion process.

The teacher/ researcher had some concern that utilizing problem-based learning would result in not being able to "cover" necessary material during the school year. However during the 2nd semester, without a conscious effort on the teacher's part, the Precalculus classes were no longer behind. As the year concluded, it was noted that the concepts covered by the students were almost identical to those that had been covered the past 5 years. As the teacher reflected on how the class made up those two weeks, it became apparent that throughout the 2nd semester, the many word problems that we studied did not take as much in-class time for the students as had been necessary in previous years (Thiel & Alagic, 2004).

The following table (Table 3.) shows the progress made during one semester.

	Female 10	Male 10	Female 11	Male 11	Female 12	Male 12
Function	95.0%	82.0%	84.0%	79.0%	79.0%	68.5%
Domain	77.5%	83.0%	75.5%	74.5%	66.0%	67.5%
Range	77.5%	76.0%	70.0%	71.5%	66.5%	64.0%
Inverse	85.0%	77.0%	82.0%	76.5%	71.0%	62.5%

Table 3. Percent Gain from Pre-Assessment to End-of-Semester

Students' Understanding. By the conclusion of this study most of the students had made significant gains in their ability to discuss the concepts of function, domain, range and inverse and in their ability to solve complex problems. It should be emphasized that the students' exposure to these concepts was never in isolation. The real-life problem scenarios, labs and exploration activities provided the contextual settings to aide students in their understanding. The use of ill-structured problems to facilitate problem-based learning and to examine multiple representations of a problem situation was validated. Graphing calculators provided a rich exploration environment because students were able to make a conjecture and then determine, using graphs or tables whether or not that conjecture was true.

During the second semester an indication of the students improved problem-solving skills or perhaps their improved self-confidence occurred. The event took place early in the 2nd semester when another ill-structured problem had been given to the students as a group assignment. The problem was to take at least two days for the students to complete. The teacher noted that as the groups in the first hour class worked on the problem almost no students asked the teacher questions. This seemed odd, as past experience had shown that this particular problem was difficult and that different groups usually asked many questions. Through the remainder of that days' precalculus classes the same phenomena continued to occur, few, if any questions were asked by the students. At the end of the day, one student remained after last hour class to talk with the teacher. He indicated that he did not believe that this problem was as difficult as the problems had been first semester. As soon as he had spoken those words, there was a look of understanding in his eyes, and he said, "This one isn't so hard because of all those other problems we did before". This unsolicited student comment was at least a sign that students had more confidence in their abilities to try on their own before rushing to ask questions.

Teacher-Researcher

Studying the learning process from the teacher's perspective and determining just how conceptual understandings are developed was a more difficult undertaking. From this study it appears that there were several key factors in this development:

i. The gains made in the last quarter lead the teacher to conclude that students must be exposed to not only a variety of activities, but that exposure must be continued over a longer period of time than just a few weeks.

ii. When students are "forced/guided" to write about concepts and/or to discuss a problem, the outcome (individual connections that they make) is much greater than when they just listen to the teacher and take notes. The teacher has repeatedly witnessed this when asking students individually to explain a concept or problem solving technique. As the words flow from the student's mouth, they catch their own mistakes and reassess what they are saying. They make new connections.
iii. The process of grading the explanations within 2 days of the submission was very time consuming, but it proved extremely worthwhile. The teacher was able to understand what misconceptions the students were developing and come up with an activity that could help them refocused.
iv. Teaching which enhances students' understanding takes willingness on the teacher's part to ask more questions, but be less willing to give the direct answers. Teachers must more often answer a question with another question/prompt or have ready a problem or a (calculator) example that would enable students to develop their own understanding.
v. The graphing calculator played an integral role in this research project. It was used in most every lab and exploration activity, and certainly was necessary for the solution of the ill-structured problems. The manner in which the graphing calculator is used continues to be a challenge for teachers. The teacher has to find new ways to ask questions and new ways to test student understanding. It is an invaluable tool when used in the correct manner. The teacher must understand and therefore teach students how the calculator is providing information. For example, the concept of domain and range are blurred if students explore the function $f(x) = \frac{1}{x-2}$ in a standard window. In this window, on the TI-83+, it appears that the domain is all real numbers, and the range $-11.75 \leq y \leq 7.833$. Changing to a "friendly window" or exploring the values of the function in table form, could more likely lead a student to discovering the actual domain and range for this function.

Study Results: Positive Effects and Challenges

The following seemed to be the positive effects of this study:
1. Students constructed viable understandings of key concepts.
2. Students were able to formulate and solve ill-structured problems.
3. In the long run, the number of concepts discussed, during the school year, neither increased nor decreased from previous years.
4. Students' confidence in their ability to solve difficult problems improved.

The following are recognized challenges.
1. Amount of time required by the teacher to grade written descriptions and ill-structured problems. This problem, perhaps, could be solved by utilizing a self-grading system as reported in a study by Ulmer (2001), Self-Grading for Formative Assessment in Problem-Based Learning.
2. Difficulty in finding challenging problems that are doable.
3. The logistics of long-term ill-structured problem solving in the classroom.

Teaching with Graphing Calculators: Further Deliberations

Preconceptions and Misconceptions

Concepts specific to this study:
- Even though students drew graphs, they were all linear or quadratic.
- They wrote function definitions, but all the definitions were in an f(x) form, as though an equation had to be in that form for it to be a function.
- They discussed the toolkit of functions, but made no mention of the functions used in the ill-structured problem.

- One of the most glaring was their belief that every line was a function.
- Discussions of domain and range were limited. For example, one major misunderstanding occurred when the students examined a table that describe a function relationship. Some students thought the domain and range was merely the difference between the highest and lowest x and y values, completely disregarding the fact that the tables were made of a discrete set of values.
- The conversation about inverses was similar to those individual statements made by the students on the individual assessment sheets.
- Students indicated that an inverse was a reciprocal of the function.

General Calculator Misconceptions/ Errors:

- What you see on the display screen is not always accurate. Students need to be made aware of the fallacy of believing everything that they see on the calculator screen.
- The graph of the function $h(x)=\dfrac{x^2-1}{x-1}$ appears to be the line with a domain and range of all real numbers when graphed in a standard window. When graphed with a ZDecimal, one is able to see the "hole" in the graph and relate that "hole" to the domain and range restrictions of the function.
- Be aware of behavior hidden from view. A particular graph or relationship may not show up on the screen because the student has not considered the appropriate domain and range of the relationship.
- Students must understand the number of significant digits present in the internal workings of the calculator.

  They are likely to believe that the function $g(x)=\dfrac{3}{x-1}$ will eventually take on a function value of 0, because they have traced either right or left on the graph screen and see a zero produced on the screen as the function value.

The "big" questions behind this study could also be: Having made a decision about the best curriculum (course of study) for students, how can the graphing calculator enhance that study, thus affecting the instruction strategies chosen? How is students' learning affected in a graphing calculators based learning environment? How should testing and other forms of assessment be modified?

Emerging Calculator Issues

- We must keep redefining the basic skills that are expected of mathematics students
- Calculator proficiency and an understanding of it's workings are necessary for the teacher to guide students in their learning
- We can expect students to solve more complex problems applicable to real-world situations since calculators can help with algebraic manipulations (Demana & Waits 1990)
- Student's must have use of a graphing calculator at home and school
- With graphing calculators we can exploit the power of visualization (Shultz, 1991)
- The focus of instruction is changing from mathematical/algebraic manipulation to the understanding mathematics as languages that allow for modeling problems. This is a manifestation of the changing role of mathematics teaching and learning
- Students using graphing calculators become good problem solvers and gain a deeper understanding of algebraic concepts and procedures (Demana & Waits, 1990)
- Dick (1992) adds his findings of technology's affect on students' skills. Teachers can concentrate on the problem-solving process. Students can gain access to mathematics beyond their level of computational skills Technology can be used to explore, develop, and reinforce concepts including estimation, computation approximation, and number properties. Students can experiment with

mathematical ideas and discover patterns. Tedious computations that arise when working with real data in problem-solving situations become doable with technology. Teachers must cover the necessary basic skills to understand the mathematics being used with the calculator.

- Demana and Waits (1990), "Technology empowers students to solve difficult problems."(p. 27)
- Calculators stimulate interest, understanding, and the desire to solve complex problems and find exact answers (Embse & Engebretsen, 1996)
- Stick (1997) did a study with two calculus classes, one with calculators the other without calculators. He concluded that students taught with calculators were more interested and were able to see calculus as applied in the world. Further, he found that exploring a graphical representation first, helped students transition to analytic methods much easier
- "The extensive use of the graphics calculator as a tool for learning and doing mathematics helps students whose limited computational abilities previously prevented them from advancing in the study of important mathematics" (Coxford & Hirsch, 1996, p. 25)
- Harvey (1992) categorizes tests into three parts: (a) technology-inactive - where no opportunity to use the technology exists; (b) technology-neutral - problems easily solved without technology, and (c) technology-active - use of technology is essential or greatly assists the completion of the problem.

Indispensable and dispensable concepts and skills

- Proficient use of the calculator is now an indispensable skill for all students. These skills should be taught as needed, not in isolation. Graphical and numerical solution should, where possible, be encouraged.
- Graphing calculator removes the constraints with which teachers and textbooks relied on artificial nice examples and exercises; approximate answers are more realistic in real-world situations (Dick, 1992)
- Students need to understand factoring and be proficient in general factoring techniques. They no longer need to spend days factoring difficult polynomials.
- The time spent on proving difficult trigonometric identities can be reduced.
- Students need to be exposed to a wider variety of functions; piecewise, higher degreed polynomials, trigonometric functions, inverse trigonometric functions, logarithms and exponentials, as well as the composition of functions.
- Emphasis should be on understanding, not just doing the same kind of problems as were practiced in homework. (Ex: Students practice the concept of transformation of functions. In an assessment situation one can give a domain and range for a function and ask how a particular transformation affects the domain and/or range.)
- Exponential growth/decay should involve a wide variety of real-world problems with “not nice” solutions.

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Appendix

Cassette Tape Problem (Thiel & Alagic, 2004)

A new company is producing a cassette tape of a popular group and wishes to determine the selling price that would result in maximum profit for the company. The data below represents the number of people who would be willing to pay a particular maximum cost for the tape. We collected this data from a class survey.

Data Collection:

	Maximum Price of Tape	Number Willing to Pay this Price
$ 0.00		
$ 5.00		
$ 6.95		
$ 7.45		
$ 7.95		
$ 8.25		
$ 8.75		
$ 9.00		
$ 9.25		
$ 9.50		
$10.00		
$10.50		
$11.00		
$12.00		
$13.00		
$13.50		
$14.00		
$15.00		
$16.00		

Some questions you may want to consider.

1. How many people would really be willing to pay $12.00 or $10.50 for a tape?
2. How would you represent a relationship between the price of a tape and the number of tapes sold?
3. How is profit determined? Cost? Income?

During the next several weeks your group will design and carry out a plan to answer the above question. While some time in class will be devoted to working on this project, your group will need to plan on out-of-class time also. The presentation of your groups' solution will be in the form of a Study Works document. Accompanying this document will be supporting work as indicated in the rubric for this problem

Bungee Egg-Drop Lab

Introduction: In many industrial, engineering, and business applications it is sometimes necessary to develop a mathematical model to predict how a system, structure, economy, etc. will perform. This mathematical model is based on a set of sample data that has been collected. The model that is developed is then used to predict behavior in new situations. In this activity your group will need to come up with a mathematical model (an equation) to describe the amount of stretch there is in a bungee cord or varying length. You will be provided with a participant (egg) and harness (netting), bungee cord (several rubber bands), and meter stick. Your goal is to develop a model that can be used to predict the number of rubber bands needed to provide a "safe jump" from a height to be determined later in the class. Of course, part of the three of bungee jumping is to see how close the participant can come to the ground without actually contacting the ground.

I. Collect your data and develop your model here. Show all data and work that goes into your development.
II. Explain the meaning of the slope and y-intercept of your model in terms of the bungee jumping problem. Do the paper clip and the netting have anything to do with either of these values?
III. Testing your model. Height your group draws: ___________

Calculations used to determine the number of rubber bands required

IV. Evaluation: Scoring will be as follows for *Successful Jumps*:

100	The "jump" is within 5 centimeters of the ground.
95	The "jump" is 5.1 to 10 centimeters from the ground.
90	The "jump" is 10.1 to 20 centimeters from the ground.
85	The "jump" is 20.1 to 30 centimeters from the ground.
80	The "jump" is 30.1 to 40 centimeters from the ground.
75	The "jump is more than 40 centimeters from the ground.
**	The "jump that is closest to the ground will receive an extra 10 points.

Scoring for *Not-So-Successful Jumps*:

85	A minor impact is made with the ground. (Small crack or can hear it touch.)
75	Impact with the ground would result in a fairly large crack or egg shattering.

Transformation Lab

Sketch the following pairs of functions on the same graph. Answer the questions following each group of graphs. Use a window of [-5,5] and [-5,5]. Your calculator should be in radian mode.

Investigation 1:

a) $f(x) = x^2$
$g(x) = f(x) + 3$

b) $f(x) = |x|$
$g(x) = f(x) - 4$

c) $f(x) = \cos(x)$
$g(x) = f(x) + 2$

Write a general statement explaining how the graph of $f(x)$ differs from the graph of $g(x)$, where $g(x) = f(x) \pm c, c > 0$.

Investigation 2:

a) $f(x) = |x|$
$g(x) = f(x+3)$

b) $f(x) = \sqrt{x}$
$g(x) = f(x-5.5)$

c) $f(x) = x^3$
$g(x) = f(x+2)$

Write a general statement explaining how the graph of $f(x)$ differs from the graph of $g(x)$, where $g(x) = f(x \pm c), c > 0$.

Investigation 3:

a) $f(x) = \text{int}(x)$
$g(x) = .5f(x)$

b) $f(x) = \cos(x)$
$g(x) = 3f(x)$

c) $f(x) = \sqrt{x}$
$g(x) = 2f(x)$

Write a general statement explaining how the graph of $f(x)$ differs from the graph of $g(x)$, where $g(x) = cf(x), c > 0$. Be sure to give a complete explanation.

Investigation 4:

a) $f(x) = x^2$
$g(x) = f(3x)$

b) $f(x) = \text{int}(x)$
$g(x) = f(.5x)$

c) $f(x) = \sin(x)$
$g(x) = f(2x)$

Write a general statement explaining how the graph of $f(x)$ differs from the graph of $g(x)$, where $g(x) = f(cx), c > 0$. Be sure to give a complete explanation.

Investigation 5:

a) $f(x) = x^2$
$g(x) = -f(x)$

b) $f(x) = |x|$
$g(x) = -f(x)$

c) $f(x) = \cos(x)$
$g(x) = -f(x)$

Write a general statement explaining how the graph of $f(x)$ differs from the graph of $g(x)$, where $g(x) = -f(x)$.

Investigation 6:

Rewrite each function as a multiple transformation of its related toolkit function using the same notation as above. Explain the transformations that took place. Graph each function to check your answer.

a) $g(x) = \sqrt{x-5} + 7$

b) $g(x) = (x+1)^2 - 3$

c) $g(x) = 3\sin(2x)$

c) $g(x) = -3|x-4|$

Volume Lab

You are given a piece of cardboard 20” by 32”. With the cardboard you are to make the largest possible box by cutting congruent squares out of each corner and folding up the sides. See diagram to the right.

1. Write a function V(x) (in factored form) which would represent the volume of the box.
2. What does x represent? What does V(x) represent?
3. Use the graphing capabilities of your calculator to sketch V(x). Include in your sketch all important characteristics including intercepts.
4. V(x) is a cubic function and therefore all real numbers are algebraically legitimate inputs and outputs. V(x), however, models a physical phenomena where they may be limitations on the domain and range of the function. Determine the apparent domain and range for the physical phenomena modeled by V(x).