# A Case for CAS

Josef Böhm & Guido Herweyers
T$^3$-Austria & T$^3$-Belgium Flemish


In my function as T$^3$-coordinator for Austria I received a mail in spring 2002. The sender was Hubert Colombat, the responsible person for T$^3$-Europe.

For the few who don´t know about T$^3$ = TTT: Teachers Teaching with Technology is a TI-sponsored activity to promote introducing technolgy in mathematics education.

-----Ursprüngliche Nachricht-----
**Von:** Colombat, Hubert [mailto:h-colombat@ti.com]
**Gesendet:** Donnerstag, 25. April 2002 08:19
**An:** "Josef Boehm (E-mail)" <
**Betreff:** Interest group on CAS - IMPORTANT

Dear Josef,

We really want to support this idea of " interest groups " but assuming there is a detailed project with clear objectives behind it. The project needs to gather a group of maybe 5 European educators from different countries; I will be able to support the organisation of a " kick-off 2-day meeting" with the group somewhere in Europe and then to give the required support on a technical standpoint if needed. I have in mind five subjects including CAS and I want to progress on this !

Could you tell me if you have some interest on this and in case of "yes", if you can work on a proposal within the next two weeks. Thank you.

Best regards, Hubert

Ian Forbes from Scotland contacted me, we worked out a common project proposal on – generally spoken – CAS.

What is CAS, that´s the question??

What can we connect with the three letters **C A S**?



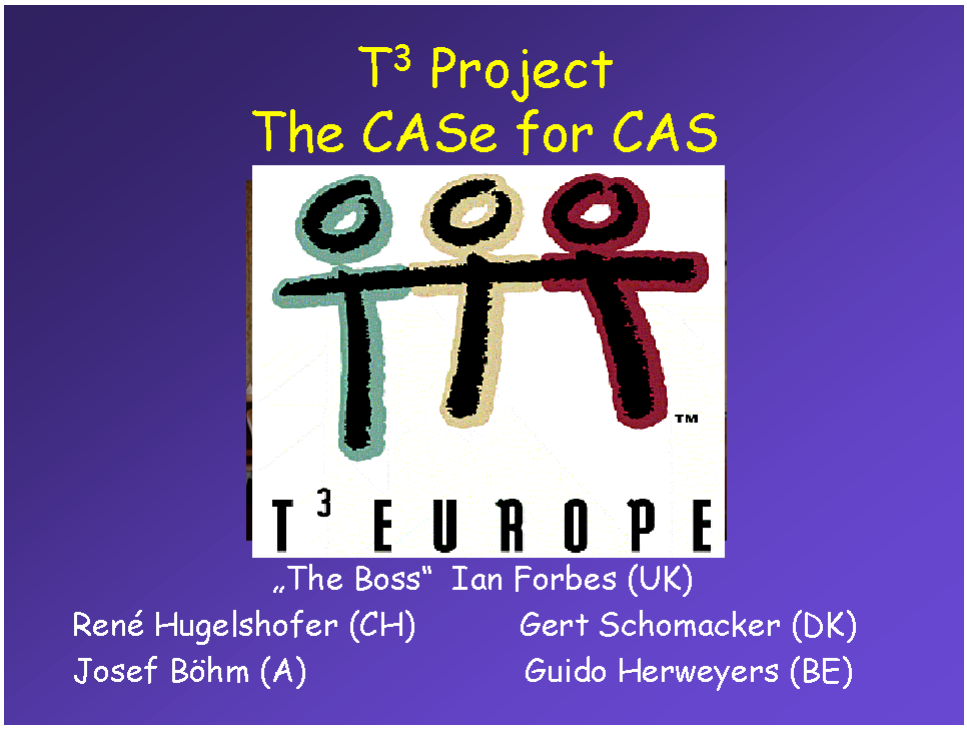

For some us it may be – see above!

Technology enthusiasts and propagators will agree with this explanation



This seems to be more approriate

And this should be our goal (including the teachers




Since 2002, many of the $T^3$ coordinators have been working on collaborative projects such as $T^3$ online courses and TI InterActive!. So why not a collaborative CAS Group? CAS presents some interesting challenges to the teaching, learning and assessment of mathematics, and indeed is banned in the national examinations in some countries such as Scotland and parts of Germany. If teaching and learning with CAS is to be promoted, then it has to be embedded in the 'high stakes' assessment, which is at the end of a course of study. How can a country be persuaded to release a ban on and adopt CAS? How can assessment be adapted to take into account the challenges of CAS? The paper aims to provide persuasive arguments for the former question and provide full and illustrative examples of the latter.

You can see the members of the group during a hard working session at a nice place in Brussels (thank you Hubert for paying).
The members from left to right are.Ian Forbes (Scotland), René Hugelshofer (Switzerland), Gert Schomacker (Denmark), Josef Böhm (Austria) and Guido Herweyers (Belgium). Unfortunately there were no women, but Lynda Ball from Australia did the important proof reading job.
It´s a pity that we must miss my co-speaker Guido. He is not in the best physical shape in the moment. He worked hard in the group and we all hope that he will recover as soon as possible.


There are many arguments for and again the use of CAS. The project sets out to provide an argumentative case for the use of CAS.
The paper summarises the research evidence to support the case for the use of CAS ...

... with the experience of teachers from different countries that already use CAS in teaching learning and assessment.

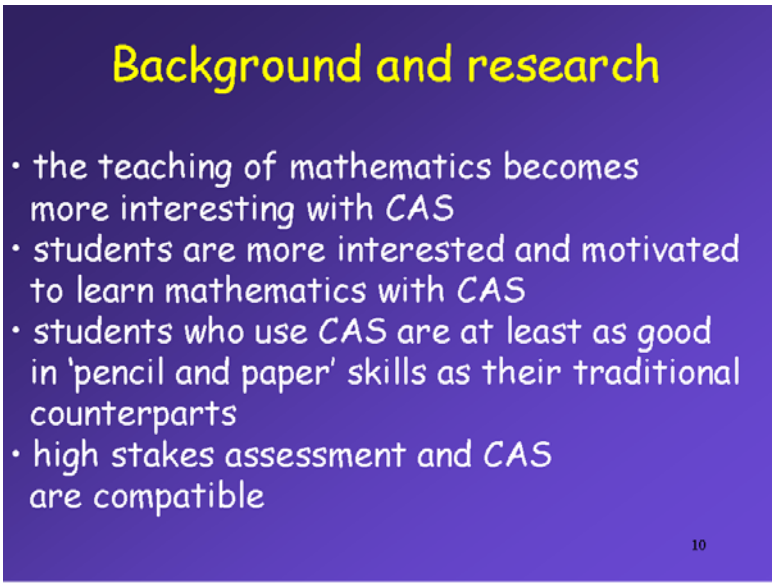
We propose a scheme for classifying our questions with respect to the role a Cas plays in the solution.

This is followed by lesson ideas and teaching sessions ...

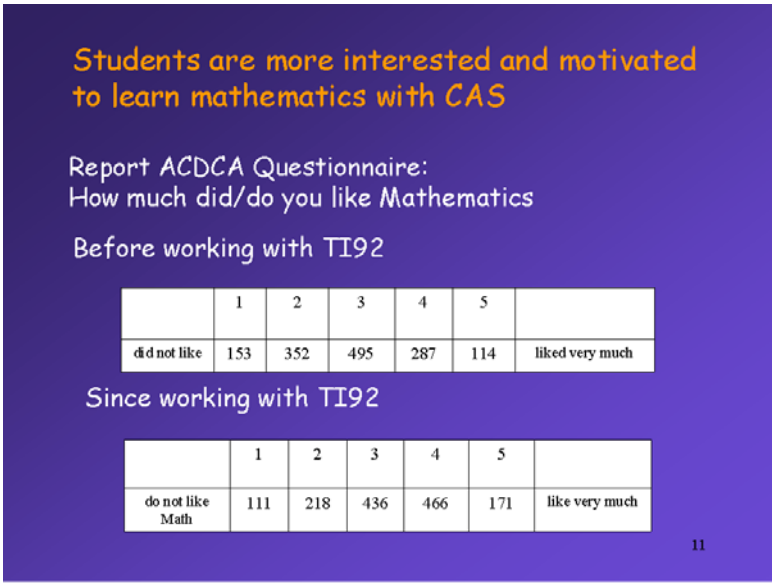
... and the adaptations which have to be made to question types in order to assess suitably

The final section of the paper outlines the arguments for and against the use of CAS ("the Devil's Advocate"), providing what we believe is a convincing case for the use of CAS in the teaching and learning of Mathematics.

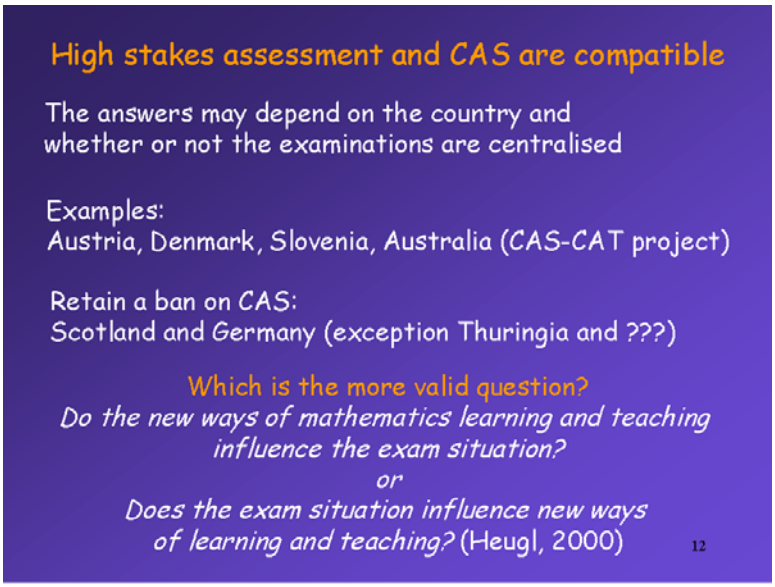
The research results are indicating four key issues:



There are a lot of studies supporting this point of view:
ACME, Scotland
USA
Thuringia (Germany)
ACDCA (Austria)




This is part of one of the ACDCA-reports comparing the students´ attitude towards mathematics before and after working with CAS.




Assessment examples will be presented.

There are changes in German federal states (Schleswig-Holstein, ...)

## Background and research

***On fluency with algebraic concepts/principles***

*Decide when it might be useful to use this concept/principle*

*Have the symbolic skills to do it correctly*

*Know what the significance of the end product is*

*Too much time has been spent traditionally on the 2nd of these and too little time on the first and last. (Kissane a.o. 1997)*

Students who use CAS are at least as good in PAP skills as their traditional counterparts.
(For many people traditional includes the use of a GC. There are two interesting studies published in the last issue of the "International Journal of Computer Algebra in Mathematics Education" on this issue.)
Kissane, Bradley and Kemp define flunency with a particular algebraic concept by three distinctive characteristics.

Helmut Heugl summarises the results of four nation wide ACDCA-projects:

## Background and research

***CAS-supported Math Education encourages the 4-key qualification***

- *Subject Competence*
- *Methodological Competence*
- *Social Competence*
- *Personal Competence*

*much better than the traditional math education.*

## Background and research

***The last word in this section goes to Australia:***

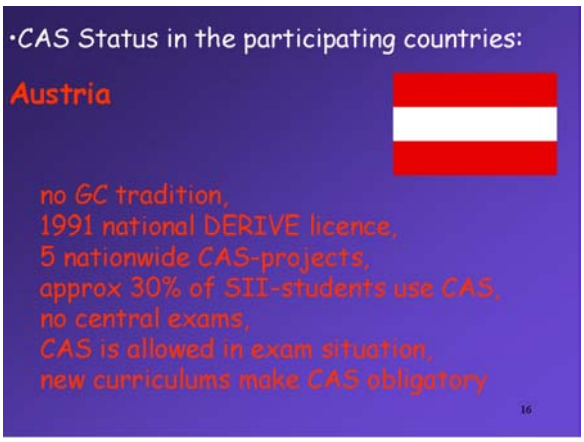
*What are the Future Goals for Mathematical Proficiency in the CAS – Age?*

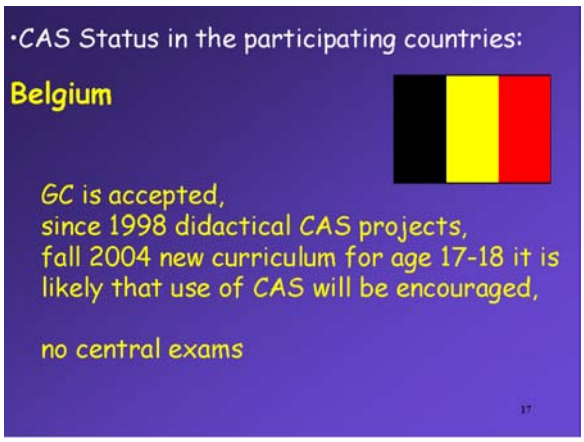
*Careful Consideration should be given to the Balance between by-hand and by-CAS procedures*

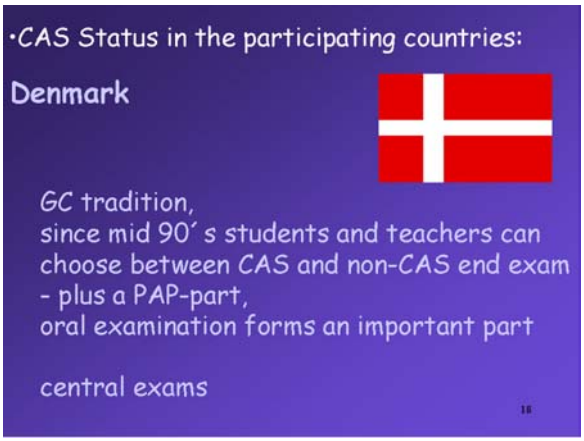
*What should Students record when Solving Problems with CAS?*

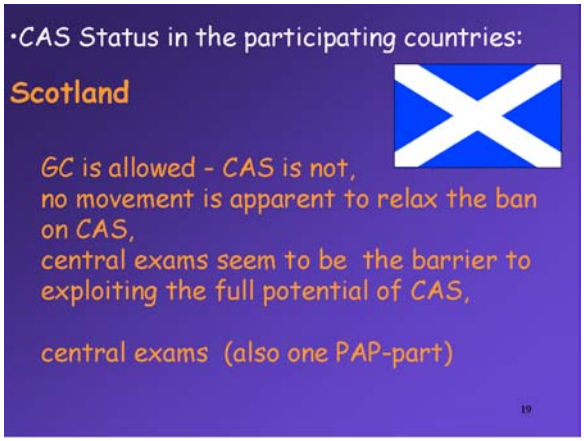
*.... a very good Understanding of Mathematics is required in order to use CAS to solve problems ...*

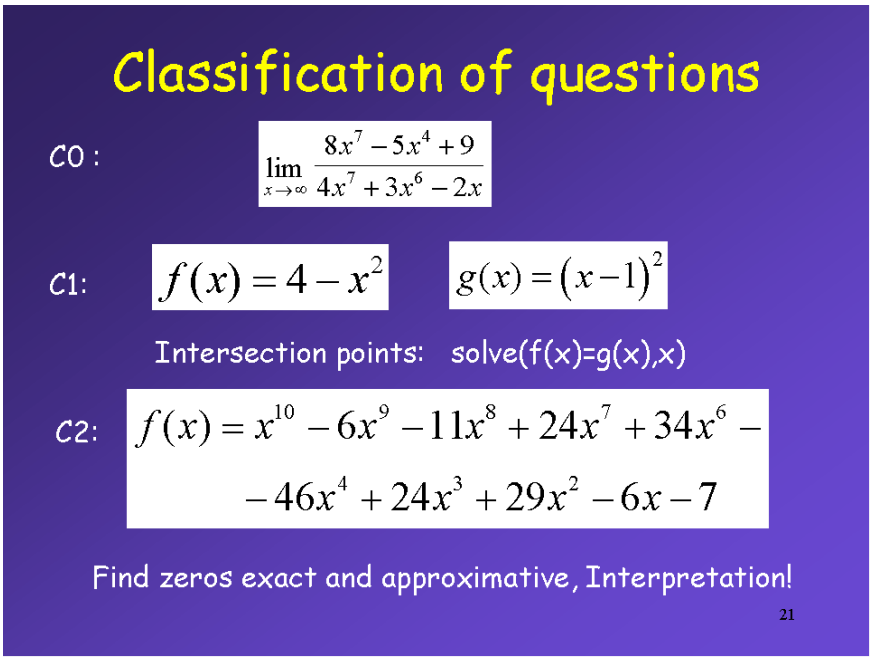
We compare the CAS-status in our home countries:



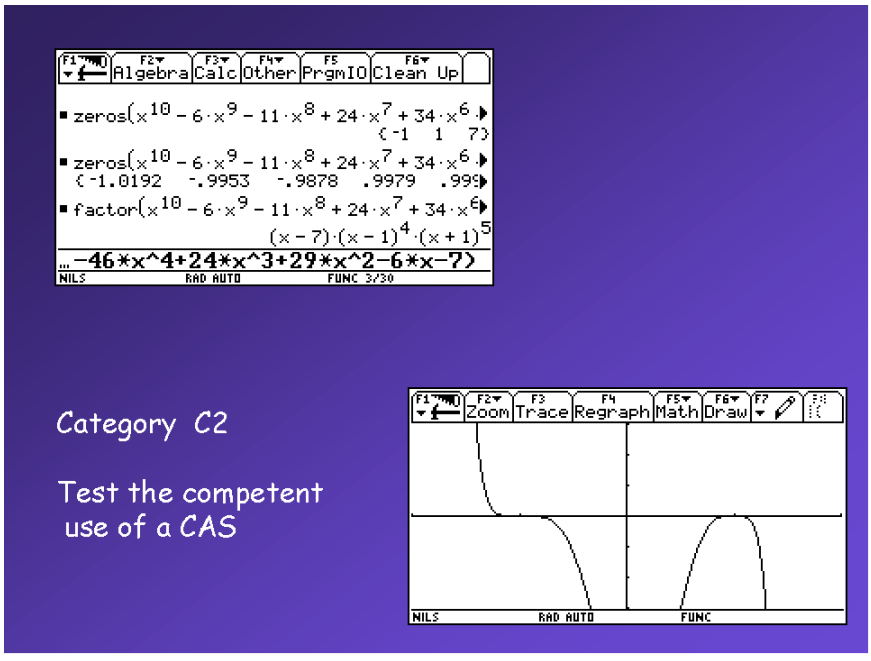




We distinguish five categories (Matia Lokar, Slovenia)

**C0** **Exercises where the use of CAS is of little or no help. More importantly, the typing would take more time than solving the problem by-hand (this of course depends on the students' skill).**

**C1** **Traditional exercises (by-hand or using scientific calculators) that are solved faster or even trivialised by CAS**

**C2** **Exercises that essentially test the ability of using CAS competently.**




Explain the different results for the zeros.
(In approx-mode the device works numerically and finds!! three zeros clos to –1 instead only one. The graph shows that we have a threefold zero. And for $x = 1$ we have a fourfold zero! The student must be able to interpret the results wrt to the MODE setting.

C3 : Introduce parameters into the example for C1

$$g(x) = (x-a)^2; \qquad f(x) = b - x^2$$

For which values of $a$ do graphs of $g(x)$ and $f(x)$ have just one intersection point?

What is the locus of all intersection points of families $g$ and $f$?

C4: CEBOTAREV Conjecture

Factor $x^n - 1$ for $n = 1,2,..,10$. All coefficients of the polynomial factors are $\pm 1$.

Is this true for all exponents $n$?

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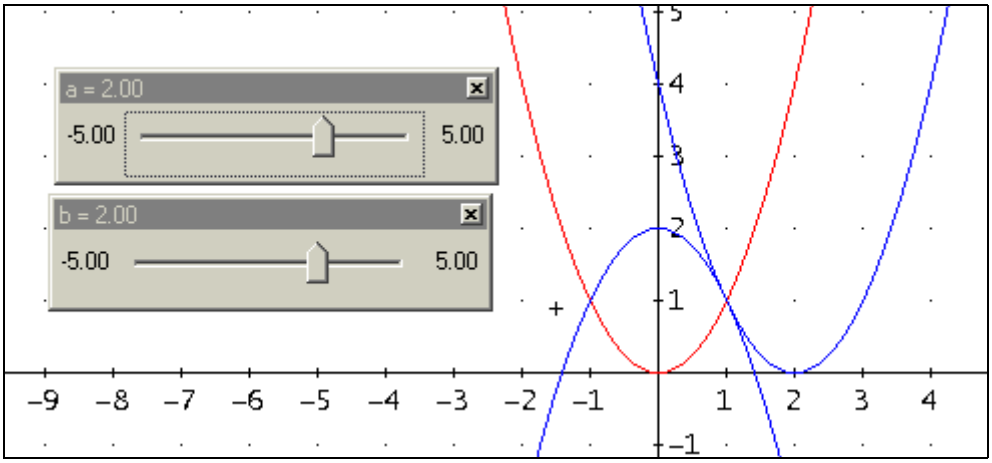
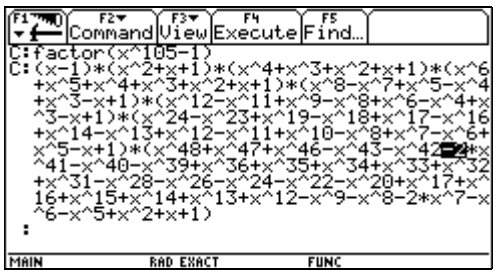
We distinguish five categories

**C3** **Exercises starting from traditional ones that are extended to CAS-exercises (e.g. by including formal parameters or using realistic data).**

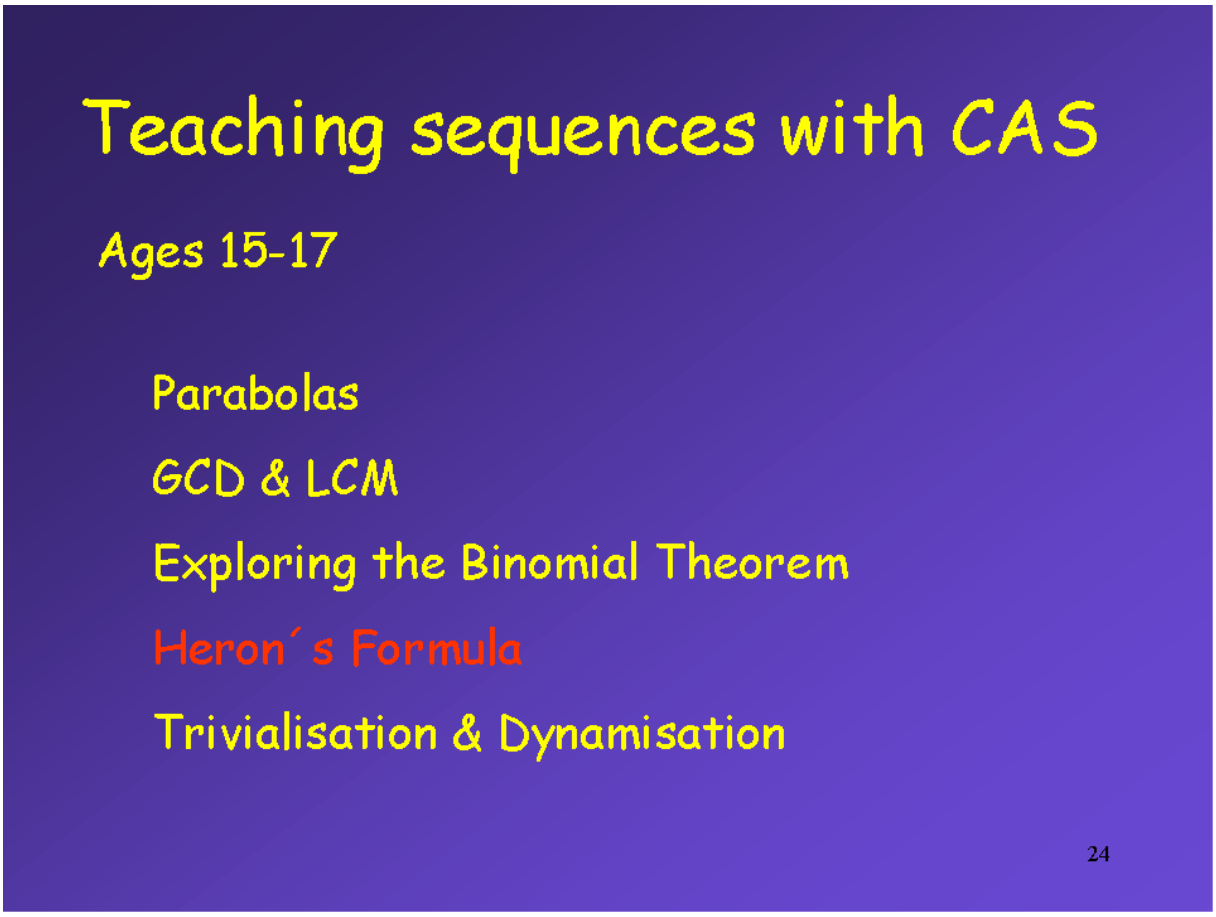
**C4** **Exercises that are difficult or time consuming to solve without CAS or those that can only be solved with the aid of CAS.**

This example for a C3 task is an opening of the C1 rated problem from above. We generalize the parabolas by introducing parameters. TI-InterActive! and Derive 6 provide nice visualizations.




It is true until $n = 104$!!




A major part of our product is a selection of teaching sequences which should underline our point of view, how the use of CAS can change math education in a very positive and inspiring way – without neglecting the "indispensable manual skills".

**Parabolas (Denmark)**

**GCD & LCM (Austria)**

**Exploring the B.T. (Austria)**

**Trivialisation. and Dynamisation. (Switzerland)**

**Heron´s Formula (Switzerland)**

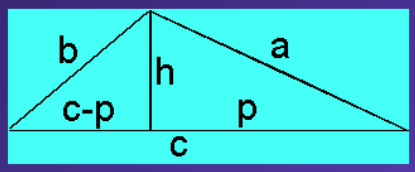
All examples were not designed just for this book, but originate from the authors´ experience as teachers.

We did not only present the teaching sequences, but also assessment examples which were connected to the teaching units.

You can also find student´s solutions.

Have a more detailed look to one sequence.

# Heron‘s Formula




$$\sqrt{s(s-a)(s-b)(s-c)} \mid s=\frac{a+b+c}{2} \rightarrow \text{heron}(a,b,c)$$

- Example of a functional object
- Calculate the side c of the triangle with the other sides a=7, b=5 and height h=2.

$$\text{solve}(\text{heron}(7,5,c)=\frac{c\cdot 2}{2},c)$$

$$\text{result}: c=\sqrt{6\cdot(11\pm\sqrt{105})}$$

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The formula of Heron is well known. René Hugelshofer tested this teaching sequence in the 9th school year.

The additional condition for the half circumference of the triangle can directly be inserted into the formula.

You can store the formula as a function with 3 variables!

**You can use this function as an object**

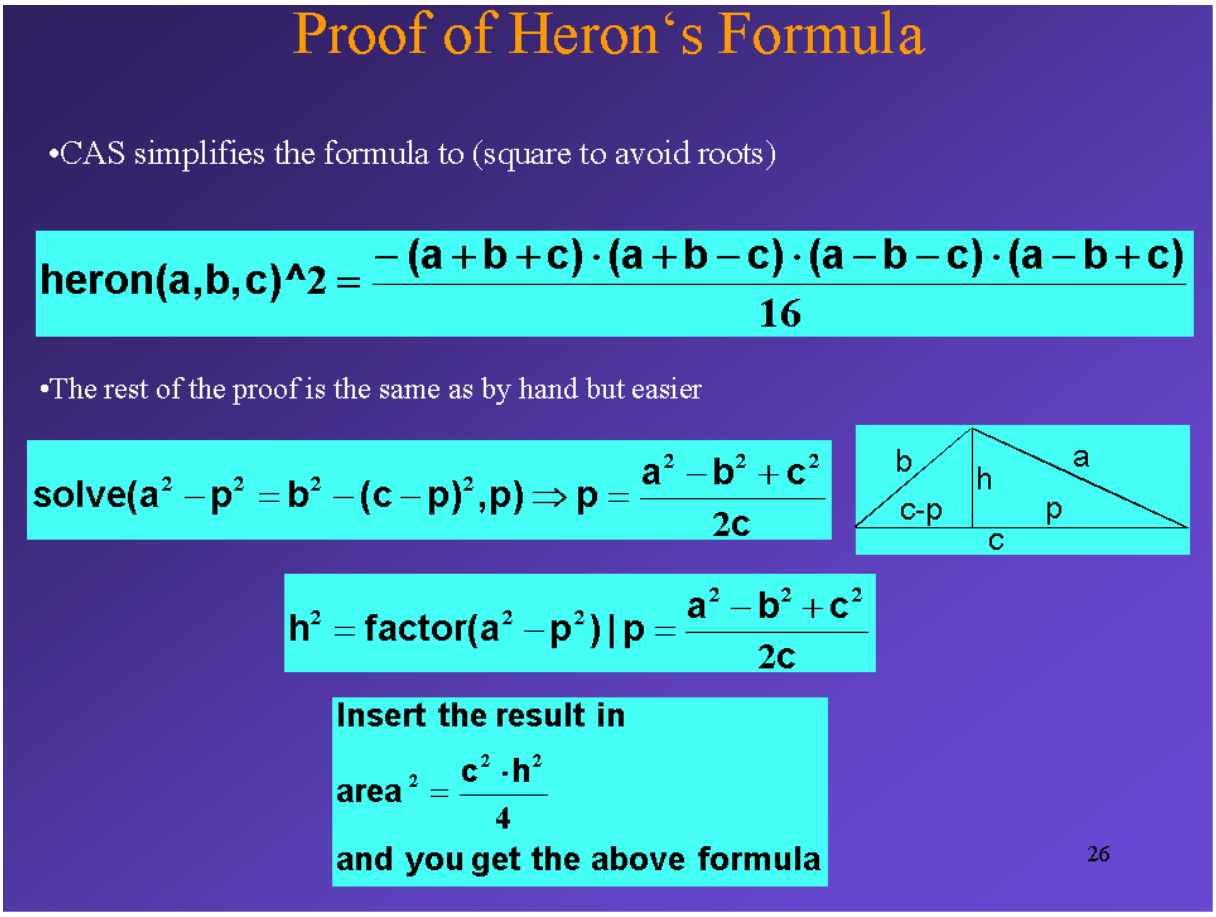
A simple example will show you the use of Heron as an object.

a, b, c have the role of placeholders in a term. As c is not given it takes the role of an unknown variable.

This problem is **trivialised with CAS** – but the result shows that by hand it would be a difficult exercise (biquadratic equation of order 4)!

**Using functions as a module is a very efficient method by using CAS.**

But we can go on ...




**First let us proove Heron's formula with CAS**

An elementary algebraic proof of Herons formula is rather sophisticated and can hardly be done by students without much help. The following proof was done individually with the worksheet you will find on the CD or in the book.

If you recall the formula **heron(a,b,c)** (squared to avoid roots) you get the simplified formula with the additional condition inserted.

Now you can proceed as by hand: Express $h^2$ in two ways with the Pythagorean theorem and solve the equation with respect to p.

Factor the easier of the two expressions for h-square and insert p.

Insert this expression for $h^2$ in the usual area formula.

You will get exactly the above formula.

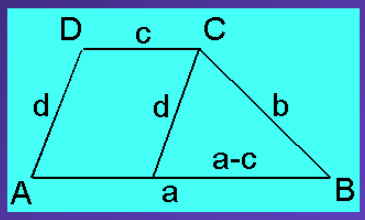
Students did this proof in the 9th school year.

## Exercise 1

define a function trapez(a,b,c,d)

```
solve(heron(abs(a-c),b,d)
=h*abs(a-c)/2,h)
```

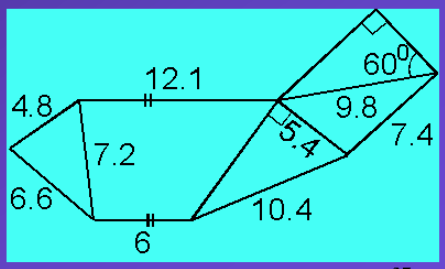
$$\frac{(a+c)\cdot h}{2}$$




## Exam question C1

•Consolidation of property:
A farmer can exchange his land for a quadratic one. What is the length of the square's side?

•Result: Just add the modules heron and trapez with the entries in the figure and take the root.

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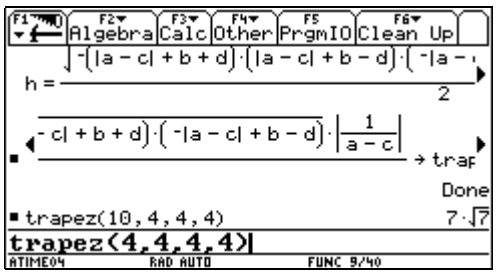
In a first exercise students have to construct their own functional object named trapez for the area of a trapezoid from given sides a, b, c, d.

The formula is rather complicated and can hardly be calculated by hand.




The formula seems to work – or not??

For the special trapezoid (4,4,4,4) the answer is undef. Can you give reasons for this strange behaviour?

In the exam students get a first question which requires these objects. But of course with some additional difficulties.

A farmer's land is divided into 5 pieces. In a consolidation of properties he can exchange it for a quadratic one. What is the length of the square's side?

Some sides are not given. The rest is the root of a sum of the objects heron and trapez.

Here again the parameters are used as placeholders!

## Exercise 2

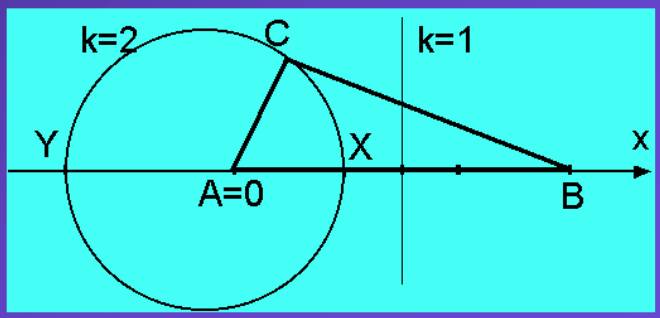
a) Does a triangle exist with sides 3, x, 2x and area A=10?
solve(heron(3,x,2x)=10,x) false

b) For which A does a solution exist?

With trial and error students find out:
0?A?3: 2 solutions
A=3: 1 solution
A>3: No solution

•Let students draw some possible vertices C.

•Geometric interpretation: Apollonian circle over side 3 with ratio 1:2




c) Generalisation:
Sides 3, x, k·x
ratio 1:k

a) In a further example we introduce parameters in Heron's formula. Consider a triangle with sides 3, x, 2x and area 10. This is not possible!

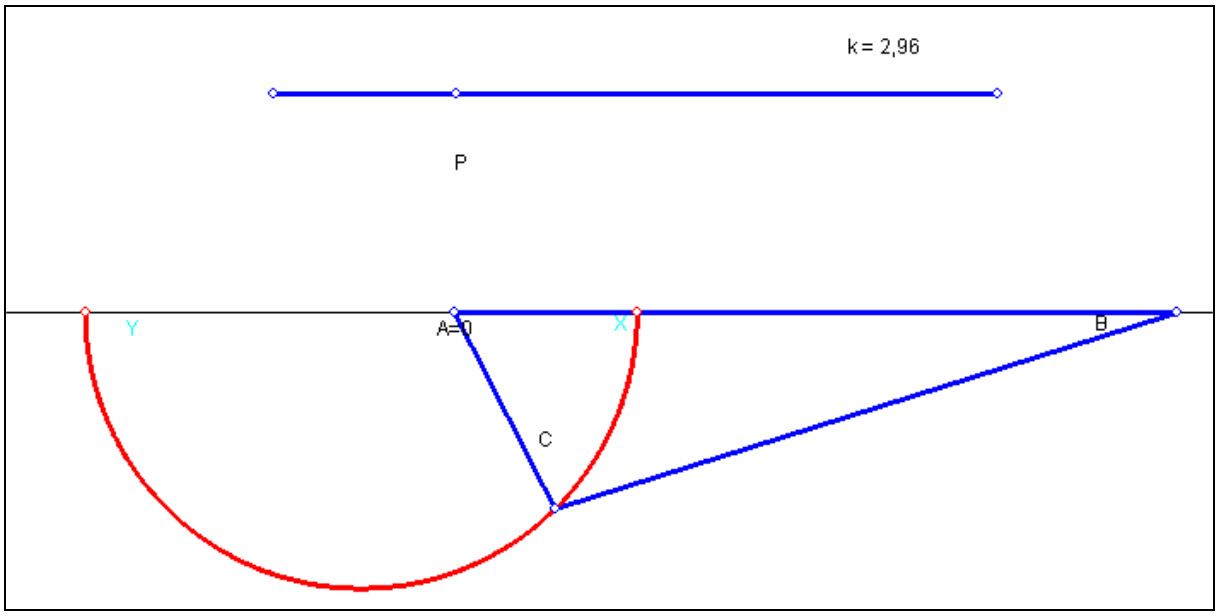
b) Of course the next question is: For which A does a solution exist? Let students find with trial and error possible areas.

Instead of calculating some triangles by hand (simple skill training) one example of this kind will bring more mathematics in your students mind.
Students can also draw the function heron(3,x,2x) to get the range of possible solutions or draw the possible vertices C and guess that they are lying on a circle.

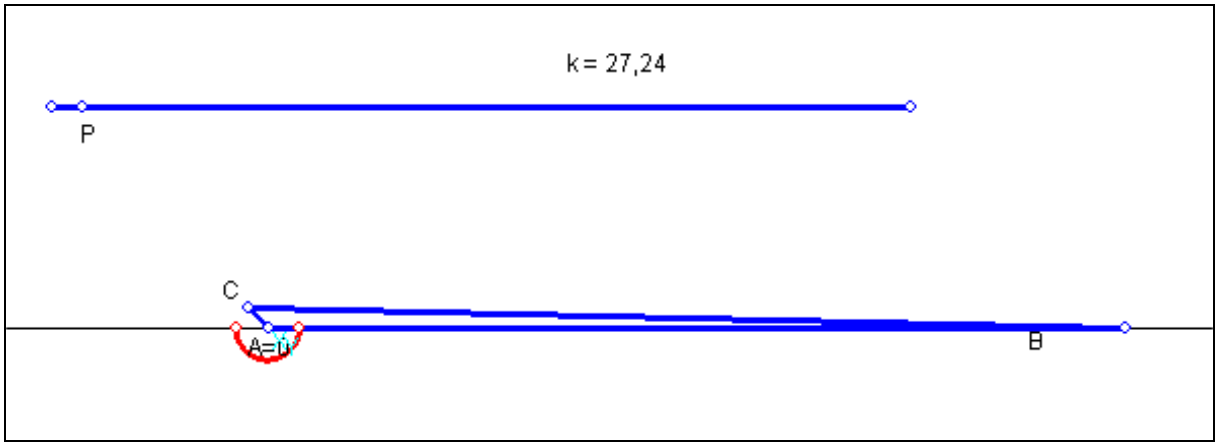
c) By parametrisation of the ratio you get new dynamic exercises in algebra similar (simlar) to dynamic geometry programs.
If you determine the points X and Y (heron()=0), you get the apollonian circle.

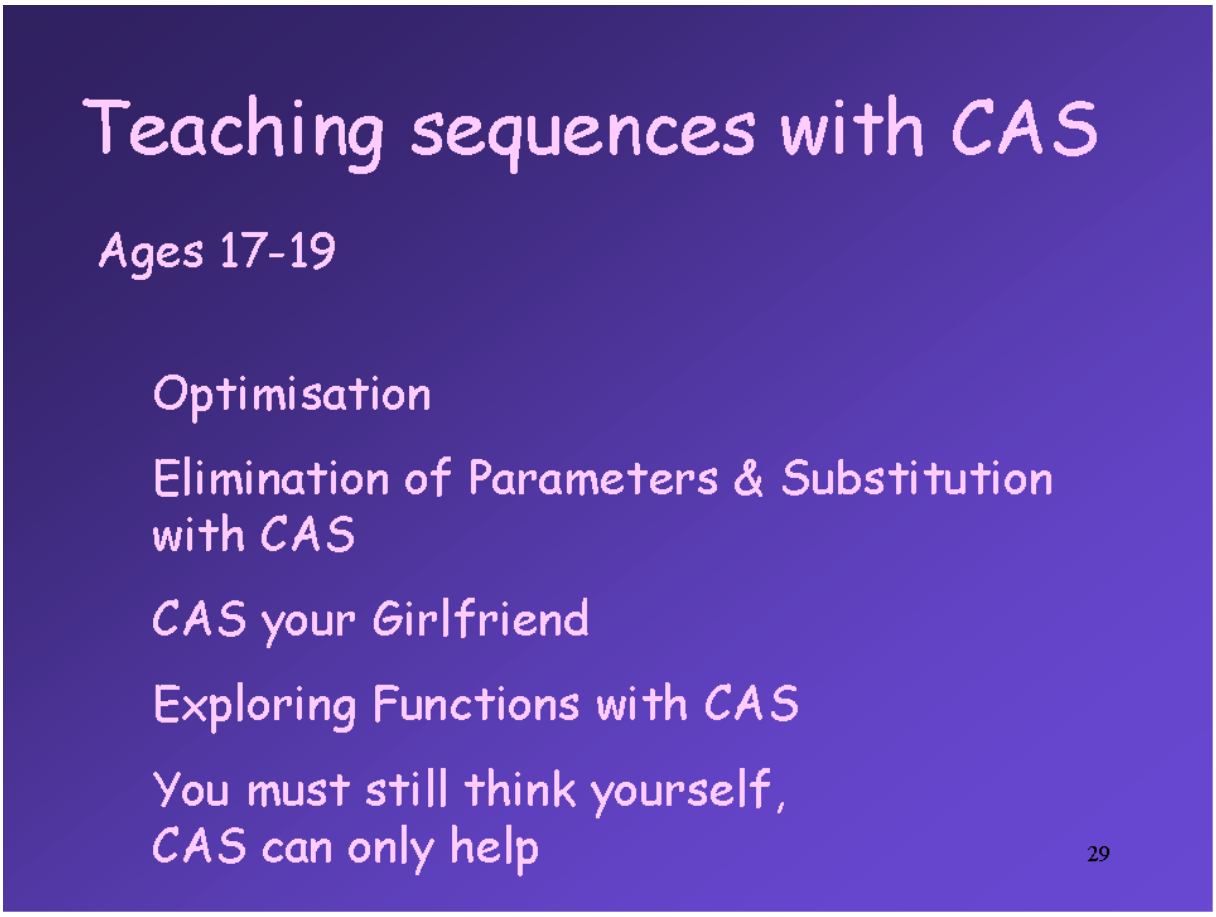
With dynamic geometry programs you visualise a process. Whereas by parametrisation with CAS you have to analyse the process and than interpret the result.

I will also visualise this process with Cabri.




Students will perceive it as an “infinite” broadening of their mind, if you take the limit k->¥. And if you come from the other side and pass through infinity, students’ minds will reach the limit of their imagination.Parametrising is another very powerful method using CAS.






We recommend to study our examples – and enjoy them.

**Optimisation (Denmark)**

**Elimination of Parameters ... (Belgium)**

**CAS your girlfriend (Denmark)**

**Exploring Functions with CAS (Belgium)**

**You must still think yourself ..... (Denmark)**

Just a short remark on the last unit:

Discussing this example during our meeting in Paris (dark cellar room) we – Gert and I – made some interesting additional findings – so we also were tempted into "thinking ourselves".

# One Guest Contribution

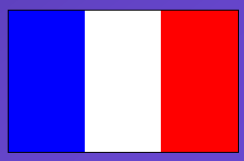
Take any three points A, B, C lying on an equilateral hyperbola and find the orthocenter O of triangle ABC.

What can you notice?

Try to proof this in general form!

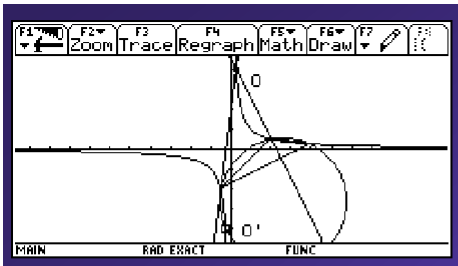
Draw the circumcircle of ABC and find point O′, lying symmetric to O wrt the origin.

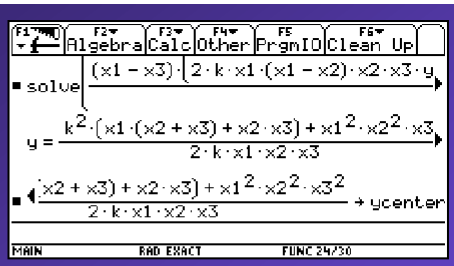
Do you have any conjecture? Prove it!!



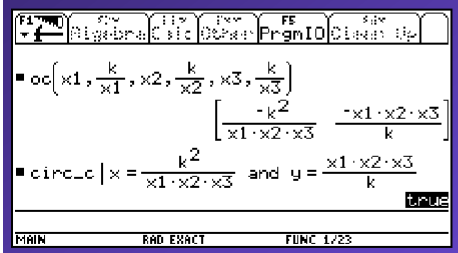
(*M. Gouy, G. Huvent & A. Ladureau*, France - Merci!!) 30

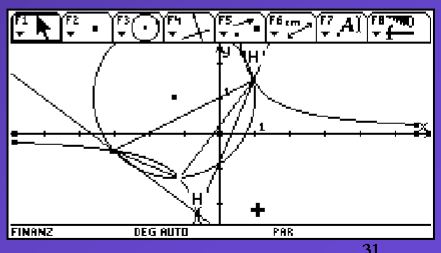
Hubert sent this contribution for possibly integrating it into our work. I contacted Michel and we produced an article for the DERIVE & CAS-TI Newsletter and finally included the CAS-part into our Case/CAS-book.
It is an excellent example for an appropriate use of CAS.




Dynamic Geometry meets CAS C3 - C4!!

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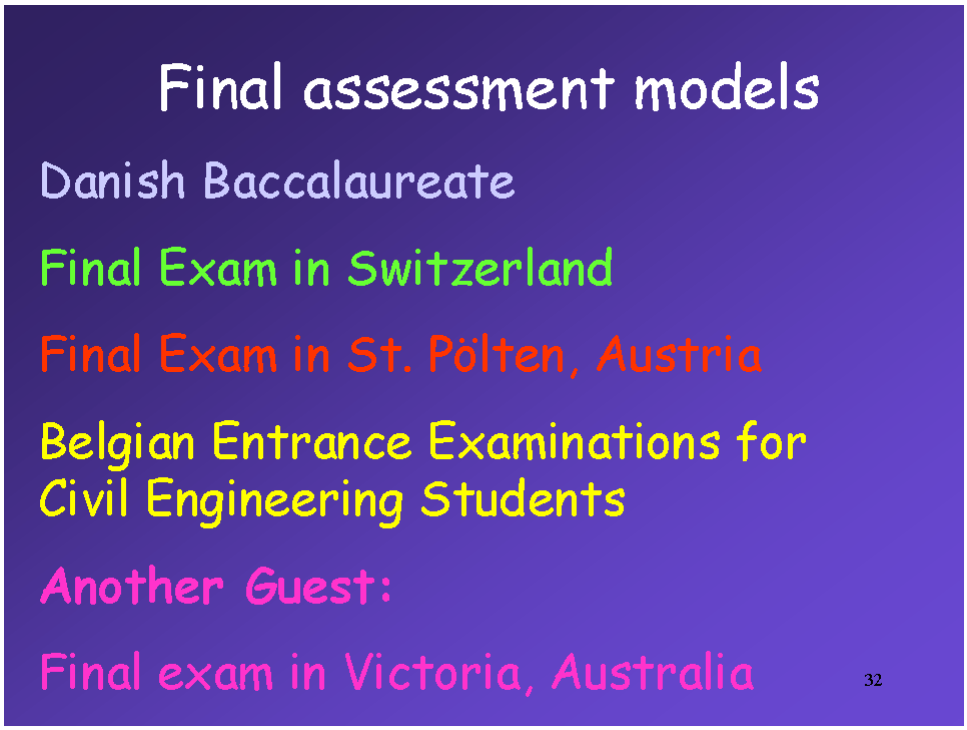
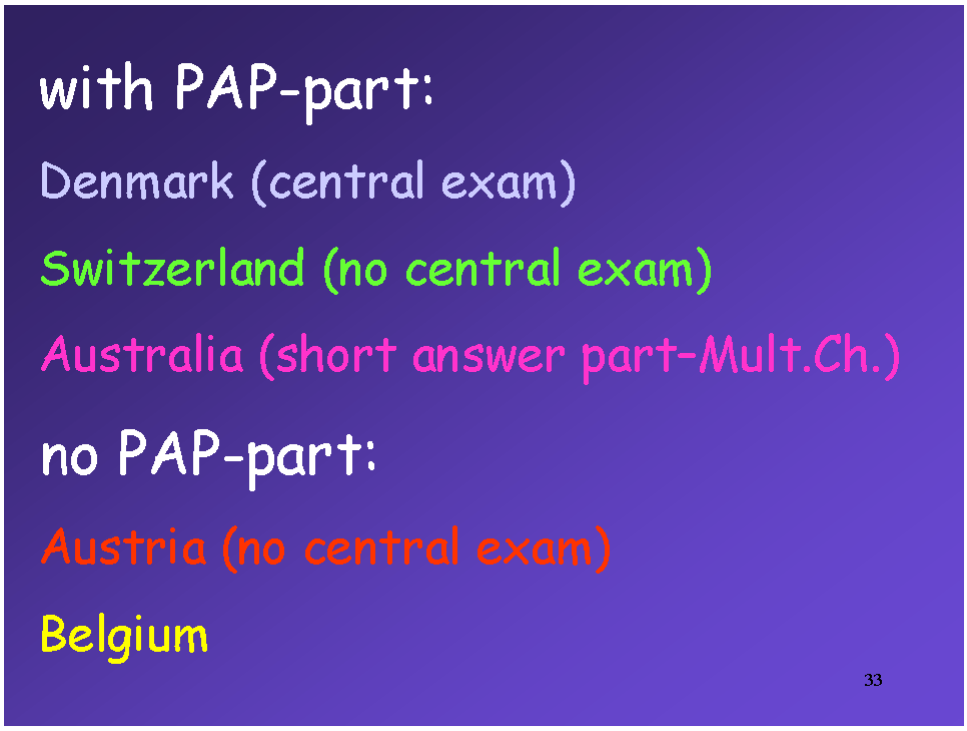
This is a fine example to combine the power of Dynamic Geometry with CAS.
The strategy for proving the conjecture is not difficult, but calculation is very time consuming, boring and offers so many (too many??) opportunities for making mistakes.

I´ll show the dynamic geometry realisation on the handheld TI. (Cabri is on the CD.)




"CAS and assessment, the never ending question – and discussion

Let´s compare the final exams in our five countries:




And allow to demonstrate some examples

I start with an example from Belgium (entrance examination)

## from Belgium

Consider the following system of linear equations in the real variables :

$$\begin{cases} a \cdot x + y + z = 1 \\ x + a \cdot y + z = a \\ x + y + a \cdot z = a^2 \end{cases}$$

For which value(s) of the real constant $a$ is there

a) No solution?

b) Exactly one solution? Find this solution.

c) An infinite number of solutions? Find the solutions.

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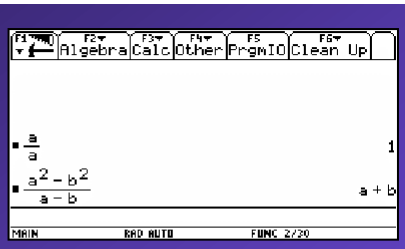
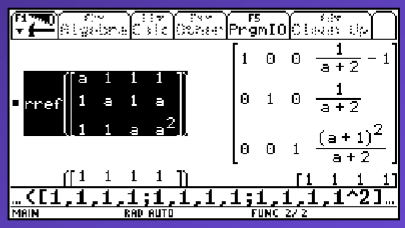
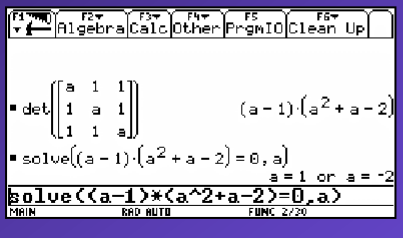
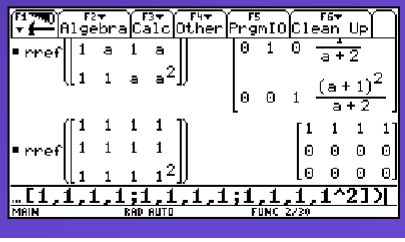
**Solution:**
First of all, be aware of the fact that a CAS cancels symbolic factors in the numerator and denominator, without mentioning that the symbolic factor may not be zero!
To detect the different cases, first calculate the zeros of the determinant of the matrix $A$ of the system.

**Category C3**




Allow me to remind you:

**C3: Traditional problems sre extended to CAS - problems**

## from Switzerland

**PAP-example:**

(3) $f_n(x) = n^2 \cdot x^3 - n \cdot x^2 + x \quad (n \in \mathrm{N})$

a) Show that all inflection points $W_n$ lie on a straight line.

b) Examine whether the tangents of inflection are parallel, coincide or intersect.

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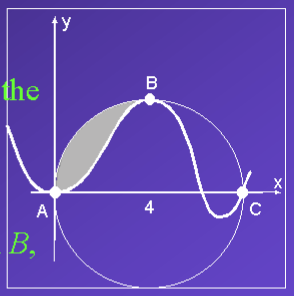
Traditional manipulation skills are required

## CAS-part Category C3

Find a polynomial $f$ with smallest possible degree which satisfies the following conditions:




The graph (which possibly does not correspond to the accompanying sketch)

- has a horizontal tangent in the origin,
- touches circle $K$ with radius 4 and centre (4|0) in $B$,
- has an inflection point in $C$,
- the shaded area amounts to 6 (integral between $A$ and $B$)

Outline the real curve as well as the circle

Choose a parameter $k$ (instead of 6) for the area. Can $k$ be arbitrary? Give reasons for your answer. Sketch the curve for $k = 0$.

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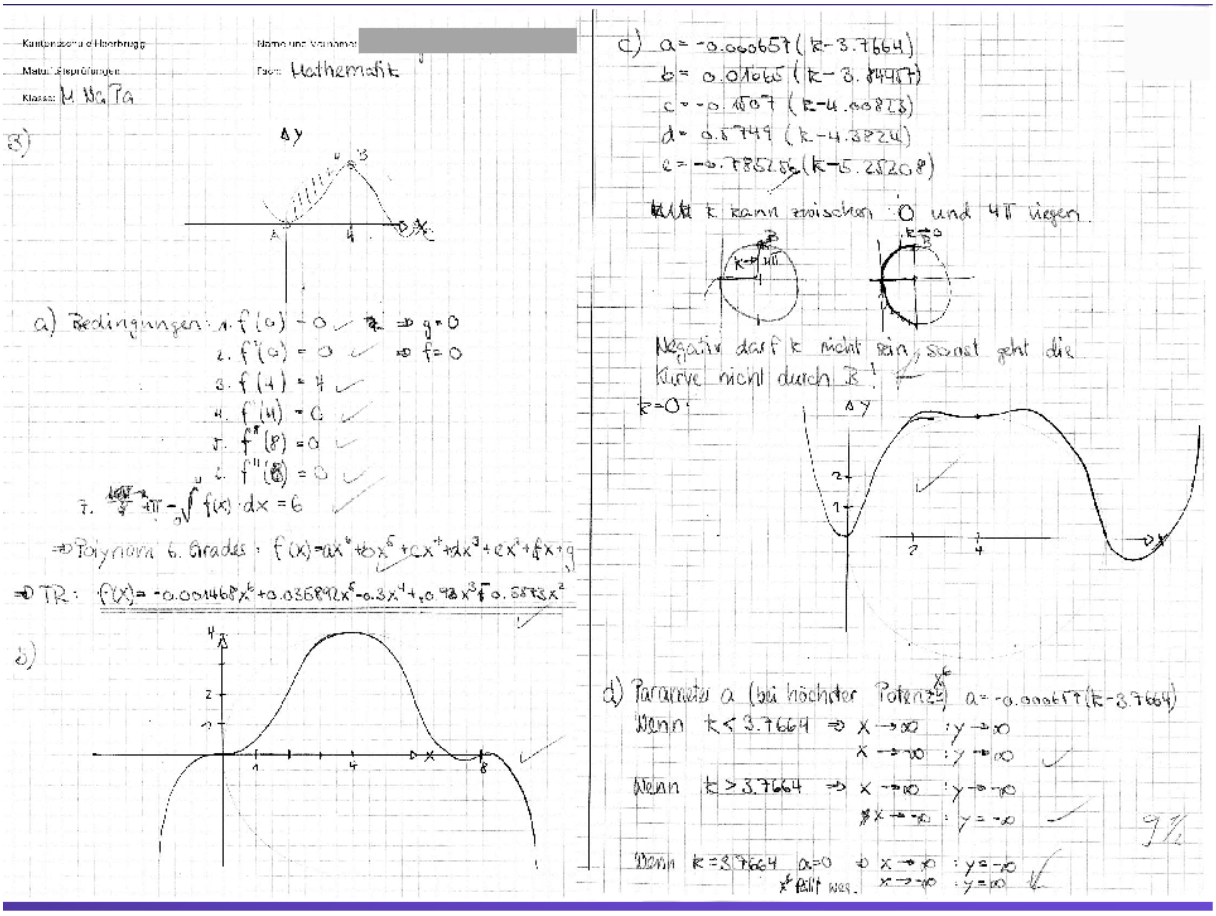
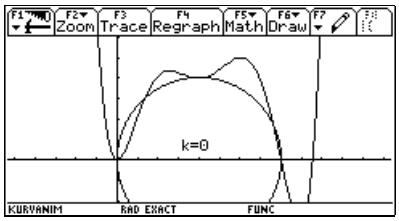
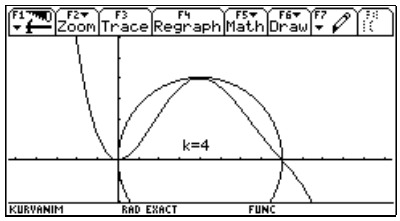
Discuss asymptotic behaviour for $x \to \pm\infty$ for all possible $k$.

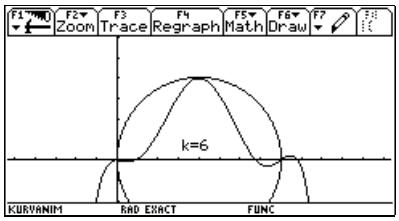
Raised on C3 level a traditional "inner mathematic" problem acquires some dynamic aspect. René provided a "dynamisation" on the TI.

On the next page you can find an original student solution and screenshots of the animation.




## from Denmark

PAP-example:

Calculate the limits:

$$\lim_{x\to 0}\frac{1-x^2}{4-2x} \quad \text{and} \quad \lim_{x\to 0}\frac{(2+3x)^2-4}{6x}$$

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**No additional comment is necessary!**

## non PAP-example:

**Question 3 (15%),** [Extra CAS-question in brackets]

**A line *l* is given by** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

**Find an equation for the plane which passes through line *l* and point *Q*(4|-2|5).**
**A sphere *K* is given by the equation**

$(x-4)^2+(y+2)^2+(z-5)^2=11$. **[$(x-a)^2+(y+2)^2+(z-5)^2=11$ with $a>2$]**

***K* and *l* have exactly one common point *P*.**

**Find the coordinates of *P*.** **[Find *a* and the coordinates of *P*]**

**Find the equation of the tangent plane of *K* in point *P*.**

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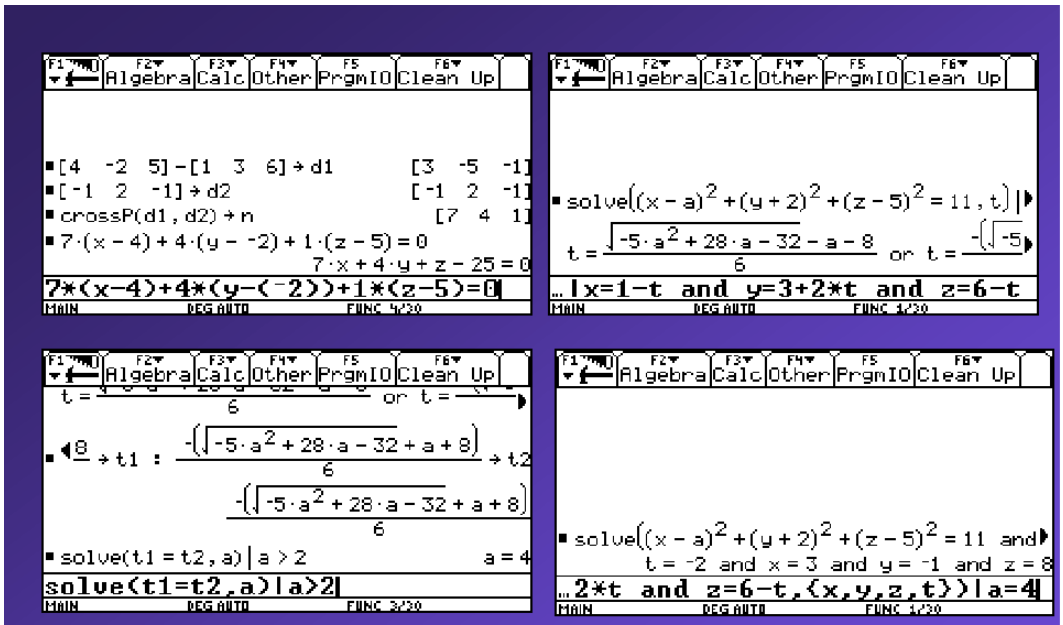
The CAS examples don´t differ too much from the non-CAS. They are only a bit „heavier“.

This is not my personal favour how to set CAS-supported problems, but it might be a starting point.




Computer Algebra at its best

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Many thanks to Lynda Ball and Hugh Leigh-Lancaster for their permissiion to include this very informative section.
In the book you can find the complete assessment (CAS vs non CAS with credits).

## from Australia (Victoria-CAS-Project)

### Part of question 2

**Non-CAS-Assessment**

**... 20% of the yellow butterflies are Jojos and the other 80% are Fhaisis ....**

**Find the probability, correct to three decimal places that a random sample of 10 yellow butterflies from this region will contain exactly 4 Jojo butterflies.**

**(2 marks)**

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## Part of question 2

**CAS-Assessment**

**Let $X$ be the random variable with values equal to the distanc in metres of Fhaisi butterfly from an old tree.**

**The probabilitydensity function of $X$ is**

$$f(X)=\begin{cases}\dfrac{2x}{a^2} & 0\le x\le a,\ a \text{ is a constant}\\ 0 & \text{otherwise}\end{cases}$$

**i) It is found that the mean distance of a butterfly from the old tree is 150m. Show that the value of $a$ is 225.**

**ii) Find the probability, correct to three decimal places, of a Fhaisi butterfly being within 200m of the old tree**

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**(4 marks)**

Before closing the presentation with possibly the most important part we would like to express our gratefulness to some people who are responsible in a high degree for this project:

Lynda Ball (proof reading + adding valuable comments)

Koen Stulens (doing all the final layout)

Andrea Forbes (designing the cover)

T3 Germany (publishing the book)

Our institutions (leaving us joining the meetings)

Our families (many hours absence physically and mentally)

Sylviane Trila-Sertori (organizing wonderful venues, ..... )

Hubert Colombat (making all this possible)

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and special thanks to René for his support in the last days.

## Argumentative discussion

a) Is there not a danger that we may repeat with CAS the errors we made in taking for granted the use of four function calculators – but on another level?

b) What evidence is available which proves the students do not become over reliant on CAS to the detriment of their basic skills?

c) Enthusiasts identify the benefits and possibilities of CAS, but what are the dangers or drawbacks?

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Questions like these were around us and in us during all our meetings.
It was important for us to justify our pro CAS arguments – not only out of our belly but based on real data, studies, etc.

## Advocatus Diaboli

d) What is about the gap between gifted and w students? Will mathematics be made to seem difficult and thus discourage students?

e) Where is the evidence to show that students ARE more motivated when using CAS (as opposed to graphing calculators)?

f) What can be done with CAS that cannot be done with a graphing calculator?

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The questions were posed by the Advocatus Diaboli". This was – who else – our boss, Ian.

The co-authors of this book received these questions from the Advocatus Diaboli (= Boss = Ian Forbes).

You are friendly invited to read their answers

and finally agree with them saying:

This is really a

**CASe for CAS**

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