Using a CAS as a Pedagogical Tool in a US High School Regular Level Precalculus Course

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INTRODUCTION

In the United States, a computer algebra system (CAS) is a new form of technology being used as a pedagogical tool in school mathematics, primarily with students between the ages of 12 and18 years. Few American teachers are experimenting with a CAS perhaps due to its comparatively high price tag, sometimes novel syntax, time-demanding learning curve, and perceived revolutionary threat to the current mathematics curriculum. Those US teachers using a CAS have learned a great deal from their counterparts in other countries who have launched into a CAS as a pedagogical tool much earlier and, currently, more extensively.

CAS-enabled calculators are presently acceptable on the calculator-active portions of the AP Calculus exams, and thus CASs are primarily used in calculus courses. It seems that most of what is done is merely use the symbol manipulating calculator to perform some of the calculus skill work. We at Glenbrook South High School (located in a suburb north of Chicago) wanted to investigate how a CAS might be used to develop mathematical concepts and help students use this technology to learn mathematics at the precalculus level. Thus, five years ago we began using TI-92 calculators in a regular level precalculus course taken by high school juniors ages 16 and 17 years. We did not substantially alter the curriculum nor did we attempt to affect each topic. Even so, we encountered several issues that arose as a result of using this new tool minimally. This paper will share our experiences with a CAS regarding basic implementation issues, content development, student reactions, and classroom observations.

SOME BASIC CONSIDERATIONS FOR IMPLEMENTATION

♦ TYPE OF CAS

We chose to use the Texas Instruments 92 calculator for a variety of reasons. Some were practical while others were more pedogogical.

- **It was difficult to schedule classes in the already crowded computer labs.**
- **Students already were familiar with the TI-92 from previous use in geometry classes.**
- **The QWERTY keypad made it easier than other models to input algebraic symbols.**
- **The TI-92 had a bigger display than other models, thus making it easier to read quickly especially while walking around the room.**
- **Each calculator could be used as a display model, readily showing student work.**

- **KEYS and COMMANDS**

We did not want students to use a CAS without thinking but rather for students to learn more mathematics and to understand the mathematics in a more comprehensive way. We wanted students to think and reason about the symbols that were entered and the results displayed on the screen. Thus, we decided to do the following:

- **Avoid the use of the Algebra menu on the TI-92 and type in the commands letter by letter.**

  The F2 menu contained some of the more "powerful" features of a CAS such as the SOLVE command. Since students did not have constant access to the CAS, they did not have time to explore the menus.

- **Provide clear instructions on how to use a CAS**. For example, putting the calculator in the exact versus approximate mode eliminated decimals and forced radical notation.

- **Keep the introduction of new commands to a minimum in any one lesson.** For instance, the EXPAND command was used rather than the PROPFRAC command when simplifying the division of polynomials.

- **Use a CAS at least twice in a chapter for students to better retain key strokes and outputs**.

- **CAS ACCESS**

Primarily because of the price, having access to a CAS whenever desirable was not always possible. Students had already purchased TI-83 calculators, and our curriculum was not CAS-ready to require the purchase of another calculator. So several factors needed to be considered.

- **Use a CAS only in the classroom with calculators owned by the mathematics department.**

- **Within the lesson development activity, most of the time there were problems to be done both with and without a CAS.** (See Example 1 on log properties.)

- **All of the CAS investigations and problem work needed to be done in class.**

- **A CAS was used to introduce the mathematical concepts.** We had no time to teach any topic twice – using traditional methods and then with the CAS technology.

- **ASSESSMENT**

- **We allowed a CAS on any test or quiz for lessons that included CAS activities**.

- **We did not alter questions substantially because many of the skills (like long division of polynomials) we did not expect students to do by hand and rather with a CAS instead**. It was too much at one time for us to write both CAS lesson development and major changes in assessments.

CONTENT DEVELOPMENT

Even though a CAS is often used merely to relieve the drudgery of long or tedious computations, we wanted to explore the potential of using a CAS to help develop understanding of mathematical ideas and theorems. The following are the areas where we incorporated CAS:

- **Making conjectures from patterns:** discovering logarithmic properties, investigating the Binomial Theorem, exploring the Division and Remainder Theorems, and factoring sums and differences of powers.
- **Equivalence transformations:** solving nth root equations, solving exponential equations, deriving inverse functions, finding parts of triangles using the distance formula, proving trigonometric identities
- **Multiple Representations:** exploring odd and even functions, finding composite and inverse functions, and building connections with the Factor Theorem.

In the following examples of how CAS is used in each of these areas, I will also highlight the benefits of a CAS.

♦ PATTERNING

Perhaps the most typical use of a CAS is for students to generate patterns and use induction to generalize a property. We employed this technique for a variety of topics including the properties of logarithms. A portion of that lesson development is shown below in Example 1. Using a CAS enabled students to “discover” the properties, provided us with more time for reflection statements, and generated class discussions because a restricted domain was necessary for a CAS to simplify logarithmic expressions with variables. However, we did not figure out why the domain of only one variable needed to be specified.(See Figure 1)

Part 1: The sum of logarithms
Simplify each expression using a CAS

1) ln(2)+ln(3)	**2) ln(12)+ln(6)**
3) ln(5)+ln(7)	**4) ln(29)+ln(x)**
5) ln(y)+ln(15)	**6) ln(2x)+ln(13)**

Try to simplify each of the expressions WITHOUT a CAS

8) ln(x)+ln(y)	**9) ln(2x)+ln(3y)**
10) ln(x)+ln(y)+ln(z)	**11) ln(4x+ln(5y)+ln(2z)**

Example 1

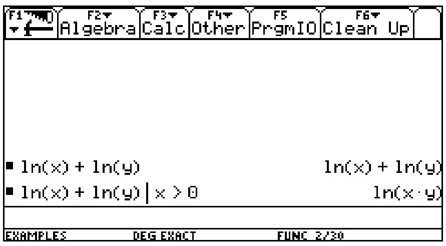


Figure 1

While using a CAS to develop the concept on the binomial theorem, we were able to write examples with higher exponents because the CAS reduced the time-consuming “messy” algebra. (See #6 in Example 2 below)

A. EXPAND each of the following:

1. $(x + y)^1 =$
2. $(x + y)^2 =$
3. $(x + y)^3 =$
…
6. $(x + y)^{10} =$

B. Write only the coefficients of the terms of each expanded polynomial listed at the left.

1.
2.
3.
…
6.

Example 2

The minimal higher-order assessment that we did do was provoked by the above lesson. (See example below.)

E.g. Find the values for A, B, C, and D such that

$$(Ax - By)^4 = Cx^4 - 756x^3y + 2646x^2y^2 - 4116xy^3 + Dy^4$$

Example 3

Other concepts that we developed through patterning were the Division and Remainder Theorems. Using a chart like the one at the right, students observed relationships among the various components of a division problem.

Complete the table below.

dividend f(x)	**divisor**		**quotient q(x)**	**remainder**		**function value**
	d(x)	degree		r(x)	degree	
$3x^2 - 16x - 35$	$x - 7$	1	3x	0	0	f(7) = 0
$x^3 + 2x^2 - 14x - 3$	$x - 3$	1	x^2+5x+1	0	0	f(3) = 0
$x^2 + 3x + 7$ Expand or propfrac	$x + 2$	1	x+1	5	0	f(-2) = 5

Because the CAS required a command for the division of certain polynomials and then returned unfamiliar outputs, class discussions erupted. (See Figure 2 at the right.) Through further examples with non-linear divisors students induced the two theorems.

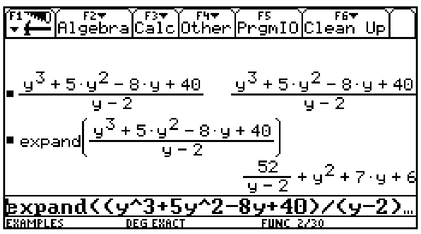
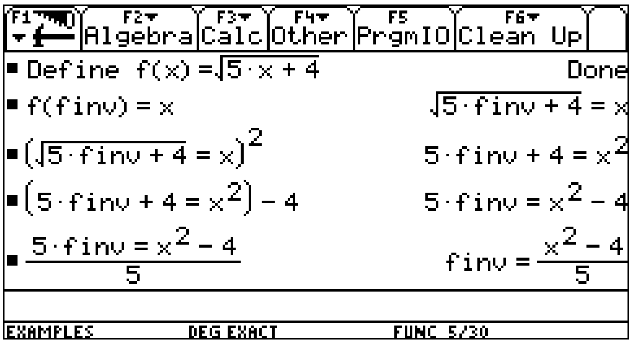


Figure 2

♦ EQUIVALENCE TRANSFORMATIONS

One of the equivalence transformation lessons involved finding the inverse function in a step-by-step procedure that helped make explicit the generation of the inverse function. Students used the concept that $f \circ f^{-1} = x$ and renamed f^{-1} as finv. (Refer to Example 5 below.) There were at least two advantages in using a CAS in this way. The use of CAS ensured correct algebraic manipulations so that weak algebra skills did not get in the way when teaching higher order concepts, such as when finding the inverse of a function. The second benefit was that it left a record of the computations involved in finding the inverse function. This allowed students to see the individual steps brought together into one function.



Example 5

While teaching the lesson on nth roots, we used a CAS to practice solving literal equations. Since these students are in the regular level precalculus, their algebraic skills were not proficient. Thus, once again weak algebraic skills did not hinder progress in solving these equations. (Refer to Example 6 and Figure 3) Some students even used the “such that” command (an important qualifier in proof) to eliminate the absolute value in the final answer.

E.g. Solve for r.

$$f = \frac{GmM}{r^2}$$

Example 6

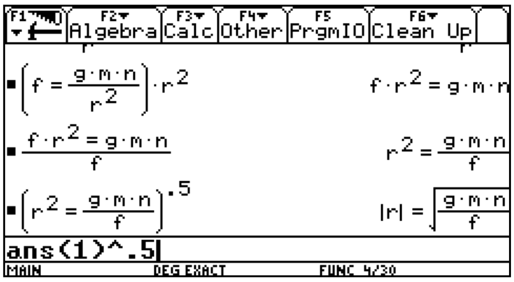
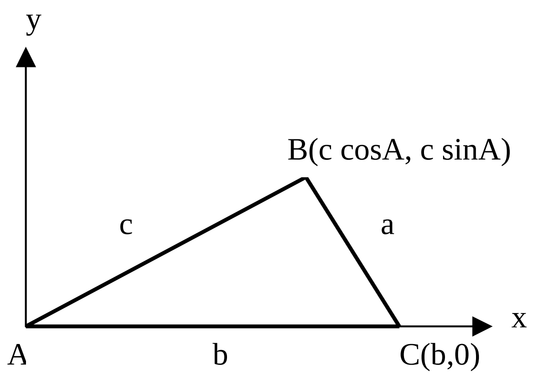


Figure 3

Even though Example 7 below does not truly exhibit equivalence transformations, it does involve using the SOLVE command to find missing parts of triangles without the Laws of Sines and Cosines per se. Here is where using a CAS enabled students to use previous knowledge (the Pythagorean Theorem and the distance formula) and apply it to a new situation instead of just memorizing a new formula. It also gave the students an opportunity to actually derive the Law of Cosines formula.

Solving Triangles with the Distance Formula

We can use a CAS to solve triangles given three pieces of information for the triangle as shown below, place the triangle in a coordinate system, and use the distance formula.



Recall that the distance formula for 2 points D (x_1, y_1) and E (x_2, y_2) is :

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = DE$$

Therefore:

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = DE^2$$

In terms of our diagram,

$(c \cos A - b)^2 + (c \sin A - 0)^2 = a^2$

Case 4: Given two angles and the included side (ASA).

If m$\angle$A = 38, $\angle$C = 27, and b =10, find a and c.

1. Draw a diagram of the problem and label the coordinates of the three vertices.

2. Write the coordinates of B in two different ways, one using trig functions of A and the other using trig functions of C. (10 – a*cos27°, a*sin27°) = (c*cos38°, c*sin38°)

3. Write two equations equating the values of each of the respective coordinates that you listed in part b.
   10 – a*cos27° = c*cos38° and a*sin27° = c*sin38°

4. Solve the above system of equations using the CAS and the SOLVE command. Make sure you are in approximate mode.(see figure 4)

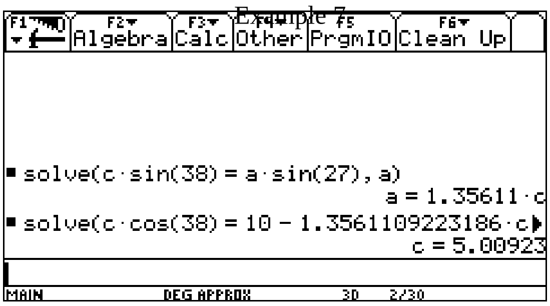
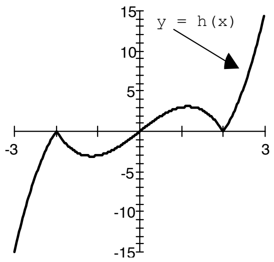


Figure 4

♦ **MULTIPLE REPRESENTATIONS**

We found the study of odd and even functions to be a good example of using a CAS with multiple representations. We used tables, graphs, and the symbol manipulator to provide various ways to consider the definition of odd and even functions. With the tables and graphs we illustrated f(-x) = f(x) for even functions by noting that for opposite domain values the functional value was the same. The CAS calculator was brought into play to compute algebraically f(-x), which the students compared with f(x) for the entire domain. The same kind of thing was done for odd functions. The CAS computed f(-x), which the students then compared with -f(x). The multiple representation approach helped students better understand by illustrating and highlighting features of a mathematical idea in various ways. A portion of the lesson worksheet illustrates these ideas. (See Example 7)



For #11 & 12 use
the graph at the right.

11. Test h(1) & h(-1), h(2) & h(-2) and h(3) & h(-3).
Does the function seem to meet the criteria
h(-x) = -h(x)?

12. The function rule is $h(x) = x\,|x^2 – 4|$.
DEFINE h(x) with a CAS and check h(-x) = -h(x).

13. Translate these symbols into words:
k(-x) = -k(x)

14. Define the following functions using a CAS and determine whether they are <u>**odd, even or neither**</u>

$f(x) = x^2 + 4x + 4$	**f(-x) = ______**
g(x) = |x|x	**g(-x) = ______**
$h(x) = -\|x^2 - 1\|$	**h(-x) = ______**
$j(x) = x^3 - 4x$	**j(-x) = ______**
m(x) = x + 1/x	**m(-x) = _____**

Example 7

The numerical evaluation from the graph as in question 11 above, helped to illustrate that the opposite values in the domain are transformed by *h* to opposite function values. The numerical evaluation was intended to clear up any errors in the estimation of the function values. The checking of the symbols should look very much like the function notation statement about odd functions, h(-x) = -h(x). Question 13 from Example 7 was there to see if students could formulate an abstract statement about odd functions.

<u>STUDENT COMMENTS and CLASSROOM OBSERVATIONS</u>

It was very exciting to observe student reactions and read their comments about using a CAS. Some findings were not exactly what we expected.

♦ **Initially, about half of the students preferred to solve the problems using pencil and paper**. This was quite a surprise to us considering how anxious the students were initially to work with a CAS. However, they said that it was sometimes too cumbersome to type in the symbols or too frustrating to try to make sense of the unfamiliar outputs. They felt more comfortable doing the problems using pencil and paper and thus took less time to find solutions. Ultimately, they felt more in control of the solutions. Some even felt that not doing the problems by hand was cheating. Using a CAS made them feel dumb since students in other courses did the same thing without a CAS. However, over time more and more students felt comfortable with a CAS and preferred to use it.

- **The other half of the students preferred to solve the problems using a CAS.** Students said that they still had to know what to tell the CAS to do, but the CAS did most of the "algebra" work for them. They also indicated that a CAS on the TI-92 enabled them to go back to previous steps and retrieve them easily rather than re-keying the entire line over again or even erasing when using pencil and paper. A few students made a rather astute comment. They noticed that when solving equations, if they got their final equation to read "x =" they knew that their "algebraic steps" were correct. There was no way that the calculator could display "x =" and perform incorrect equivalence transformations. This was not like the uncertainty that exists when doing the problems by hand where students needed to always check if their answer was correct. Other students said that they just enjoyed the novelty of playing with a new "toy."

- **Students who knew the mathematical concepts but not the symbolic skills were still able to solve the problems correctly.** Several students knew what properties to apply to solve equations but would often make mistakes when working the problems using pencil and paper. One student said, "I know that I need to take the log of both sides of this equation, but I get messed up when I do that on paper." He knew the process but lacked the skill. A CAS enabled students to apply their conceptual understanding to solve the problem.

- **Some students were able to determine when a CAS was helpful to use.** Sometimes it occurred with students who knew the process but became anxious and confused in a testing situation. A case in point was one bright girl who did not like to use a CAS because she could do things faster using pencil and paper. However, on a test when she was given a "messier" equation to solve, she found herself making several mistakes and not getting an answer that made sense to her. So after several attempts at this, she finally reached for her TI-92 and used equivalence transformations to arrive at an answer that made sense to her. The CAS gave her an alternative way of dealing with the situation.

- **Using a CAS allowed more time for reflection on each topic**. Since the CAS eliminated most of the messy and time-consuming algebraic skills, there was time for students to actually reflect on what they were doing instead of just hurrying through a problem set.

- **A CAS prompted good discussions on rather "dry" topics.** Even though teachers think that everything in mathematics is fascinating, students often have a different opinion. Long division of polynomials is such an example However, because of some unfamiliar output generated by a TI-92, lots of discussion ensued. What was the quotient? Remainder? Why did the TI-92 write it in that order? The class became alive with comments analyzing the answers.

- **CAS also enabled the exploration of more complicated problems on each topic since the time and messiness of the algebraic skill were a mute point**. Students even made up their own messy problems to test their conjectures.

- **Students increased their mathematical knowledge**. Amazingly enough, there were students who could verbalize their solutions better as a result of working with a CAS. In particular, one very average precalculus student who originally had substantial difficulty solving literal equations, was able to dictate in front of the entire class how to solve a similar problem on the white board after working with a CAS for several days. The class as a whole had a higher average on the hardest chapter test of the semester (composition, inverses, and transformations of functions) as compared to those students who in the past few years had never used a CAS.

More US teachers are becoming interested in learning about using CAS as a pedagogical tool in secondary mathematics. The US has two endeavors that are trying to activate interest in the use of CASs. There is now an incorporated consortium of teachers, Mathematics Educators Exploring CAS, that has formed in the northern part of the state of Illinois. In addition, a USACAS conference is held each year in June. Both entities have similar goals: primarily to provide opportunities for exchange of ideas about teaching and learning with computer algebra systems, but also to support research in the pedagogical use of a CAS. Hopefully the consortium and the conference will influence curriculum, assessment, and pedagogy of US mathematics classrooms.