Cognitive Tools for Exploring Linear and Exponential Growth

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Abstract. Empowering teachers through the use of technology in mathematics exploration, open-ended problem solving, developing conceptual understanding, and interpreting and communicating about mathematics is at the heart of BRIDGES professional development (http://education.wichita.edu/alagic/bridges/BRIDGES.htm). Throughout the BRIDGES activities, concrete experiences have been provided to explore technology-based representations of data, graphs and functions. A particular focus of the professional development has been two-dimensional: (a) deepening teachers' understanding of linear and exponential growth via technology-based representations, and (b) providing effective contexts for students' learning from the same technology-based representations, considering the fact that they do not have teachers' standard representations in their toolbox. A variety of cognitive tools (spreadsheets, graphing calculators, hand-helds (Personal Digital Assistants – PDAs) have provided for (a) visual and graphical multiple representations interconnected with appropriate simulations, (b) meaningful explorations of a variety of cases in a smaller amount of time than if standard representations had been used and (c) nurturing learning environments that support priorities of teaching for understanding and teaching for transfer. This paper will report on lessons learned in attempts to explicate some bridges between classical and technology-based representations, as well as on teachers' views about indispensable and dispensable mathematical abilities and skills related to the concepts studied.

Introduction

Findings reported here are part of research carried out during first and second year implementation of the professional development grant for middle school mathematics teachers, *BRIDGES: Connecting Mathematics Teaching, Learning and Applications*, supported by No Child Left Behind Funds (Neal, Alagic, & Krehbiel, 2002; Alagic, Krehbiel, & Palenz, 2003). An overarching goal for BRIDGES participants is to develop a deeper understanding of how to integrate standards-based school mathematics into the larger world of mathematics including its applications in real world settings. Furthermore, the idea that teachers should know school mathematics at a deep level and should have an understanding of its place in the real world (Newman, 1956; NCTM, 2000; NRC, 2000) is undergirding the BRIDGES philosophy. With that idea in mind, the appropriate curriculum has been selected from the *Modeling our World* (Garfunkel, Godbold, & Pollak, 1998).

This paper reports about a study carried on within the BRIDGES professional development program. The goal of the study was to explore how available technology-based tools can (a) deepen teachers' understandings of linear and exponential growth via technology-based representations and (b) provide for students' learning from the same technology-based representations, considering the fact that students do not have the teachers' (standard) representations in their toolboxes. Findings from the first year (which focused on spreadsheets as cognitive tools) are reported in Alagic &

Palenz (2004). The current paper expands those findings in two directions: (a) teacher's explorations of exponential and linear growth in the second year of the project, and based on those experiences, (b) use of other cognitive tools in exploring exponential and linear growth.

The first section of this paper briefly introduces the professional development setting and philosophy as they relate to this study. The Cognitive Tools section gives a concise summary of the literature review related to spreadsheets, graphing calculators, dynamic geometry, and handhelds. It is followed by a section Context: Modeling Our World that describes more specifically the context and triggers of the study. Cognitive Tools in Action portrays multiple representations of linear and exponential growth examples generated with different cognitive tools implemented during the professional development sessions. The final section, Lessons Learned elaborates on some aspects of pedagogical content knowledge related to linear and exponential growth: preconceptions and misconceptions about exponential growth and selection of appropriate representations.

BRIDGES professional development

Context: Middle School Mathematics Teachers Participating in a NCLB Grant

BRIDGES professional development for middle school mathematics teachers (Neal *et al*., 2002; Alagic *et al*., 2003) is grounded on two models reflecting (a) practice-based professional development connecting professional development activities of teachers to the actual classroom work of teachers (Smith, 2001), and (b) immersion in inquiry focusing on teachers' exploration of real-life problems that challenge their (mathematical) reasoning (Loucks-Horsley, Hewson, Love, & Stiles, 1998; Loucks-Horsley & Mastsumoto, 1999; Heaton, 2000).

Teachers need successful experiences and ongoing pedagogical and technological support to do mathematics in environments supported by diverse technologies (Dreyfus & Eisenberg, 1996). To empower teachers, BRIDGES activities provided concrete experiences in the use of technology in mathematics exploration, open-ended problem solving, interpreting mathematics, developing conceptual understanding and communicating about mathematics. Furthermore, BRIDGES is based on the belief that teachers must become and continue to be mathematics learners, if they are going to teach for understanding. As Schifter & Fosnot (1993, p. 26) point out: "perhaps more important for [the teachers] than their investigation of any specific content area is the process of active self-reflection. By analyzing together their experience of the just-completed mathematics activity, teachers begin to construct an understanding of how knowledge develops and the circumstances that stimulate or inhibit it." More details about the context and underlying philosophy of technology integration in BRIDGES activities can be found in Alagic & Palenz (2004).

Philosophy: Teaching for Conceptual Understanding

Conceptual approaches encourage learners to explore mathematics and solve problems by working from their own understanding (Thompson, Philipp, Thompson, & Boyd, 1994). Learners have to be able to represent a concept in multiple ways and establish interrelationships among multiple representations to accomplish learning goals (Lesgold, 1998). Learners' explorations in both selecting representations and making connections among them when building conceptual understanding of mathematical ideas are of paramount importance for meaningful learning (e.g., Greeno & Hall, 1997). Teachers' selection of multiple representations may provide connections between concrete and abstract representations in order to reach conceptual understanding. Providing opportunities for students to build their own representations supports even further conceptual orientation in learning. New representations are essential components of a learning environment in which learners are required to think harder about the topic being studied and to generate ideas that would be impossible without these new representations. Integration of technology-based representations may maintain this creative thinking and may sustain the conceptual orientation (Alagic, 2003). In addition, Perkins' (1993) six priorities for teachers who are teaching for understanding in technology-oriented classrooms constitute an organizational framework for implementation of BRIDGES.

National Educational Technology Standards for students and teachers present an additional ongoing challenge and opportunity for BRIDGES teachers (International Society for Technology in Education [ISTE], 2000). That challenge can be sometimes narrowed down to providing an effective context for students' learning from technology-based representations for which they do not have teachers' standard representations. Advantages for students include meaningful explorations of a variety of cases in a smaller amount of time, opportunities for independent and small group discoveries, Concerns include a blurring of the distinctions between exact and approximate values or solutions, a lack of clarity regarding how different representations connect with each other mathematically, even while these representations are being used simultaneously to explore a problem.

Cognitive Tools

Cognitive tools, according to Derry (1990), are both mental and computational devices that support, guide, and extend the learners' thinking processes. The principle that learners need to create their own understanding can effectively be supported with appropriate cognitive tools. These tools give the learner a way (often visual) of representing their understanding of a new concept/phenomena and how it relates to their existing understanding of the same idea. Intellectual efforts of the learner can go further than just processing information. Cognitive tools main "goal is to make effective use of the intellectual efforts of the learner" (Jonassen, 1992; 1996, p.10). For example, graphing calculators and spreadsheets as cognitive tools for creating representations are often used for exploring mathematical phenomena and building conceptual understanding of mathematical ideas.

Well-designed cognitive tools can: (a) facilitate development of knowledge representations, (b) engage learners in critical thinking about the subject, (c) support learners to acquire skills that are generalizable and transferable to other contexts, (d) be simple but powerful in order to encourage deeper thinking and processing of information, and (e) be relatively easy to learn (Jonassen & Reeves, 1996; Alagic, 2003).

Salomon, Perkins, and Globerson (1991) explain the distinction between traditional learning applications of technologies and their use as cognitive tools as the effects OF technology versus the effects WITH technology. When students work WITH appropriate technology, instead of being controlled by it, they enhance the capabilities of the tools, and the tools enhance their thinking and learning. The result of that kind of "intellectual partnership" is that "the appropriate role for a computer system is not that of a teacher/expert, but rather, that of a mind-extension cognitive tool" (Derry & LaJoie, 1993, p.5).

Spreadsheets

The amount of research and literature about various cognitive tools is growing. Spreadsheets are taking a special place. First, they are readily available to computer users and second, the richness of this tool is still far from completely explored in terms of its potential use in mathematics education. The fact that an online journal, Spreadsheets in Education (http://www.sie.bond.edu.au/), is available is yet another confirmation that spreadsheets are becoming more and more used as cognitive tools by different types and different levels of learners. Vockell & van Deusen (1989) describe spreadsheets as tools that involve users in designing new rules based on the given sets of rules. That indicates the potential of spreadsheets both in problem solving and problem posing. Spreadsheets as cognitive tools for developing conceptual understanding of mathematical phenomena support a variety of means for storing data, calculating, and representing information. Manipulation of the numerical content can be done in a number of different ways, starting from simple calculations and function applications to generating tables and a variety of graphical representations for stored data. Encouraging learners to manipulate spreadsheets and generate different representations, supports their reasoning and understanding of the concepts, relationships and procedures.

As data is represented, manipulated, reflected on and quantitative information speculated about, the underlining mathematical thinking is made explicit (Beare, 1992). The calculational and graphical capacities of spreadsheets support students' problem-based learning and open-ended investigations and provide a context to engage students in analyzing and connecting multiple representations in all three modes: geometric, computational and algebraic. Considering implications of alternatives may activate higher-order thinking (Sounderpandian, 1989). Sutherland and Rojano (1993) investigated how pre-algebra students could use spreadsheets to represent and solve algebra problems. The study was conducted simultaneously in Britain and Mexico. During a 5-month period, students moved from a cause-effect local numerical view of algebraic relationships to general rule-guided relationships, symbolized both in the spreadsheet and in algebraic notation. These studies provide just a few small insights about the effectiveness of spreadsheets as cognitive tools. Teachers' understanding and use of spreadsheets can extend from kindergarten up, through all the grade levels, from simple 4-pane magic squares, multiplication tables, and graphing to complex applications, and further into still (for many teachers) uncharted areas, such as conditional formatting (Abramovich & Sugden, 2004).

Graphing Calculators

Maxwell (1994) showed that students who were taught pre-calculus using graphing calculators outperformed students who were taught without the use of graphing calculators. Harvey, Waits, and Demana, (1995) determined that prolonged use of graphing calculators leads students toward better understanding of mathematical functions. Students who use graphing calculators display better understanding of function and graph concepts and have improved problem solving (Smith, 1997).

Smith and Shotsberger (1997) considered the effect of integrating the graphing calculator into a semester-long college algebra course. The project focused on the extent to which student achievement, attitude, and problem-solving methods were effected by the use of the graphing calculator. They indicate that graphing technology may have even greater benefits for some special populations. A study by Quesada and Graham and Thomas (1994) found that the use of graphing calculators was effective in improving students' understanding of mathematical concepts which were previously difficult to comprehend. Thiel & Alagic (2004) reported about a study with two intertwined aspects. The first one was enabling students to build a stronger conceptual understanding of functions and related concepts and the second was to study what conditions provide for a successful learning environment utilizing graphing calculators. The key features identified and applied were: (a) long-term exposure to ill-structured problems, (b) writing about the concepts, (c) the teacher answering questions with appropriate questions/prompts to provide for scaffolding, (d) cooperative learning and (e) the teacher's proficiency with graphing calculator. Through these features the students developed a deeper understanding of the concepts and they were more willing to attempt complex problems. Also, their communication skills improved. The study indicates that problem-based learning in a technology-oriented environment provides appropriate conditions for developing critical thinking and communication skills.

Dynamic Geometry

Virtual tools provided through dynamic geometry software are replacing traditional tools of Euclidian geometry. As a cognitive tool, dynamic geometry allows the user to make quick and accurate constructions. These constructions are dynamic in the sense that they can be manipulated, dragged and animated; parts can be measured; collected data can be graphed; changes/processes can be observed. Dynamic geometry tools are engaging students in interactions with multiple visual representations of mathematical concepts offering exciting new opportunities –for active learning in geometry. In contrast with static figures constructed using straightedge and compass, dynamic geometry figures can be easily manipulated. One can consequently observe invariant properties while changing variant properties by dragging. Data produced during this process is then available for analysis. A static/motionless learning environment that emphasizes proofs discovered by others is being replaced with an environment that can facilitate active mathematical inquiry. The main contribution of dynamic geometry seems to be its potential for interactivity provided by 'dragging' action. After completing a construction, the learner can drag various elements to observe geometric representations and conceptualize geometric properties. However, exactly how students grasp geometric ideas from the dynamics of observed figure is not always clear (Goldenberg & Cuoco, 1998)

Validity of constructions is definitely very much different than in paper-pencil constructions. The figure might appear well-constructed, but the consistency with geometrical theory needs to be confirmed; "drag test" has to be performed before proceeding with other explorations and explanations. Keeping the connectedness/robustness of the figure intact is already described in the literature (e.g., Balacheff & Sutherland, 1994; Jones, 2000; Laborde, 1993).

Personal Digital Assistants (PDAs)

Handheld devices are one of the more recent developments in educational computing. Their introduction is of particular interest because of their portability and potential to be more integrated with the learning activity itself than desktop computers (Soloway, Norris, Blumenfeld, Fishman, &Marx, 2001). It appears from the literature that there is a lot of interest and enthusiasm for the use of mobile devices in learning (e.g., Roschelle, 2003; Soloway et al., 2001). Many teachers and educators have adopted the handheld as a note-taking and record keeping device, allowing students to collaborate with others by sharing data through built-in infrared communication capabilities. The ease of having a portable space for data collection is one of the advantages of handhelds.

Perry (2003) reported that students adapt easily to the use of handhelds. But without rethinking the rationale in their use, data collection and analysis may actually have the undesired effect of reinforcing a pencil-and-paper approach without modifying the underlying learning structures. Perry suggests that teachers are not always as enthusiastic as their students, not just because of their novelty but because of the lack of simple and clear educational software applications.

Context: Modeling Our World

During the first year of BRIDGES, we used the text, *Mathematics: Modeling Our World* (Garfunkel et al., 1998), to challenge teacher-participants with real world problems whose solutions required the application of significant mathematical concepts and related technology expanding on middle and high-school mathematics curriculum. Deepening teachers' understanding of linear and exponential growth via technology-based representations became a focus of this study somewhat unexpectedly (Alagic & Palenz, 2004). There were two triggers: teachers' somewhat narrow current understanding of linear and non-linear graphs and opportunities provided for exploring these concepts in a technology-oriented environment.

Problem Triggers: All graphs and data sets were assumed to be linear

There were two explicit situations that initiated this refocusing "on our feet". The first one had to do with an assignment of making up real-life situations to match simple graphs presented to teachers. The second one came from a middle school classroom, as reported by the teacher, and had to do with graphing collected data. The sequence of activities, reflections and analysis of subsequent investigations are presented in more detail in Alagic & Palenz (2004). A brief summary of the triggers is provided here for the purpose of continuity of this paper.

When teachers shared their stories and interpretations to accompany given graphs, it was obvious that the stories were the simplest possible cases of time-distance or time-growth graphs. The stories accounted for the increasing or decreasing aspects of the graph, but they did not take into account other significant mathematically observable features, such as the rapid acceleration of an exponential curve.

The second event occurred when one of the teachers shared results of his implementations of BRIDGES curriculum with his 7th graders. All students had correctly gathered data indicating exponential growth, but all of the graphs they developed pictured linear growth. The participating teachers did not readily identify the mistake, indicating that they, too, were not experienced with making the necessary distinctions between different sorts of increasing graphs. When instructors probed, the teacher explained how he "helped" students "linearize" their graphs, attributing non-linearity to imprecision in drawing.

Because of these "triggers", we added more explorations of linear versus exponential growth into our on-going workshops in order to provide for teachers' richer collection of appropriate representations.

Exponential Growth Stories

The first of these stories was used during the first summer-workshop. The second one was introduced in the fall in order to develop a richer multiple-representations collection for exponential growth.

Moose Return (Garfunkel et al., 1998). In late 1980 some moose began to reappear in New York's six-million-acre Adirondack State Park wilderness area. By 1988, experts estimated that 15 to 20 moose were in the park. By 1993, this number had increased to between 25 and 30 moose. A survey conducted by the New York State Environmental Conservation Department (ECD) found that the public favored an increase in the moose population. The ECD determined that a plan to move 100 moose into the park over a 3-year period would cost $1.3 million. Students are to pretend that they are the commissioner of the ECD responsible for making a recommendation to the governor about their findings. Their consideration of the given information leads to mathematical models for growth of the moose population. Analysis of the first model proposed (linear) raises questions as to its validity for the known and predicted behavior of moose. Further information leads to an exponential growth model.

The King's Chessboard (Birch, 1988). The King insisted that a wise man choose a reward. The wise man asked that he be given rice each day for the 64 squares on the King's chessboard. For the first day, he wanted 1 grain of rice; each following day the rice would double, for as many days as there are squares on the chessboard.

One grain became two and then four; grains became ounces; ounces became pounds; a bag became two bags; … The Grand Superintendent informed the King and the King summoned his royal mathematicians. He discovered that he had promised the wise man approximately three hundred billion tons of rice. There was not enough rice in the King's world to fulfill the promise.

Cognitive Tools in Action: Generating Multiple Representations

Spreadsheets

Exponential growth stories became a convenient background to explore further and converse about exponential growth, as well as to investigate the appropriateness of different cognitive tools for such explorations.

DAY	# OF GRAINS OF RICE		graph below is scatter plot, so days form x-axis				
1	1		click on graph and then change the number of days by dragging in table				
2	2						
3	4						
4	8						
5	16						
6	32						
7	64						
8	128						
9	256						
10	512						
11	1024						
12	2048						
13	4096						
14	8192						
15	16384						
16	32768						
17	65536						
18	131072						
19	262144						
20	524288						
21	1048576						
35	17179869184						
36	34359738368		How much rice is this?				
37	68719476736		Estimates ranged from 15 to 50 grains of rice per cubic centimeter.				
38	1.37439E+11				(a) 15 grains	6.1489E+17	cm^3
39	2.74878E+11				(b) 50 grains	1.8447E+17	cm^3
40	5.49756E+11						
41	1.09951E+12		vol of earth				
42	2.19902E+12		4/3 * Pi * (4000 mi)^3 *(1.6 km/mi)^3 * (100000 cm/km)^3				
43	4.39805E+12		1.10E+27	cm^3			
44	8.79609E+12		surface of earth				
45	1.75922E+13		4 * Pi * (4000 mi)^2 * (1.6 km/mi)^2 * (100000 cm/km)^2				
46	3.51844E+13		5.15E+18	cm^2			
47	7.03687E+13		What part is land? Rice fields?				
48	1.40737E+14						
49	2.81475E+14						
50	5.6295E+14		Grand Canyon				
51	1.1259E+15		446km x 29km x 1.5km				
52	2.2518E+15		1.9401E+19	cm^3			
53	4.5036E+15						
54	9.0072E+15						
55	1.80144E+16		Students' first guesses:				
56	3.60288E+16		Rice would fill Grand Canyon 5 times?				
57	7.20576E+16		about 2% of the earth's volume?				
58	1.44115E+17						
59	2.8823E+17		No--rice would fill 2% of Grand Canyon?				
60	5.76461E+17		3.8802E+17	cm^3			
61	1.15292E+18		or 10% of surface of earth, 1 cm thick				
62	2.30584E+18		5.15E+17				
63	4.61169E+18		or 6-8 billion boxcars				
64	9.22337E+18		3.6E+17	cm^3			

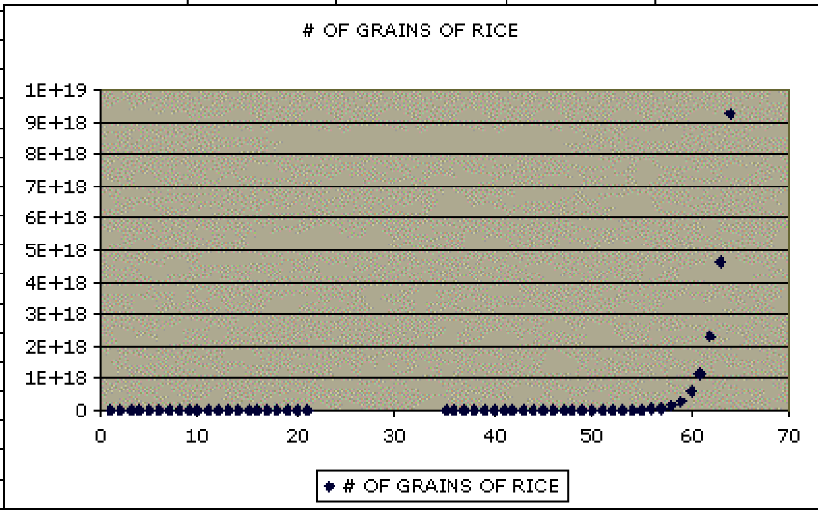

Figure 1. Estimating amount of the Wise Man's rice

For example, after some hands-on preliminary exploration of the number of grains of rice in a given volume, we used spreadsheets on laptop computers and graphing calculators to consider the quantity of rice that the King promised to the wise man. Yes, we knew the number, but what does 300,000,000,000 tons of rice represent in some observable way. Technology made it possible to explore the data quickly. Beginning explorations (see Figure 1.) related to the amount of rice that the King promised to the Wise man and different types of estimates that teachers explored to better understand the meaning of large numbers and exponential functions. Additional what-if questions provided context for further exploration: What if the amount of rice was tripled each day? What if we apply this idea in a different context; what context might be interesting? Money-related questions came quickly into the discussion.

It is important to mention here that some teachers needed more time to get comfortable with spreadsheets and valued opportunities to explore more than one example.

After explorations using these growth stories, further teacher-participants' examples included more references to exponential growth. They came up naturally and the teachers seemed to be more comfortable with the ideas. In one presentation a pair of teachers introduced exponential decay as it related to the amount of a drug or medicine in the body and the time it takes for the substance to decompose. One of them introduced a concrete example of tearing a sheet of paper in half and half again and again and again. In a spreadsheet, we calculated the number of tears and the area of the remaining paper, and plotted this information. Using graphing features of spreadsheets to plot two separate data lines, we could see that the number of tears is a linear function while the area of the remaining paper is an exponential function. A scatter plot (of tears vs. area) puts the two columns together as ordered pairs for one (exponential decay) function.

Graphing Calculators

Some of the teachers expressed interest in exploring similar problems using graphing calculators. We did multiple approaches to finding the value of a house if it starts at $83,500 and is appreciating at a rate of 6%: 1st we just did one year at a time in the calculation mode; 2nd we made two lists and created years 0 through 10 and values of the house at the end of each year. Instead of calculating the 2nd list one number at a time, we created a function [$83500*1.06^x] and then used it to calculate all of list 2 as a function of list 1. We plotted the function, choosing an appropriate window, and also plotted the two lists and practiced tracing and getting specific values through the trace mode.

In several sessions working with graphing calculators and spreadsheets, teachers discussed the need to introduce technological abbreviations for scientific notation. Students need to explore the meaning of expressions such as "2.3556 E17". Related topics include scientific notation, number of decimal places and significant digits, and display options and control for numerical expressions in spreadsheets and calculators.

Dynamic Geometry

One week of the summer activities was devoted to dynamic geometry and discussions about traditional versus technology-based development of geometry concepts. Teachers explored a variety of Euclidean geometry examples and generated graphs in the dynamic software environment but neither teachers nor instructors made a connection to possible geometric representations of exponential and linear growth.

The last day activities included an example about tearing the paper to demonstrate decay. That brought to our attention an additional possibility for representing these functions. Visual representation of tearing paper in half can be demonstrated using dynamic geometry environment. For exponential decay, a rectangle could be repeatedly cut in half and the areas of the resulting pieces could be measured. Different ways of tearing the paper (or halving the rectangle) could be compared. Does it matter if we cut the rectangle in half vertically or horizontally? What happens to the area if we cut both the length and the width in half?

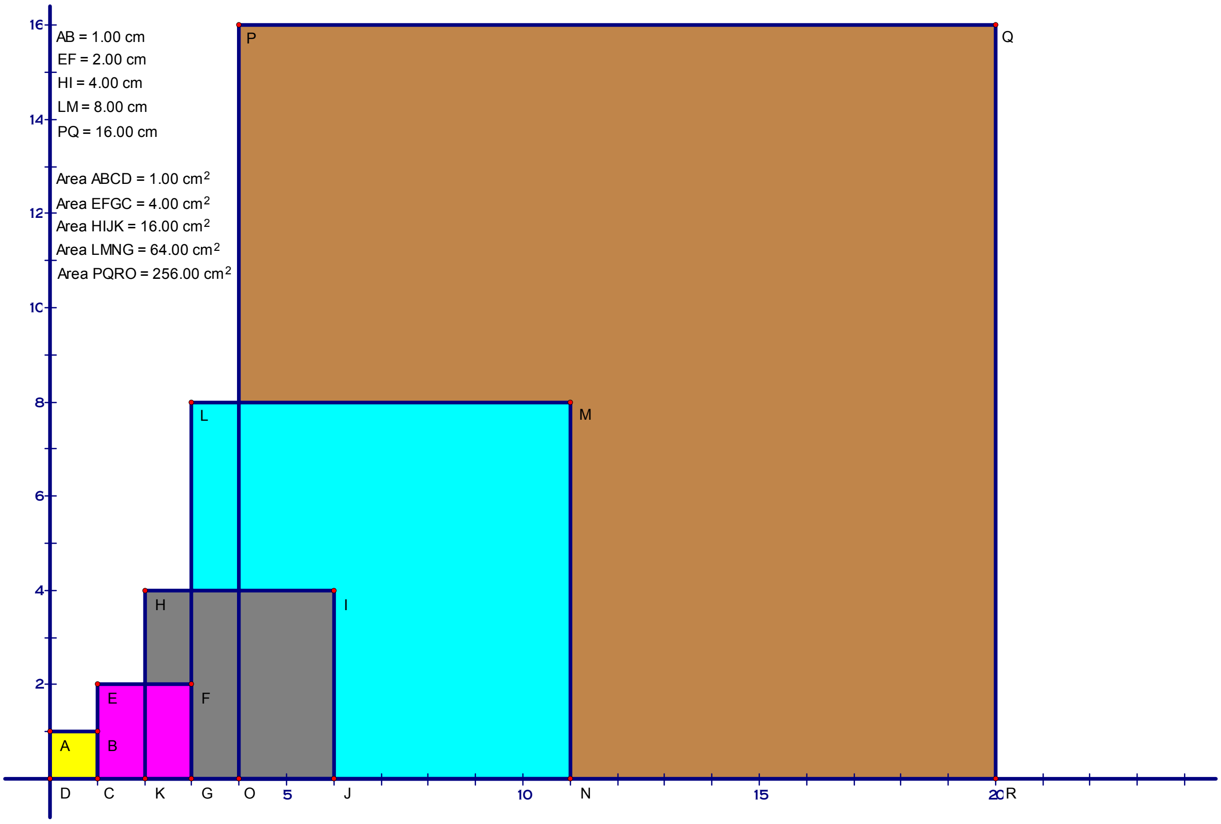

Figure 2. Geometer's Sketchpad representation of doubling side of unit square

Bisecting a piece of paper again and again is possible only for a certain finite number of steps. In the same manner we can demonstrate that within dynamic geometry. After that we need our abstract thinking skills to visualize what is happening and to verbalize our understanding of exponential decay. It is a little bit more complex to demonstrate doubling a piece of paper or doubling sides of a square to generate a sequence of squares demonstrating exponential growth. But dynamic geometry can help to examine situations of that kind. For example, Figure 2 demonstrates some of steps necessary to explore the question: What happens to the length of a side and the area of a unit square if the sides of a unit square are repeatedly doubled?

Lessons Learned: Pedagogical Content Knowledge

Comprehending the blending/interplay among subject knowledge, understanding of students, and understanding of teaching techniques is an attribute of pedagogical content knowledge (PCK).Shulman (1987) identifies PCK this way: "the key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy" (p. 15). An effective teacher needs to have well-established PCK to be able to design learning situations appropriate for the needs of students.

Teaching and learning for conceptual understanding in a technology-oriented environment makes elevated demands on teachers' pedagogical content knowledge in the area of selection of appropriate representations as part of the instructional design (Alagic, 2003). Greeno and Hall emphasize the importance of providing opportunities for students' exploration in selecting representations. In the presence of technology, "the ability of students to operate within and between different representations of the same concept or problem setting is fundamental in effectively applying technology to enhance mathematics learning" (Demana & Waits, 1990, p.218). With multiple contexts, students are more likely to abstract the relevant features of the concepts and develop a more flexible representation of knowledge. Research has also shown that developing an assortment of representations enables learners to think flexibly about complex domains and enhances their conceptual understanding.

Technology-augmented activities support mathematical connections in at least three ways: they (a) link multiple representations of the same mathematical idea, enhancing the context for reflective abstraction; (b) interconnect mathematical topics and (c) connect mathematics to real-world phenomena. Appropriate selection and use of cognitive tools can make it easier for learners, both teachers and students, to bring together multiple representations via intermediate representations or explicating links among different representations of some mathematical concepts (Alagic, 2003).

This section is a reflection on the teachers' development of some PCK components emerging during BRIDGES activities. The most significant component considered is the teachers' understanding of the representations they explored and the connections among those representations, especially when the cognitive tool has been changed. The focus is on representational properties arising through the use of cognitive tools and emerging challenges.

Mathematics Knowledge: Misconceptions about Exponential Growth

The mathematical underpinnings of the BRIDGES consist of (a) solving real-life based problems, (b) creating mathematical models, and (c) enhancing teachers pedagogical content knowledge related to the middle school mathematics. In earlier sections, emergence of the topic for this study *exploring exponential and linear growth* has been described in detail, including the fact that teachers' knowledge of exponential growth and related concepts required a well-structured intervention. In agreement with the philosophy of teaching for conceptual understanding (Alagic, 2003) facilitators focused on providing variety of experiences through which several different representations for exponential and linear growth were generated and connections among them were studied. In addition to already mention misconceptions, there is a tendency to connect points on the graph; discrete exponential sequences versus continuous exponential functions need to be carefully considered.

Technology Integration Knowledge: Selection of a Cognitive Tool

Technology integration was an underlying assumption during the initial design of BRIDGES, and its effective presence was an ongoing opportunity and challenge. Teachers' previous experiences with technology could be described as basic. All of them expressed willingness and interest to learn how to integrate technology even beyond what is available at the moment in their classrooms.

In general terms, selection of a cognitive tool can be considered as at least two-layered. The tool appropriate for the task is often not determined just by "the best tool for the task". A teacher's selection of the cognitive tools in current education is probably more dependent on many outside factors (district and school decisions) than on the teacher's familiarity or competence with a given tool. To give an example, our local district is currently investing a lot of money in getting classroom sets of handhelds. We will not argue about the validity of that decision; "technology is getting into students' hands". But it is clear that curriculum choices will depend on what a student can and cannot do with a handheld. To elaborate further about the determining factors in selecting cognitive tools, spreadsheets are widely available, assuming that schools have computers or laptops, and we know that handhelds have spreadsheets and that graphing calculator software appropriate for handhelds exists. But that is not the case with dynamic geometry software, especially not in middle schools. Teachers' competence becomes the second layer. Not just competence in using the available tool but, mathematical competence, willingness and enthusiasm to work with new tools and to transform mathematics related PCK into *mathematics and technology related PCK.* BRIDGES leadership had an opportunity to use a large variety of cognitive tools, most of the things available at this point in time. Our exploration included not only creating multiple representations with a given tool but creating multiple representations with multiple cognitive tools. That enhanced discussions about selecting "the best tool" for a given task and possible pitfalls of various tools.

Classical and Technology-Based Representations

Geometric, numeric, algebraic and verbal representations of certain concepts may be supported with well-chosen cognitive tools better than with traditional means. Technology-based representations often appear concrete when exploring abstract ideas. Cognitive tools allowed teachers to explore and recognize the differences between linear and non-linear graphs and with some guidance, their vocabulary expanded to allow for descriptions and discussions about these differences and features. The tools provided for (a) visual and graphical multiple representations interconnected with appropriate simulations, (b) meaningful explorations of a variety of cases in a smaller amount of time than if standard representations had been used and (c) a nurturing learning environment supporting Perkins' (1993) priorities, such as making learning a thinking-centered process, providing for rich, ongoing assessment, supporting learning with powerful representations, and teaching for transfer. In that manner, teaching for conceptual understanding philosophy was carried throughout the study (Alagic, 2003)

By repeating the cycle of using cognitive tools, reflecting on their use and the way they illustrate the concepts, discovering vocabulary to describe new aspects, and then examining new real-world problems and again choosing appropriate cognitive tools, the teachers gained skills and confidence, building their mathematics related PCK in the technology – based environment.

Overcoming Preconceptions and Misconceptions

Besides the challenges described above, related to linear and exponential graphs, we dealt with other preconceptions and misconceptions.

Willingness of learners to accept at face value results generated by a cognitive tool has to be taken into careful consideration. The ability to evaluate effectively the quality of the generated output is dependent on the learner's own level of mathematical understanding (Goos, Galbraith, Renshaw, & Geiger, 2003).

The technological orientation of the environment affects teachers' actions, the feedback that is provided to the user/student and the interpretation the student creates based on that feedback. An example that explicates that very well is that of problem solving in the dynamic geometry environment. An assignment solved using dynamic geometry may require different strategies when done with paper and pencil (see Laborde, 1993).

"Zooming" is a popular technological tool to explore graphs. Graphs of linear and exponential functions gave us an opportunity to consider the advantages and disadvantages of zooming. Teachers were able to see that as we zoom in and out on an exponential graph, its shape remains essentially the same—with a relatively flat part curving into a steep rise. Also, as they zoomed out, the details and sharp rise of a section of the graph were lost—variations disappearing in the loss of detail which happens with zooming out. A follow-up activity involved beginning with a similar data table and graph, either in Excel or on a graphing calculator, and then trying to make the graph appear to have a steeper rise or a flatter section by changing the window (by zooming in or out of the graph). The lesson here is two-fold: zooming can reveal hidden detail in a graph or zooming can disguise the shape of a graph.

In the cognitive tools environment, problem solving should be used to explore, investigate and discover mathematics, not to reinforce theory delivered by the teacher. There is a caution to go with that. The distinction between illustration and justification/proof gets often distorted. Allen (1996), in reference to dynamic geometry, asserts that a person's problem solving ability depends on how much mathematics he/she understands and suggests that the almost automatic performance of routine tasks is fundamental to the development of mathematical understanding.

During this study some pitfalls in using dynamic geometry have been observed. The subsequent literature review confirmed the following:

- There is a tendency of students to draw pictures rather than construct mathematical objects; the distinction between *drawing* and *figure* is described by Laborde as "*drawing* refers to the material entity while *figure* refers to a theoretical object" (1993, p.49)
- Learners often get 'stuck' somewhere between a drawing and a figure (Hölzl 1995, 1996)
- Mathematical arguments are often just explanations of what is visible on the screen; the 'dynamic' nature of the software is influencing the form of explanation given by the students (Jones, 2000)
- Stability/robustness of figures is linked with using points of intersection to try to hold the figure together (Jones, 2000)

Ideas for further exploration:

- Exploring sequences that are linear, quadratic, cubic, n^p, etc., compared to exponential; specifically, to see that the differences are constant for linear, linear for quadratic, etc., while they are again exponential for exponential sequences
- Link exponential growth and decay by exploring the ways different numbers get represented on the calculator screen—e.g. looking at 3^n to see when the calculator converts from integer to scientific notation. How big a power will it take? Then do 3^{-n} and ask the same question. This activity would provide an opportunity to assess whether students really know what the E22 (or whatever) means in their calculator or spreadsheet display
- Can we control the output appearance; force scientific notation or a particular number of decimal places for the answers on the screen? Can we do that as a general setting? Can we have the calculator rewrite an answer just given in a different format?
- How do we get fractions? That is very helpful for teachers grading or creating problems which use fractions. This also brings up another discussion of exact versus approximate decimal representations of fractions.
- Exploratory "What if" questions have always been a good teaching and learning tool. With cognitive tools available, opportunities for new challenging "what if" questions and conjectures are emerging.

Closure

Teachers taking their new knowledge to the classroom need opportunities to practice with it and to reflect on both their lessons and students' experiences with new tools. Many teachers will not share cognitive tools with students until they are confident that they can deal with "how-to" questions and guide students successfully. BRIDGES has provided a supportive learning environment and forum for sharing and reflecting about successes and challenges, including ongoing sharing and discussion opportunities using yet another technological tool, "Blackboard" collaborative learning web-based software.

Being proficient with cognitive tools takes time. This investment makes sense only if the learning goals accomplished are proportional to the effort. Therefore in the technology-oriented context, selection of a problem becomes even more relevant. The rich, open-ended, real-life problem, *Moose Return,* was a worthwhile task for BRIDGES participants. It provided a starting point for a sequence of explorations, both in terms of related real-life problems and mathematical abstractions. *The King's Chessboard* allowed for exploring new representations of the concepts studied. All these activities supplied opportunities for deepening participants' conceptual understanding of exponential growth and their ability to distinguish between linear and non-linear growth/graphs. Opportunities for modeling, simulation, mathematical abstractions and transfer unfolded (*e.g.*, a packaging problem was brought into the discussion).

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