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Approximations of intégrals

(Bernard Egger)

A first way of finding the approximate value of an integral is, when that is possible, to replace the function by a power series expansion
Derive allows to show rather easily what such an expansion means.

We will consider only one simple case here: that of the function defined by

f(x)=sin(x)+cos(x)

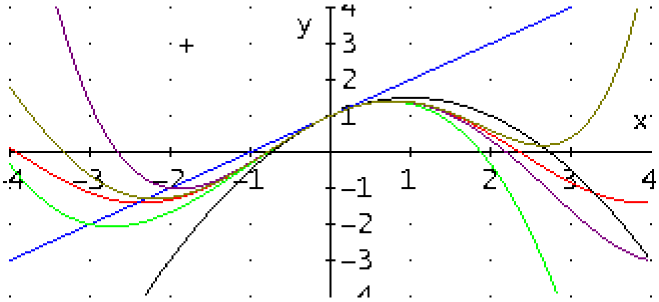
```
f(x) := SIN(x) + COS(x)
```

Derive obviously makes it possible to find a Taylor expansion to a high order number generated by this function:

$$\mathrm{TAYLOR}(f(x),\ x,\ 4) = \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2} + x + 1$$

We can then plot on the same graph these different polynomial functions as well as the f function curve. We will note that the curves are nearer and nearer to that of f. We can obtain all these curves with only one instruction.

```
k := 0

PROG(k := k + 1, TAYLOR(f(x), x, 0, k))
```



But Derive 6 makes it possible to highlight this proximity more dynamically.
To do this, we build a function depending on a parameter. While varying this parameter, we will observe a continuous "polynomial" deformation.

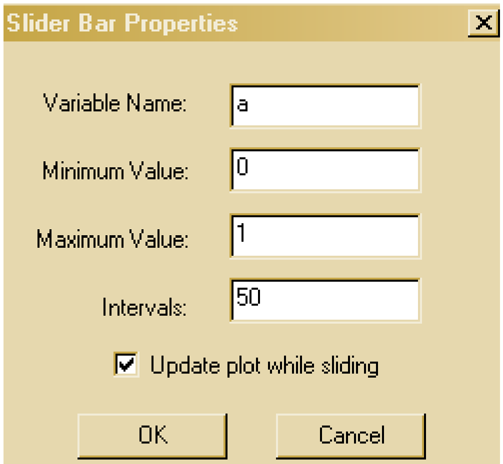
To simplify our work, we write functions which will highlight this continuous variation.

```
InputMode := Word

p1(x) :=
  If x < 0
     0
     If x ≥ 1/3
        1
        3·x

p2(x) :=
  If x < 1/3
     0
     If x ≥ 2/3
        1
        3·(x - 1/3)

p3(x) :=
  If x < 2/3
     0
     If x ≥ 1
        1
        3·(x - 2/3)
```

Then a function g defined in the following way :

$$g(x) := 1 + x + p1(a)\cdot\left(-\frac{x^2}{2}\right) + p2(a)\cdot\left(-\frac{x^3}{6}\right) + p3(a)\cdot\frac{x^4}{24}$$

We open a new 2D_plot window and we associate the variable to a slider bar with the following properties



We start by plotting the graph of g then that of f (the order is important). We can then modify the parameter value in the slider bar to see the curve of g changing with continuity. Graphically, we can make the assumption that the function g when a = 1 gives a good approximation of f on the interval [- 1,1].

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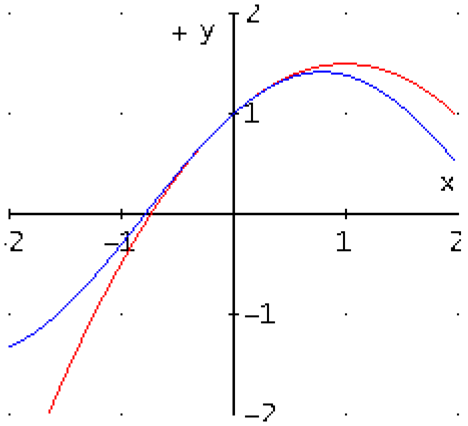

$$\text{SUBST}(g(x),\ a,\ 1) = \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2} + x + 1$$

We are thus being able to approach the integral of f on this interval by that of g. We have:

$$\int \left(\frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2} + x + 1\right) dx = \frac{x^5}{120} - \frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} + x$$

$$\int_{-1}^{1} \left(\frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2} + x + 1\right) dx = \frac{101}{60}$$

$$\int_{-1}^{1} \left(\frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2} + x + 1\right) dx$$

1.683333333

$$\int_{-1}^{1} (\text{SIN}(x) + \text{COS}(x))\ dx = 2\cdot\text{SIN}(1)$$

$2\cdot\text{SIN}(1)$

1.682941969

Traditional methods of approximations and their speed of convergence

We will look briefly at some traditional methods of approximate-calculation integrals and will compare their speed of convergence with two examples .

Traditionnal methods

Rectangle approximation method

These are based on two possible formulas:
either using the left points (left rectangle approximation method LRAM)

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n-1} \frac{b-a}{n} f\left(a + k\frac{b-a}{n}\right)$$

or using the right points (right rectangle approximation method RRAM)

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} \frac{b-a}{n} f\left(a + k\frac{b-a}{n}\right)$$

Derive has a built-in function which allows the left sum calculation

$$\mathrm{LEFT\_RIEMANN}\left(\frac{1}{x}, x, 1, 2, 5\right) = \frac{1879}{2520}$$

For the right sum, it is easy to build a new function :

$$\mathrm{right\_riemann}(u, x, a, b, n) := \sum_{k=1}^{n} \frac{b - a}{n}\cdot \mathrm{SUBST}\left(u, x, a + \frac{k\cdot(b - a)}{n}\right)$$

$$\mathrm{right\_riemann}\left(\frac{1}{x}, x, 1, 2, 5\right) = \frac{1627}{2520}$$

The trapezoidal method

This is defined by the formula:

$$\int_a^b f(x)dx \approx \frac{1}{2} h \sum_{k=1}^{n} \left(f\left(a + kh\right) + f(a + (k+1)h)\right)$$

with h=(b-a)/n.
The approximation provided by the trapezoidal method appears as the arithmetic mean of the approximations given by the two rectangle approximation methods.
We can thus define a function:

$$\mathrm{trap}(u, x, a, b, n) := \frac{1}{2}\cdot(\mathrm{LEFT\_RIEMANN}(u, x, a, b, n) +$$

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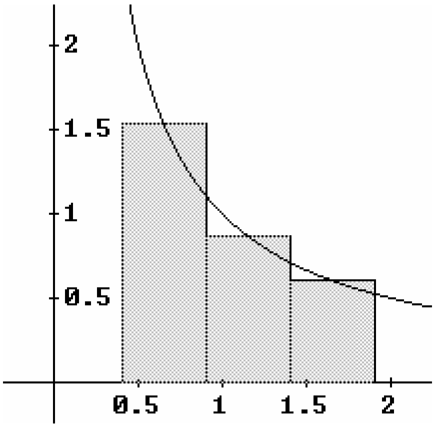
```
right_riemann(u, x, a, b, n))
```

$$\text{trap}\left(\frac{1}{x},\ x,\ 1,\ 2,\ 5\right) = \frac{1753}{1260}$$

$$\frac{\dfrac{1879}{2520} + \dfrac{1627}{2520}}{2} = \frac{1753}{2520}$$

The Midpoint Method

The midpoint method also uses rectangles, but does not provide an interval around the integral.
We obtain for example:



The formula used in approximate calculation is:

$$\int_a^b f(x)dx \approx h\sum_{k=0}^{n-1} f\left(a+kh+\frac{h}{2}\right) = h\sum_{k=0}^{n-1} f\left(a+\frac{(2k+1)h}{2}\right)$$

With Derive, we have :

```
point_milieu(u, x, a, b, n, h) :=
  Prog
    h := (b - a)/n
    RETURN h·∑(SUBST(u, x, a + (2·k + 1)·h/2), k, 0, n - 1)
```

$$\text{point\_milieu}\left(\frac{1}{x},\ x,\ 1,\ 2,\ 5\right) = \frac{479378}{692835}$$

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$$\mathrm{APPROX}\left(\mathrm{point\_milieu}\left(\frac{1}{x},\ x,\ 1,\ 2,\ 5\right),\ 5\right) = 0.6919$$

To obtain the graph, we use a small program which is written in two different times:

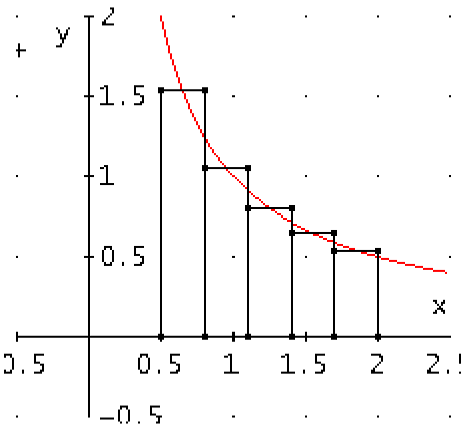
$$\mathrm{rectangle}(a1,\ b1,\ a2) := \begin{bmatrix} a1 & 0 \\ a1 & b1 \\ a2 & b1 \\ a2 & 0 \end{bmatrix}$$

The "rectangle" procedure makes it possible to plot a rectangle whose apex are the points of coordinates:
(a1,0),(a1,b1),(a2,b1),(a2,0) (indicated in the matrix).

```
graph_point_milieu(u, x, a, b, n, h, gr, k, a1, b1, a2) :=
  Prog
    gr := [[]]
    h := (b - a)/n
    k := 0
    Loop
      a1 := a + k·h
      b1 := SUBST(u, x, a1 + h/2)
      a2 := a1 + h
      gr := INSERT(rectangle(a1, b1, a2), gr, 0)
      k := k + 1
      If k = n
         RETURN gr
```

The program "graph_point_milieu" makes it possible to build a set of successive rectangles, arranged in "a large" named matrix "gr.".
On condition, of course, they you select the option "connected points ".

$$\mathrm{graph\_point\_milieu}\left(\frac{1}{x},\ x,\ 0.5,\ 2,\ 5\right)$$



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Simpson's method

We approach the integral by a sum of integrals of 2nd-degree polynomial functions.
The formula of computation is given by

$$\int_a^b f(x)dx \approx \frac{2}{3}h\sum_{k=0}^{n-1} f\left(a+kh+\frac{h}{2}\right)+\frac{h}{6}\sum_{k=0}^{n-1} f\left(a+(k+1)h\right)+\frac{h}{6}\sum_{k=0}^{n-1} f\left(a+kh\right)$$

With this formula, we can determine the link between the formula of Simpson and those of the rectangle approximation methods and the midpoint method.
We have :

$$\text{simpson}(a,b)=\frac{2}{3}\text{point\_milieu}(a,b)+\frac{1}{6}\text{left\_riemann}(a,b)+\frac{1}{6}\text{right\_riemann}(a,b)$$

With Derive, we obtain

$$\text{simpson(u, x, a, b, n, h)} := \frac{2}{3}\cdot\text{point\_milieu(u, x, a, b, n, h)} + \frac{1}{6}\cdot\text{(LEFT\_RIEMANN(u, x, a, b, n, h) + right\_riemann(u, x, a, b, n, h))}$$

$$\text{simpson}\left(\frac{1}{x},\ x,\ 1,\ 2,\ 5\right) = \frac{48408065}{69837768}$$

$$\text{APPROX}\left(\text{simpson}\left(\frac{1}{x},\ x,\ 1,\ 2,\ 5\right),\ 5\right) = 0.69315$$

Brief summary

Let us examine the 5 methods seen for integral calculation and compare the results obtained with the value that gives Derive for ln2. We have:

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$$\text{APPROX}\left(\begin{bmatrix} \text{LEFT\_RIEMANN}\left(\frac{1}{x}, x, 1, 2, 5\right) \\ \text{right\_riemann}\left(\frac{1}{x}, x, 1, 2, 5\right) \\ \text{trap}\left(\frac{1}{x}, x, 1, 2, 5\right) \\ \text{point\_milieu}\left(\frac{1}{x}, x, 1, 2, 5\right) \\ \text{simpson}\left(\frac{1}{x}, x, 1, 2, 5\right) \\ \text{LN}(2) \end{bmatrix}, 5\right) = \begin{bmatrix} 0.74563 \\ 0.64563 \\ 0.69563 \\ 0.6919 \\ 0.69315 \\ 0.69314 \end{bmatrix}$$

Simpson’s method seems to be most precise, the trapezoidal and midpoint methods are equivalent, and finally rectangle approximation methods seem the
We can look at that more closely least precise.

$$\text{APPROX}\left(\begin{bmatrix} \text{LEFT\_RIEMANN}\left(\frac{1}{x}, x, 1, 2, 5\right) - \text{LN}(2) \\ \text{right\_riemann}\left(\frac{1}{x}, x, 1, 2, 5\right) - \text{LN}(2) \\ \text{trap}\left(\frac{1}{x}, x, 1, 2, 5\right) - \text{LN}(2) \\ \text{point\_milieu}\left(\frac{1}{x}, x, 1, 2, 5\right) - \text{LN}(2) \\ \text{simpson}\left(\frac{1}{x}, x, 1, 2, 5\right) - \text{LN}(2) \end{bmatrix}, 5\right) = \begin{bmatrix} 0.052487 \\ -0.047512 \\ 0.0024877 \\ -0.0012391 \\ 3.1245\cdot10^{-6} \end{bmatrix}$$

Is it possible to specify the precision of these methods?
Of course, there are traditional demonstrations which allow us, under certain conditions to study this precision formally. But initially, it can be interesting for students who do not know the problem, to have a more experimental approach.
This is the goal of the second part of the paper.

Study of the speed of convergence

We will mainly use the example which we have already met several times before, by preserving the same step in each case.
This step is shown relatively precisely in the first cases and only evoked for the others.

Left rectangle approximation method

Here, we define a new function which measures the variation in absolute value between the result returned by the LRAM for a partition which gives a number of subintervals k and the value given by Derive for ln(2)

$$\text{pr\_lr}(k) := \left| \text{LN}(2) - \text{LEFT\_RIEMANN}\left(\frac{1}{x}, x, 1, 2, k\right) \right|$$

$$\text{APPROX}(\text{pr\_lr}(10), 20) = 0.02562422261$$

$$\text{APPROX}(\text{pr\_lr}(15), 20) = 0.01694429046$$

$$\text{APPROX}(\text{pr\_lr}(20), 20) = 0.01265620123$$

To study the evolution of the precision in accordance with the number of subintervals, it is undoubtedly easier to use to the natural logarithm (and even the opposite of this logarithm since they will only be negative numbers).

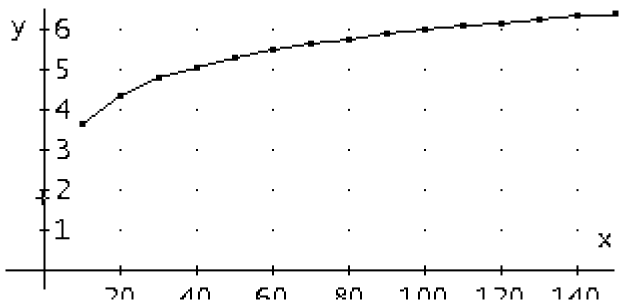
$$\text{log\_pr\_lr}(k) := - \text{LN}(\text{pr\_lr}(k))$$

$$\text{APPROX}(\text{log\_pr\_lr}(10), 3) = 3.66$$

$$\text{APPROX}(\text{log\_pr\_lr}(20), 3) = 4.37$$

We can calculate a table of the values for a few values of k

TABLE(log_pr_lr(k), k, 10, 150, 10)



We plot these points in an adapted cartesian coordinate system (X-coordinates between -10 and 150, Y-coordinates between -1 and 7).
We obtain a curve which ressembles a logarithmic curve.
We then find an "equation" of this adjustment using the function FIT.

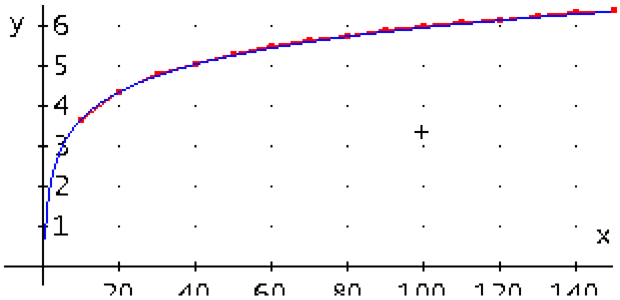
APPROX(FIT([x, a·LN(x) + b], TABLE(log_pr_lr(k), k, 10, 150, 10)), 3)

$$1 \cdot \text{LN}(x) + 1.35$$

What is the quality of this adjustment? We will examine the question on a graphic point of view.

TABLE(log_pr_lr(k), k, 10, 150, 10)

1·LN(x) + 1.35



The adjustment seems to be of very good quality.

If we name p(k) precision as function of k (p(k)=pr_lr(k)), we have roughly

p(x) :=

(let us notice that Derive forces us to define a function p to be able to use it formally)

− LN(p(k)) = LN(k) + 1.35

APPROX(EXP(− LN(p(k)) = LN(k) + 1.35), 3)

$$p(k) = \frac{0.259}{k}$$

This important result shows that in this particular case (but we know that this result spreads) the error made by rectangle approximation method is inversely proportional to the number of subintervals.
Then we start the same thing again with another function.

$$\int_0^1 COS(x)\ dx = SIN(1)$$

$$APPROX\left(\int_0^1 COS(x)\ dx\right) = 0.8414709848$$

pr_lr_cos(k) := |SIN(1) − LEFT_RIEMANN(COS(x), x, 0, 1, k)|

$$0.9910073905 \cdot LN(x) + 1.514564526$$

APPROX(FIT([x, a·LN(x) + b], TABLE(− LN(pr_lr_cos(k)), k, 10, 150, 10)), 3)

$$0.994 \cdot LN(x) + 1.5$$

We find a result of the same magnitude as the precedent.

<u>Trapezoidal method</u>

We use the same step again with the approximation that the trapezoidal method gives us.

We define the function:

$$pr\_trap(k) := \left| LN(2) - trap\left(\frac{1}{x}, x, 1, 2, k\right) \right|$$

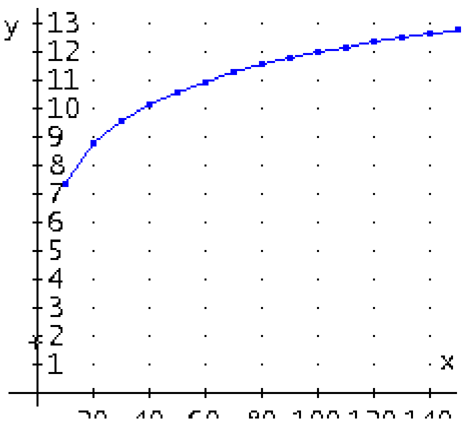
We have for example the following results:

$$APPROX(pr\_trap(10)) = 0.0006242226157$$

$$APPROX(pr\_trap(20)) = 0.000156201234$$

We find the highest degree of accuracy of the trapezoidal method compared to the rectangle approximation method.
We can calculate a table of the values

TABLE(− LN(pr_trap(k)), k, 10, 150, 10)



The shape of the curve is "a logarithmic curve".
We thus try another adjustment of this type for the opposite of the logarithm of the precision (-ln(pr_trap(k))

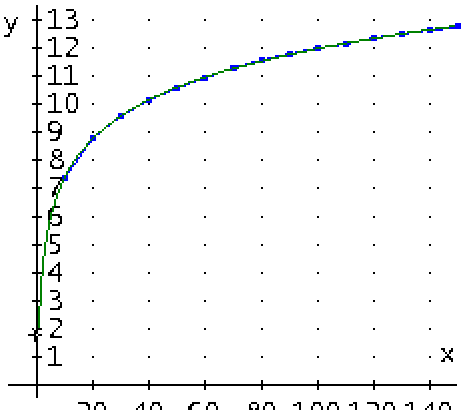
FIT([x, a·LN(x) + b], APPROX(TABLE(− LN(pr_trap(k)), k, 10, 150, 10)))

$$\frac{287937 \cdot LN(x)}{143992} + \frac{6018175}{2169432}$$

This can be rounded-off to 10^-2

$$\mathrm{APPROX}\left(\frac{287937\cdot \mathrm{LN}(x)}{143992} + \frac{6018175}{2169432},\ 3\right) = 2\cdot \mathrm{LN}(x) + 2.77$$

We can once again "check graphically" the quality of the adjustment



We again find a relation between p(k) and k

$$-\ \mathrm{LN}(p(k)) = 2\cdot \mathrm{LN}(k) + 2.77$$

$$\mathrm{APPROX}(\mathrm{EXP}(-\ \mathrm{LN}(p(k)) = 2\cdot \mathrm{LN}(k) + 2.77),\ 3)$$

$$p(k) = \frac{0.0626}{k^2}$$

As for the rectangle approximation method, we can study the other function now.

$$\mathrm{pr\_trap\_cos}(k) := |\mathrm{SIN}(1) - \mathrm{trap}(\mathrm{COS}(x),\ x,\ 0,\ 1,\ k)|$$

$$\mathrm{FIT}([x,\ a\cdot \mathrm{LN}(x) + b],\ \mathrm{APPROX}(\mathrm{TABLE}(-\ \mathrm{LN}(\mathrm{pr\_trap\_cos}(k)),\ k,\ 10,\ 150,\ 10)))$$

$$\frac{4165219\cdot \mathrm{LN}(x)}{2082564} + \frac{1232506}{463817}$$

$$\mathrm{APPROX}\left(\frac{4165219\cdot \mathrm{LN}(x)}{2082564} + \frac{1232506}{463817},\ 3\right) = 2\cdot \mathrm{LN}(x) + 2.65$$

Once again the order of magnitude is identical to that found with the other function.

We shall finish by Simpson’s method

<u>Simpson's méthod</u>

We repeat the same step.

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$$pr\_simp(k) := \left|LN(2) - simpson\left(\frac{1}{x}, x, 1, 2, k\right)\right|$$

$$APPROX(pr\_simp(10)) = 1.941055719 \cdot 10^{-7}$$

$$APPROX(pr\_simp(20)) = 1.218847226 \cdot 10^{-8}$$

The Simpson's method seems to be really the most precise.
We can calculate values of opposite of logarithms of this function which are then plotted.

TABLE(– LN(pr_simp(k)), k, 10, 150, 10)

We find again the same logarithmic curve shape.

FIT([x, a·LN(x) + b], APPROX(TABLE(– LN(pr_simp(k)), k, 10, 150, 10)))

$$\frac{1769473 \cdot LN(x)}{445353} + \frac{3764901}{594470}$$

$$APPROX\left(\frac{1769473 \cdot LN(x)}{445353} + \frac{3764901}{594470}, 3\right) = 3.97 \cdot LN(x) + 6.33$$

We deduce that

$$- LN(p(k)) = 3.97 \cdot LN(k) + 6.33$$

APPROX(EXP(– LN(p(k)) = 3.97·LN(k) + 6.33), 3)

$$p(k) = \frac{0.00178}{k^{3.97}}$$

The precision is thus approximately proportional to the inverse of k^4.
If we look at this result with the other function:

$$pr\_simp\_cos(k) := |SIN(1) - simpson(COS(x), x, 0, 1, k)|$$

FIT([x, a·LN(x) + b], APPROX(TABLE(– LN(pr_simp_cos(k)), k, 10, 150,
10)))

$$\frac{1149027 \cdot LN(x)}{295984} + \frac{5820094}{681195}$$

$$APPROX\left(\frac{1149027 \cdot LN(x)}{295984} + \frac{5820094}{681195}, 3\right)$$

$$3.88 \cdot LN(x) + 8.54$$

This result is equivalent to the precedent.