On the CAS and the coordination of semiotic registers

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1. Introduction

The importance of the semiotic representations and their relations with the cognitive processes have been shown by many researches in Mathematics Education (Artigue, D'Amore, Duval, Gagatsis, Mackie, Pavlopoulou, Tall). Deep learning, that is the conceptual acquisition of a concept, occurs when the pupil is able to pass from a representation in a given register to another one in another register or in the same register. The attention devoted to such topic comes from the ascertainment, remarked by Duval, that in Mathematics the conceptual acquisition of an object is necessarily based on the acquisition of one or more semiotic representations. In particular citing the words of Duval (1995b), we want to underline the importance of such topic for the teaching-learning of Mathematics: *«… the coordination of registers is the condition for the mastery of the comprehension because it is the condition for an actual diversification between the mathematical objects and their representation. It constitutes a threshold whose overcoming radically changes the attitude in front of a kind of activity or a domain (conscience of the overcoming of a threshold, initiative and self control in the development of the procedures…). Now, such coordination has nothing spontaneous. At different levels of learning it is possible to individuate the permanence of a subdivision of the representation registers among themselves. An important factor of such phenomenon of subdivision is the not-congruence between a representation to be converted and the representation of the chosen register».*
From the above considerations some didactical proposals arise. In this paper we want to show how CAS, with direct and active involvement of the student, can improve learning in the above sense. This is because such environments are multiple representation systems, symbolic, graphical, numerical, parametric, logical, … Students are often in front of diverse answers to the same questions (for example solving systems of linear equations in Derive can be done by SOLVE or SOLUTIONS or simply by PLOT) so they are stimulated to concentrate their attention to the meaning of the results obtained by the computer, to establish links among different ways of seeing same formal expression which acquire different meaning in diverse contexts. The ability to recognize such different representations and their common properties conduces to construct the "abstract" concept of a mathematical object or process. Such abstraction is foster by CAS use.

2. Representations and learning in Mathematics: theoretical framework

Looking at mathematics books, we see a wide range of utilised representations: figures, graphs, tables, natural language, symbols. The aim of the books' authors is to make the content as much clear as possible, assuming that the main characteristics of the human intelligence is just the fact that humans are able to use different types of semiotic registers, so it is considered "natural" to handle with multiple representations. On the other hand it is to

be underlined that different descriptions (algebraic, graphical, verbal) make evident various information about the presented concept: the nature of the semiotic register chosen to represent a content imposes to select some significant element of the content to be represented. This means that *each representation is cognitively partial w.r.t. it represents* (Gagatsis, 2003) and different registers presents various aspects of the content, being complementary each other.
The possibility of switching from a register to another one involves functions of economy of treatment, because we can choose the representations more suitable depending on the problem we handle with, for example a graph gives more information than the related algebraic or verbal versions.
As pointed out from Duval, each knowledge is not separable from representations phenomena, that is the comprehension depends on a representation activity, that is conceptualisation (*noesis*) is strictly linked to representation processes (*semiosis*): *«The characteristic feature of mathematical activity is the simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from oneregister to another. To be sure, depending on the domain or the phase of problem-solving one register may explicitly dominate, but there must always be the possibility of passing from one register to another. One can therefore advance the following hypothesis - or in mathematical terms "conjecture" - : comprehension in mathematics assumes the coordination of at least two registers of semiotic representation»* (Duval, 2001).
On the cognitive hand, it is important to stress the importance of the functions related to the representation: the *treatment*, that is the capability to make transformations on the representations staying in a given fixed register, and the *conversion*, that is the passage from a representation in a register to another one in another register, conserving the reference to the same object.
Such activities, we can call *coordination of semiotic registers* and that consists in using spontaneously and quickly different semiotic registers to represent an object, is well known to be indispensable for effective mathematics learning, because only the manipulation of various representations allows to distinguish an object from its representations, that is an essential property of mathematical concept.
According to D'Amore (2001), *"construction of the knowledge in Mathematics"* means exactly the union of the three actions:
- *to represent* the concepts;
- *to treat* the representations obtained within a given register;
- *to convert* the representations from a register to another one.
So the teacher has to foster the students in order they reach a unique "multilinked representation" (Tall, 1991) of the mathematical objects. Just the construction and the recognition of the links and the common properties among the various representations lead the student to construct and own the "abstract concept" which is underlying all the representations (Mackie, 2002).

3. The use of CAS to promote multilinked representations

In this section we try to give an answer to the following question: how the new technologies can foster an effective learning process taking into account the essential functions of construction and manipulation of representations in various semiotic registers?

Handling with mathematics, we concentrate our attention on the Computer Algebra Systems (CAS) (see also Ferrari, 2003).
First of all we make evident that the CAS are for their own nature systems with multiple representations:

- Algebraic/symbolic
- Numerical
- Figural/graphical
- Parametric

- Logical
- …

Thus involving students in activities by such tools means to put them in font of different modalities of express a unique content. For example, given a linear system, Derive offers two algebraic functions to handle them: Solve and Solutions, whose answers are very different! The first one gives an equivalent reduced system: formally we remain in the starting register, the algebraic one, but from the semantic point of view we get an important information regarding linear independence of the equations that is the minimum number of needed equations to have the same set of solutions. The second function we get the answer in another register, the geometric one, explicating the set of the solutions. Both answers are successively translated into the figural register by using the Plot function.
Then the necessity arises that the students are able to catch the equivalence of the answers, correctly interpreting them and using the one more suitable to the particular situation depending on the case. Thus it seems to be a good strategy to introduce all the semiotic registers, encouraging the students to become experienced in treating from a representation to another in the same register and in converting representations in different registers (D'Amore, 2003).
The CAS are tools including mathematical knowledge, that can be used in two way:

- Blackbox: for example considering liner systems, if we want to think about the concept of system, we are not interested in concentrate our attention on the solutions techniques, so it could be good that the CAS computes the solutions without showing the underlying process;
- Whitebox: if we want to learn the Gauss elimination method for computing the solutions of a linear system, I have the opportunity to operate on the equations so to discover the elementary operations, that are the ones transforming the system in an equivalent one, in order to get the solutions of the given system.

Certainly one of the more relevant aspects that the use of the CAS offers is the great capability of visualisation. Many studies have shown how the visualisation fosters the creation of a "multi-linked representation" (for a specific bibliography see Mackie, 2002). In particular, the CAS offer, beyond the static modality, the dynamical visualisation supplying with the possibility to "construct" the objects. In the example we show in details in the following, concerning linear systems, the fact of giving a system rather than directly the coordinates of the point means to furnish some more information, because in the writing of the system there are not only the coordinates of the point but we can discover a "method to construct" the point. This one in fact comes from three planes and, as suggested by students themselves (Albano et al., 2003), it can be "assemble step by step" by a CAS, visualising each plane for time and then their intersection that, at first time, will be a line, and successively will be the point given by the intersection between the line and the third plane. Actually such construction is not unique, because the order with which the planes can be intersect is unessential, and this means, coming back to the algebraic register, that the order of the equations is not a requirement w.r.t. the solutions of the system, that is the interchange of equations is an elementary operation. This is just an example of how the possibility of "manipulating" in some way the mathematical objects allows to investigate mathematical properties (algebraic, geometrical, logical, …) of the concepts or of the process in stake.
As final remark of this section we want to emphasise that the most interesting activity allowed by the use of the CAS is that the offered chance to manipulate, explore, conjecture and verify permits the students to move on transversal paths within the knowledge graph. This matches the recent theories according to which learning is no more a sequential process but it needs continuous reconstructions and reorganisations of the various concepts, indispensable to make able an effective flexibility among the different meanings, viewpoints and representations (Artigue, 1999).

CONTS

4. A case study in linear algebra: first explorations

Let us give an overview of the possibilities given by a CAS, like DERIVE, to treat different semiotic registers. In this section we present a case study, in which Derive is used to treat a linear system.
Concerning linear equations, we have the following ones:

- *algebraic*: solution as reduced system (SOLVE), or as parametric description (SOLUTIONS); in both the cases we work with infinity precision;
- *figural*: solution as graph (PLOT); in particular such graphical function can be managed in static and above all in dynamic way;
- *numerical*: moving the cursor on the graph it is possible to read the numeric coordinates of the points (TRACE), or in the case of a single equation the solution can be found by NSOLVE and NSOLUTIONS, working with finite precision.

Let us consider a single equation and explore which tools we have to handle it.

We can apply the two functions able to solve the equation.

#1: $\mathrm{SOLVE}([2\cdot x + y + 2\cdot z - 1 = 0], [x, y, z])$

#2: $$[2\cdot x + y + 2\cdot z = 1]$$

#3: $\mathrm{SOLUTIONS}([2\cdot x + y + 2\cdot z - 1 = 0], [x, y, z])$

#4: $$\left[\left[@1,\ @2,\ -\frac{2\cdot @1 + @2 - 1}{2}\right]\right]$$

In the first case the answer gives the same equation, in the latter case we obtain the parametric description of the equation. We can show the equivalence using the SUBSTITUTION function as follows:

#5: $$\mathrm{SUBST}\left([2\cdot x + y + 2\cdot z - 1 = 0], [x, y, z], \left[@1,\ @2,\ -\frac{2\cdot @1 + @2 - 1}{2}\right]\right)$$

#6: $$[\mathrm{true}]$$

As further representation we can have a table of vectors satisfying the equation, both in exact or finite precision:

#7: $$\mathrm{TABLE}\left(\mathrm{TABLE}\left(\left[@1,\ @2,\ -\frac{2\cdot @1 + @2 - 1}{2}\right], @1, 1, 5, 1\right), @2, 1, 5, 1\right)$$

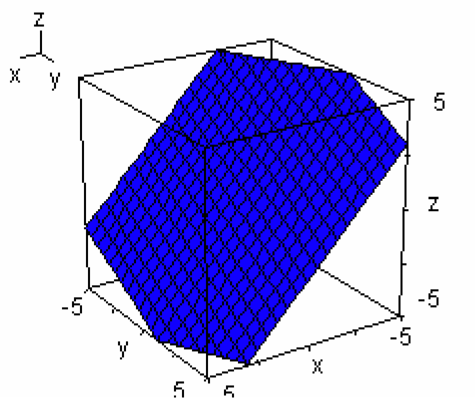
$$\left[\begin{array}{cc} & \\ & \\ 1 & \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 2 & 1 & -2 \\ 3 & 3 & 1 & -3 \end{bmatrix} \end{array}\right.$$

#8:

$$
\left|\begin{array}{cc}
 & \left[\begin{array}{cccc} 4 & 4 & 1 & -4 \\ 5 & 5 & 1 & -5 \end{array}\right] \\
2 & \left[\begin{array}{cccc} 1 & 1 & 2 & -\frac{3}{2} \\ 2 & 2 & 2 & -\frac{5}{2} \\ 3 & 3 & 2 & -\frac{7}{2} \\ 4 & 4 & 2 & -\frac{9}{2} \\ 5 & 5 & 2 & -\frac{11}{2} \end{array}\right] \\
3 & \left[\begin{array}{cccc} 1 & 1 & 3 & -2 \\ 2 & 2 & 3 & -3 \\ 3 & 3 & 3 & -4 \\ 4 & 4 & 3 & -5 \\ 5 & 5 & 3 & -6 \end{array}\right] \\
4 & \left[\begin{array}{cccc} 1 & 1 & 4 & -\frac{5}{2} \\ 2 & 2 & 4 & -\frac{7}{2} \\ 3 & 3 & 4 & -\frac{9}{2} \\ 4 & 4 & 4 & -\frac{11}{2} \\ 5 & 5 & 4 & -\frac{13}{2} \end{array}\right] \\
 & \left[\begin{array}{cccc} 1 & 1 & 5 & -3 \\ 2 & 2 & 5 & -4 \end{array}\right]
\end{array}\right|
$$

$$\left[\begin{array}{cc} 5 & \begin{bmatrix} 3 & 3 & 5 & -5 \\ 4 & 4 & 5 & -6 \\ 5 & 5 & 5 & -7 \end{bmatrix} \end{array}\right]$$

Finally we can PLOT the equation.



Let us consider a system of two equations and apply the SOLVE function.

#9: SOLVE([2·x + y + 2·z − 1 = 0, 4·x + 2·y − z − 2 = 0], [x, y, z])

#10: [2·x + y = 1 ∧ z = 0]

The answer is given in the same register of the input, that is an equivalent linear system, just simpler. Moreover SOLVE answer uses the logical connective AND.

Second, we can use:

#11: SOLUTIONS([2·x + y + 2·z − 1 = 0, 4·x + 2·y − z − 2 = 0], [x, y, z])

#12: [[@1, 1 − 2·@1, 0]]

In this case we have a different register, the parametric form.
Confronting the two answers, we can note that both point out an important information, that is the third coordinate z is 0, which is not evident in the input system form.
The linear system given by SOLVE explicit such information with the equation z=0, so even if the answer of SOLVE is of the same type of the input, it is such that some characteristics of the input system are made available.

Some open problems promptly arise:
- what does the system in stake represents?
- the three representations we deal with are equivalent? And how we can see that?
We try to give an answer in the following section.

5. Examples of treatment/conversion

First we want to explore treatment operations, that is how we can change representation in a fixed register.

o *Within the algebraic register*: If we call SOLUTIONS function on a linear system we have a parametric description of the system:

#13: SOLUTIONS([2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0, x − 2·y − 2·z − 1 = 0], [x, y, z])

#14: $$\left[\left[\frac{3}{5},\ @2,\ -\frac{5\cdot @2 + 1}{5}\right]\right]$$

So first of all we need to prove that the Cartesian and the parametric expression are equivalent. To this aim it is sufficient to show that the vectors whose components are those given by SOLUTIONS satisfy the given system, using SUBSTITUTION function.

#15: $$\mathrm{SUBST}\left([2\cdot x + y + z - 1 = 0 \wedge 3\cdot x - y - z - 2 = 0 \wedge x - 2\cdot y - 2\cdot z - 1 = 0],\ [x, y, z],\ \left[\frac{3}{5},\ @2,\ -\frac{5\cdot @2 + 1}{5}\right]\right)$$

#16: [true]

If we call the SOLVE function for a system of two independent equations, we get another equivalent representation, consisting in different planes. This allows to introduce from the algebraic viewpoint the concept of equivalent systems.

#17: SOLVE([2·x + y + 2·z − 1 = 0 ∧ 4·x + 2·y − z − 2 = 0], [x, y, z])

#18: [2·x + y = 1 ∧ z = 0]

Using the algebraic register, we can show that the two systems are equivalent calling SOLUTION on both and observing that the new answers coincides.

#19: SOLUTIONS([2·x + y + 2·z − 1 = 0 ∧ 4·x + 2·y − z − 2 = 0], [x, y, z])

#20: [[@2, 1 − 2·@2, 0]]

#21: SOLUTIONS([2·x + y = 1 ∧ z = 0], [x, y, z])

#22: [[@1, 1 − 2·@1, 0]]

Alternatevely we can compute the set of solutions of one system and substitute in the other one.

#23: SUBST([2·x + y + 2·z − 1 = 0 ∧ 4·x + 2·y − z − 2 = 0], [x, y, z], [@1, 1 − 2·@1, 0])

#24: [true]

In the case of a system with linear dependent equations, using the SOLVE function we get an equivalent reduced system with less equations.

#25: SOLVE([2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0, x − 2·y − 2·z − 1 = 0], [x, y, z])

#26: $$\left[x = \frac{3}{5} \wedge y + z = -\frac{1}{5}\right]$$

From the algebraic viewpoint, we can show the equivalence applying SOLUTIONS to both systems. We obtain the same answer, so we have shown that the equations of the given systems are redundant.

#27: SOLUTIONS([2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0, x − 2·y − 2·z − 1 = 0], [x, y, z])

#28: $$\left[\left[\frac{3}{5}, @1, -\frac{5\cdot @1 + 1}{5}\right]\right]$$

#29: $$\mathrm{SOLUTIONS}\left(\left[x = \frac{3}{5}, y + z = -\frac{1}{5}\right], [x, y, z]\right)$$

#30: $$\left[\left[\frac{3}{5}, @2, -\frac{5\cdot @2 + 1}{5}\right]\right]$$

All the previous algebraic expressions are continuous descriptions of the system, but we can remain in the algebraic description and give a discrete explanation of the solutions. In fact, starting from the parametric form of the solutions, it is possible to construct a table of values:

#31: $$\mathrm{TABLE}\left(\left[\frac{3}{5}, @1, -\frac{5\cdot @1 + 1}{5}\right], @1, 1, 10, 1\right)$$

$$\begin{bmatrix} 1 & \frac{3}{5} & 1 & -\frac{6}{5} \\ 2 & \frac{3}{5} & 2 & -\frac{11}{5} \\ 3 & \frac{3}{5} & 3 & -\frac{16}{5} \\ 4 & \frac{3}{5} & 4 & -\frac{21}{5} \\ 5 & \frac{3}{5} & 5 & -\frac{26}{5} \end{bmatrix}$$

#32: $$\begin{bmatrix} 6 & \frac{3}{5} & 6 & -\frac{31}{5} \\ 7 & \frac{3}{5} & 7 & -\frac{36}{5} \\ 8 & \frac{3}{5} & 8 & -\frac{41}{5} \\ 9 & \frac{3}{5} & 9 & -\frac{46}{5} \\ 10 & \frac{3}{5} & 10 & -\frac{51}{5} \end{bmatrix}$$

- *Within the figural register*. Let us consider the following system:

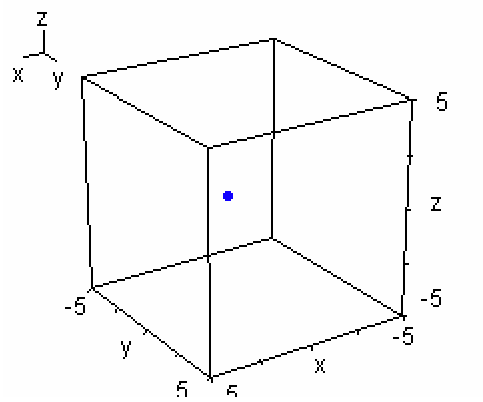
#33: [2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0, x − y + z − 1 = 0]

whose solution is a point, as shown solving it:

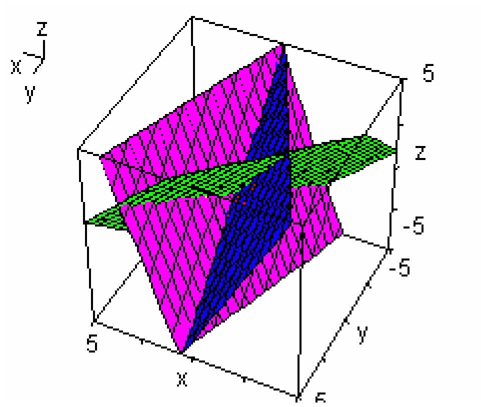
#34: SOLUTIONS([2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0, x − y + z − 1 = 0], [x, y, z])

#35: $$\left[\left[\frac{3}{5}, -\frac{3}{10}, \frac{1}{10}\right]\right]$$

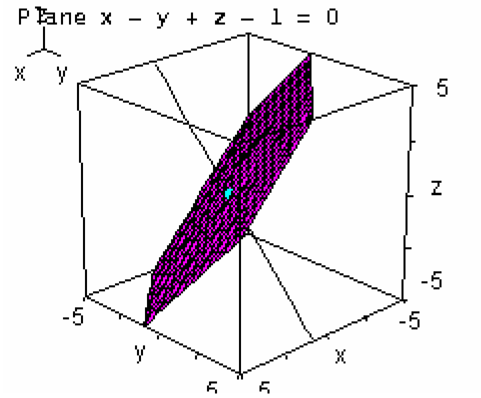
From the figural point of view, we can describe such point in many different ways:
- just as the point:

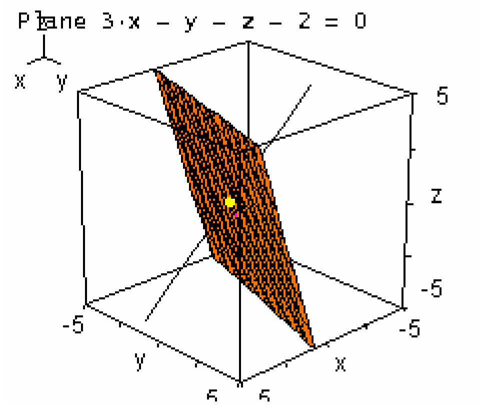


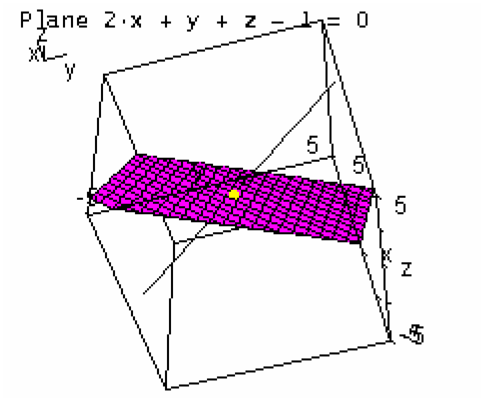
- as intersection of the three planes:



- as intersection of the line generated by two planes and the third plane. In this case we have more than one possibilities depending on the choice of the two planes:



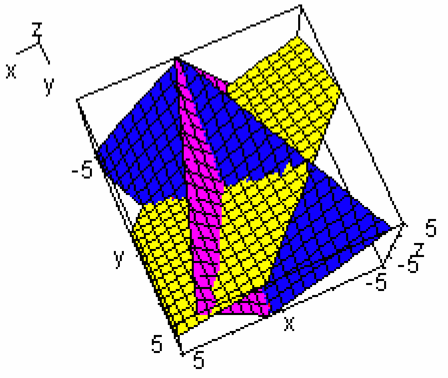




How can we change register? Let us see.

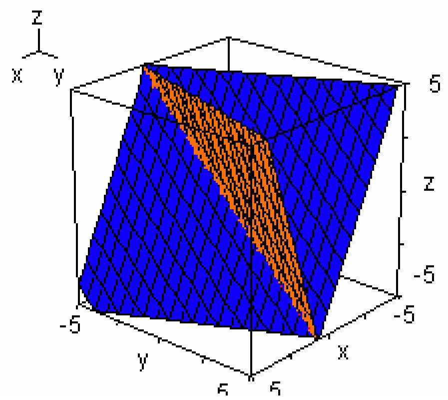
o *From the algebraic register to the static figural one*: it is sufficient to use PLOT on the system, highliting all the equations

#36: [2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0, x − y + z − 1 = 0]



o *From the algebraic register to the dynamic figural one*: here we appeal to the step by step construction. First we plot the first two planes corresponding to the first two equations of the system and we graphically can see if they intersect or not and how.

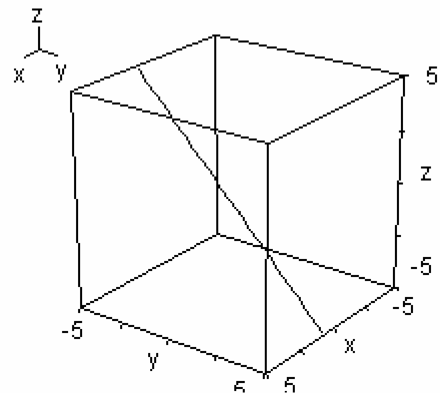
#37: [2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0]



Calling SOLUTIONS on the the system of the two previous planes, we get the parametric descrition of the line (intersection) and we can directly plot it.

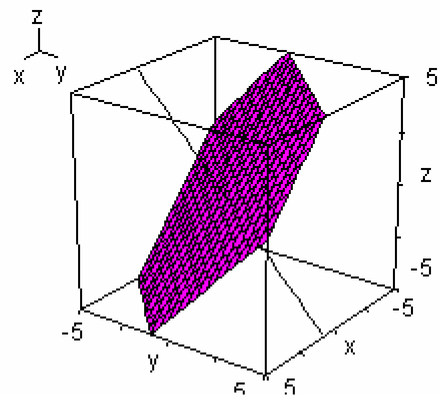
#38: SOLUTIONS([2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0], [x, y, z])

#39: $$\left[\left[\frac{3}{5},\ @1,\ -\frac{5\cdot @1+1}{5}\right]\right]$$



Finally, plotting the third equation, we have the solution of the given system.

$$\#40: \quad \left[\left[\frac{3}{5},\ @1,\ -\frac{5\cdot @1+1}{5}\right],\ x-y+z-1=0\right]$$



o *From the figural register to the numeric one*: if we move on the graph by the cursor, we can read the values of the coordinates of the points on the figure. In this case we can remark the non continuity of the representation of a line (for example) w.r.t. the continuity of the real line.

o *From the algebraic register to the numeric one*: let us consider the parametric form of the solutions of a system of two equation, we can have a table of values that are in algebraic form too, that is symbolic expressions of numbers (e.g. fraction of integers), but using APPROXIMATE we can get numeric values of the solutions.

$$\begin{bmatrix} 1 & \frac{3}{5} & 1 & -\frac{6}{5} \\ 2 & \frac{3}{5} & 2 & -\frac{11}{5} \\ 3 & \frac{3}{5} & 3 & -\frac{16}{5} \\ 4 & \frac{3}{\ } & 4 & -\frac{21}{\ } \end{bmatrix}$$

$$
\#41: \quad \left[\begin{array}{cccc} & 5 & & 5 \\ 5 & \frac{3}{5} & 5 & -\frac{26}{5} \\ 6 & \frac{3}{5} & 6 & -\frac{31}{5} \\ 7 & \frac{3}{5} & 7 & -\frac{36}{5} \\ 8 & \frac{3}{5} & 8 & -\frac{41}{5} \\ 9 & \frac{3}{5} & 9 & -\frac{46}{5} \\ 10 & \frac{3}{5} & 10 & -\frac{51}{5} \end{array}\right]
$$

$$
\#42: \quad \left[\begin{array}{cccc} 1 & 0.6 & 1 & -1.2 \\ 2 & 0.6 & 2 & -2.2 \\ 3 & 0.6 & 3 & -3.2 \\ 4 & 0.6 & 4 & -4.2 \\ 5 & 0.6 & 5 & -5.2 \\ 6 & 0.6 & 6 & -6.2 \\ 7 & 0.6 & 7 & -7.2 \\ 8 & 0.6 & 8 & -8.2 \\ 9 & 0.6 & 9 & -9.2 \\ 10 & 0.6 & 10 & -10.2 \end{array}\right]
$$

6. A new “treatment” … through the conversion

In this section we present a new method to show the equivalence of different representations in a same fixed register using a conversion operation.
As first example we consider the following:

```
#43:  SOLUTIONS([2·x + y + z - 1 = 0, 3·x - y - z - 2 = 0, x - y + z - 1
          = 0], [x, y, z])
```

#44: $$\left[\left[\frac{3}{5}, -\frac{3}{10}, \frac{1}{10}\right]\right]$$

The system and the array have the same "abstract" meaning, that is we are talking of a point, but in the latter case we have a blackbox form, whilst the first form give a whitebox method to better understand how that point is originated because the system gives a constructive definition of the point that can be explicit using PLOT function.
The equivalence of the two previous expression can be proved using the conversion from the algebraic register to the dynamical figural one, as shown in the previous section.
This gives a first example of conversion using the figural register. In fact an equation has a corresponding geometric meaning, so first of all we can use PLOT for each equation to "see" the objects we are treating. In the case of a system, also the sign of braces has a corresponding geometric meaning: if we are studying a system of three equations, in order to "see" the concept of system, we need to plot all the three equations on the same sheet, and the system is represented by the part of the graph common to all the figures.

In the following we consider we consider various cases where two algebraic representations of a linear system are proved to be equivalent taking advantage of figural representations. Let us see in details.

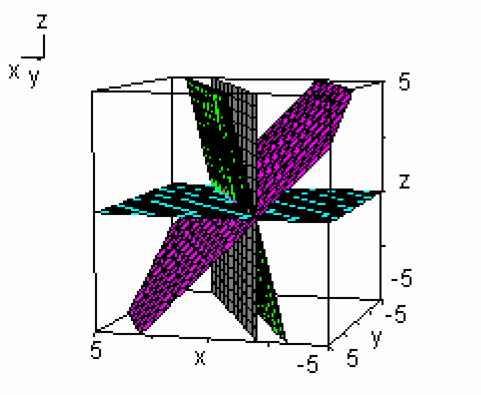
- *Two Cartesian representations*. Let us consider a given linear system and the corresponding answer of SOLVE function on it.

#45: SOLVE([2·x + y + 2·z − 1 = 0 ∧ 4·x + 2·y − z − 2 = 0], [x, y, z])

#46: [2·x + y = 1 ∧ z = 0]

In order to show that the two systems are equivalent, in traditional class we can appeal to the matrices and rank theory, show that the equations of a system are linearly dependent on the equations of the other and conversely, or simply compute the solutions of a system and show that they are solutions of the other one and conversely. All these procedures are complicated due to the involvement of advanced theory or to great amount of calculations.

Using the figural register, we can prove the equivalence showing that in both the case we get the same figural representation (in our case a line).



Note that such method is also interesting because it allows to introduce from the geometric viewpoint the concept of not unique Cartesian representation of a line in 3D-space.

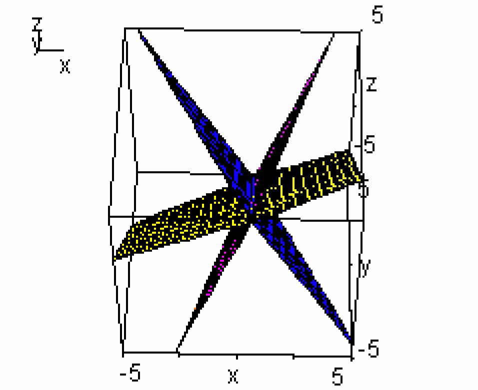
Considering system with linear dependent equations, such the following

#47: SOLVE([2·x + y + z − 1 = 0, 3·x − y − z − 2 = 0, x − 2·y − 2·z − 1

$= 0], [x, y, z])$

#48: $$\left[x = \frac{3}{5} \wedge y + z = -\frac{1}{5}\right]$$

showing the equivalence from the figural viewpoint, plotting each equation, we can note that the third one gives a plane that intersects the previous two ones in the same line where they meet each other, so the third plane is not necessary for defining that line.



On the algebraic hand this means that the third equation does not contribute to the solutions of the system of the previous equations.

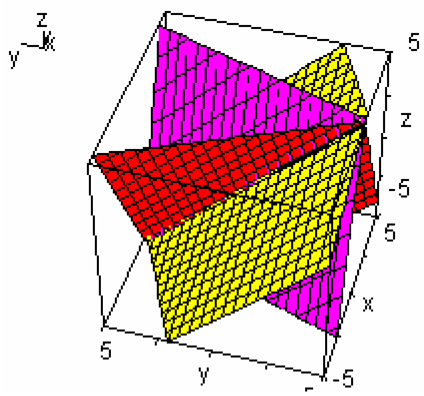
- *A Cartesian representation and a parametric one*. Let us consider the following systems:

#49: $2\cdot x + y + z - 1 = 0 \wedge 3\cdot x - y - z - 2 = 0 \wedge x - 2\cdot y - 2\cdot z - 1 = 0$

#50: $x = \frac{3}{5} \wedge y = t \wedge z = -t - \frac{1}{5}$

In traditional class, to the aim of proving the equivalence, we susbstitute the parametric equations in the other ones and we get an identity, or solve the first system and show that the set of the solutions is exactly that described by the parametric system.

Using the figural register, we plot the three planes and the points whose coordinates are given by the parametric system and immediately see that the thre planes described by the equations of the Cartesian system intersect exactly in the line given by the parametric system: then the equality!



- *Two parametric representations*. Let us consider the following parametric systems:

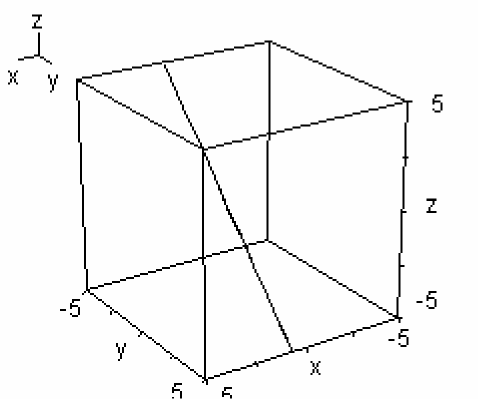
$$\#51: \quad x = \frac{3}{5} \wedge y = t \wedge z = -t - \frac{1}{5}$$

$$\#52: \quad x = \frac{3}{5} \wedge y = -1 + 2 \cdot t \wedge z = \frac{4}{5} - 2 \cdot t$$

In traditional class, we can compute the Cartesian expression of one system and then show the equivalence as in the previous case. From figural viewpoint, the equivalence is immediately proved plotting the two parametric points and simply observing that we obtain the geometrical object.

$$\#53: \quad \left[\frac{3}{5}, -1 + 2 \cdot t, \frac{4}{5} - 2 \cdot t\right]$$

$$\#54: \quad \left[\frac{3}{5}, t, -t - \frac{1}{5}\right]$$



From the geometrical viewpoint last remark allows to introduce the geometrical concept of not unique parametric representation of a line.

7. Conclusions

It is common opinion among researchers in Mathematics Education that effective learning needs the acquisition of more representations in different semiotic registers and the capabilities to manage them. In this paper we give a first example of how to foster such process in the context of CAS based teaching.

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