# The Combinatorial Matrix Approach on Symbolic Polynomial Systems

**Jianong Zhou**
NCCT Laboratory, University of Ottawa
Ottawa ON. K1N 6N5 Canada
jnzhou@hotmail.com

**Abstract** A new approach of the Combinatorial Matrix Approach for eliminating several variables simultaneously on symbolic polynomial systems is presented. It is proved that in the case of the polynomial systems with bi-variable in two degrees or with bi-variable in three degrees it is more efficient than other methods such as the Wu's Elimination and the Gröbner Bases Approach even the Dixon approach.

We know that, in computer algebra, to efficiently eliminate the variables in a symbolic polynomial system is very crucial to a computer algorithm and a software to solve the realistic problems. For example, in machine proving and in computer automated reasoning, there are a bunch of symbolic polynomial systems in which the variables needed to be eliminated. Some of the problems are difficult to be solved with the existed eliminating methods such as Wu's elimination and the Gröbner Bases Approach.

The Combinatorial Matrix Approach focuses on how to derive a linear symbolic system from a nonlinear polynomial system. Then solve the equivalent linear system instead of the nonlinear system. Several examples are given to show the efficiency of the method. Comparisons among the various methods are also made here.

**Key words:** Combinatorial Matrix, Dixon resultant, Symbolic polynomial systems, Multi-variable elimination

## 1. Introduction

In the real world, many problems can be described by algebraic polynomial equations. Especially, in the geometry area, many geometrical problems are corresponding to some polynomial equation systems. Take the Apollonius problem as an example. The Apollonius problem says: Given three circles in the plane, find or construct a circle tangent to all three (it will be discussed later). It is said that a biochemistry research problem of computing the medical axis of the space around a molecule is related to the Apollonius problem [1,2]. Besides, in many geometrical theorems involving incidence, congruence, distance and parallel relations, the conditions and the conclusions are also expressed as algebraic equations respectively [3]. Among those equation systems, there are many polynomial equation systems with symbolic coefficients. To find of the solution for the equation systems is the key to solving the problems. There exist many different techniques to solve a polynomial equation system or eliminate variables from the polynomial equation system. The earlier and more popular way to get the information from the equation system is to use the resultant computations. The most famous resultant to determine the common solution for two polynomial equations with one variable is the Sylvester resultant, which was implemented in the software Maple. By computing the Sylvester resultant successively, one can solve a multi-variable polynomial equation system by eliminating variables one at a time. But the successive eliminating method is very inefficient. Another alternative method is the Characteristic set method [4], which implements the Wu's elimination method to get a set of triangular lists of the multi-variable polynomial equation systems. Though the method is perfect in theory, in practice it doesn't seem to work well since it eliminates variables one at a time and must deal with the branch situations. So does the Gröbner bases method. The break through work was done by D. Kapur, T. Saxena and L. Yang. They proposed a method called Dixon resultant method, with which one can eliminate several variable simultaneously. Besides, the method is fully automatic and needs no human intervention. With it the time complexity can be reduced dramatically [5]. The Dixion resultant method can be applied to a system with $n$+1 *generic n-degree* polynomials [5] in $n$ variables. It gives the resultant of $n$+1 *generic n-degree* polynomials. For arbitrary set of $n$+1 non-homogeneous polynomials with $n$ variables, it gives a necessary condition for the existence of a common affine zero. Many intractable geometric and algebraic problems which once were difficult to be solved by the Characteristic set method and the Gröbner bases method were attacked by the Dixon resultant method. However, since the Dixon resultant method produces the derived set of polynomials automatically, the derived polynomials usually have higher degree. And sometimes the Dixon resultant vanishes identically without giving any information.

To improve the Dixon resultant method, this paper presents a new approach for extracting information from a given polynomial system, which is called the *Combinatorial Matrix Approach*. The Combinatorial Matrix method tries to get independent polynomials by constructing the polynomial combinations from the given system. Then it constructs a combination matrix, which is formed by the coefficients of the original polynomials and the derived polynomials. The determinant of the combination matrix is called a combination resultant. The vanishing of the combination resultant is the necessary condition for the existence of a common zero for the given polynomial system. There are several advantages of the Combinatorial Matrix Approach. First, the combinatorial matrix method can also eliminate several variables simultaneously. Second, the combinatorial matrix method can reduce the total degree of the coefficients of the determinant of the combinatorial matrix pertaining to the coefficients of the original polynomials. The reducing of the degree of the coefficients of the combinatorial resultant can contribute to the decreasing of the time complexity. Third, the combinatorial matrix method has more freedom to produce the independent polynomials, which allows one to get the polynomials with lower degrees. Furthermore, the time complexity is reduced and the efficiency to eliminate variables is increased.

In section 2 of this paper, we will review the Dixon resultant method. Then in section 3 we will introduce the combinatorial matrix approach by solving a polynomial system with 3 generic 2-degree polynomials in 2 variables and 3 generic 3-degree polynomials in 2 variables. We will give several examples by using the Combinatorial Matrix approach in section 4. Finally, the several methods will be compared in section 5.

## 2. Review of Dixon resultant method

At first we recall a well known fact that for a given polynomial system $P=\{p_1, p_2, \ldots, p_n\}$, if another polynomial system $Q=\{q_1, q_2, \ldots, q_s\}$ is derived from the system $P$, then the zero set of $P$ is included in the zero set of the derived system $Q$. That means

$$Zero(P) \subseteq Zero(Q). \tag{2.1}$$

Then instead of finding the zero set of the polynomial system $P$ directly from $P$, we turn to find the solutions from the zero set of the system $Q$ by sifting the spurious factors ( Gather-and-sift [6]).

Let $P=\{p_1(x_1,x_2,\ldots,x_n), p_2(x_1,x_2,\ldots,x_n),\ldots, p_{n+1}(x_1,x_2,\ldots,x_n)\}$ be the set of $n$+1 *generic n-degree* polynomials in $n$ variables and $d_i$ = max( $degree(p_j, x_i)$ | $j$= 1,2,…,$n$+1) for $i$=1,2,…,$n$. Define the polynomial $\Delta(x_1,x_2,\ldots,x_n,\alpha_1,\alpha_2,\ldots,\alpha_n)$ as the following determinant:

$$\begin{vmatrix} p_1(x_1,x_2,\ldots,x_n) & p_2(x_1,x_2,\ldots,x_n) & \ldots\ldots & p_{n+1}(x_1,x_2,\ldots,x_n) \\ p_1(\alpha_1,x_2,\ldots,x_n) & p_2(\alpha_1,x_2,\ldots,x_n) & \ldots\ldots & p_{n+1}(\alpha_1,x_2,\ldots,x_n) \\ p_1(\alpha_1,\alpha_2,\ldots,x_n) & p_2(\alpha_1,\alpha_2,\ldots,x_n) & \ldots\ldots & p_{n+1}(\alpha_1,\alpha_2,\ldots,x_n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ p_1(\alpha_1,\alpha_2,\ldots,\alpha_n) & p_2(\alpha_1,\alpha_2,\ldots,\alpha_n) & \ldots\ldots & p_{n+1}(\alpha_1,\alpha_2,\ldots,\alpha_n) \end{vmatrix} \tag{2.2}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_n$ are new variables. Obviously, each of $x_i=\alpha_i$ is a zero of $\Delta$, $i$=1,2,…,$n$. So we can remove the factor $\Pi_{i=1}^{n}(x_i-\alpha_i)$ from the $\Delta$ to get a polynomial:

$$\delta(x_1,x_2,\ldots x_n,\alpha_1,\alpha_2,\ldots,\alpha_n)=\frac{\Delta(x_1,x_2,\ldots,x_n,\alpha_1,\alpha_2,\ldots,\alpha_n)}{\prod_{i=1}^{n}(x_i-\alpha_i)} \tag{2.3}$$

The polynomial $\delta$ is called the **Dixon polynomial,** which is of degree $((n+1-i)\times d_i)$-1 in $\alpha_i$ and $(i\times d_i)$-1 in $x$ for $i$=1,2,…,$n$. If $x_1=c_1, x_2=c_2,\ldots, x_n=c_n$ is a common zero of the system $P$, it makes the Dixon polynomial vanish, no matter what the value $\alpha_1$ , $\alpha_2,\ldots\alpha_n$. That means that all the coefficients of the various power products of $\alpha_1,\alpha_2,\ldots,\alpha_n$ in $\delta(x_1,x_2,\ldots,x_n,\alpha_1,\alpha_2,\ldots,\alpha_n)$ vanish. If regard the polynomial $\delta$ as the polynomial of the power products of $\alpha_1,\alpha_2,\ldots,\alpha_n$, one can find that there are just $\prod_{i=1}^{n}((n+1-i)\times d_i)= n!\prod_{i=1}^{n}d_i = s$ coefficients, which correspond to $s$ equations called **Dixon Derived System**:

$$\Gamma: \{ f_1=0, f_2=0,\ldots, f_s=0 \}. \tag{2.4}$$

The variables in all of those equations are power products of $x_1, x_2,\ldots, x_n$. And there are exact

$$\Pi_{i=1}^{n} i\times d_i = n!\,\Pi_{i=1}^{n} d_i = s$$

power products of $x_1, x_2,\ldots,x_n$. So there are just $s$ equations and $s$ variables in $\Gamma$. If let $D$ be the $s\times s$ coefficients matrix of $\Gamma$ and, $v_1$ = 1, $v_2=x_2$, $v_3=x_1x_2,\ldots,v_s=\Pi_{i=1}^{n}x_i^{i\times d_i-1}$, then

$$\Gamma: \quad D\,(v_1,v_2,\ldots,v_s)^{tr}=(0, 0, \ldots, 0)^{tr}. \tag{2.5}$$

If $P$ has a common zero (say $x_1=c_1, x_2=c_2,\ldots, x_n=c_n$), so does the $\Gamma$, then the det($D$) vanishes. Therefore det($D$)=0 is the necessary condition on the coefficients of $p_1, p_2,\ldots, p_n$ for them have a common zero. The matrix $D$ and its determinant det($D$) are called Dixon Matrix and Dixon resultant.

If the Dixon matrix $D$ is singular, [5] suggested that we can take one $D$'s maximum sub-matrix, say $R$, to get the necessary condition for existing the common zero for polynomial system $P$.

The following example shows how to solve geometric problem by using of Dixon resultant method.

**Example** ( Apollonius problem ) Given three circles on the plane, find other circles tangent to all three. Suppose the equations for three circles are as follows:

$$\boldsymbol{C}_1\boldsymbol{:}\ (x-a_1)^2+(y-b_1)^2=r_1^2,\ \boldsymbol{C}_2\boldsymbol{:}\ (x-a_2)^2+(y-b_2)^2=r_2^2,\ \boldsymbol{C}_3\boldsymbol{:}\ (x-a_3)^2+(y-b_3)^2=r_3^2.$$

Let $C$ denote the solution circle with center $(x, y)$ and radius $r$. If two circles is tangent to each other, the distance between their centers is equal to the algebraic sum of their radii. Then the solution circle must satisfy the following equations:

$$\begin{cases} p_1(x,y) \equiv (x-a_1)^2+(y-b_1)^2-(r\pm r_1)^2 = 0 \\ p_2(x,y) \equiv (x-a_2)^2+(y-b_2)^2-(r\pm r_2)^2 = 0 \\ p_3(x,y) \equiv (x-a_3)^2+(y-b_3)^2-(r\pm r_3)^2 = 0 \end{cases} \tag{2.6}$$

Theoretically, there are eight cases of the solution corresponding to the eight kinds of choices of the signs on the right side of the equations. Among eight cases just one case that three circles are outside the solution circle (take all signs positive).

**Step 1** Construct the Dixon polynomial:

$$\begin{cases} p_1(\alpha,y) \equiv (\alpha-a_1)^2+(y-b_1)^2-(r+r_1)^2, \\ p_2(\alpha,y) \equiv (\alpha-a_2)^2+(y-b_2)^2-(r+r_2)^2, \\ p_3(\alpha,y) \equiv (\alpha-a_3)^2+(y-b_3)^2-(r+r_3)^2, \\ p_1(\alpha,\beta) \equiv (\alpha-a_1)^2+(\beta-b_1)^2-(r+r_1)^2, \\ p_2(\alpha,\beta) \equiv (\alpha-a_2)^2+(\beta-b_2)^2-(r+r_2)^2, \\ p_3(\alpha,\beta) \equiv (\alpha-a_3)^2+(\beta-b_3)^2-(r+r_3)^2. \end{cases} \tag{2.7}$$

**Step 2** Construct the Dixon polynomial as follows:

$$\delta(x,y,\alpha,\beta) = \frac{1}{(x-\alpha)(y-\beta)} \begin{vmatrix} p_1(x,y) & p_2(x,y) & p_3(x,y) \\ p_1(\alpha,y) & p_2(\alpha,y) & p_3(\alpha,y) \\ p_1(\alpha,\beta) & p_2(\alpha,\beta) & p_3(\alpha,\beta) \end{vmatrix} \tag{2.8}$$

Theoretically, here $d_1=d_2=2$ and s=2!* $d_1$*$d_2$=8. The Dixon matrix $D$ is $8\times 8$ matrix. However the rank of the Dixon matrix is 3. So det($D$)=0 identically. Fortunately, the Dixon matrix has a $3\times 3$ non-singular sub-matrix. The calculating result shows that the Dixon polynomial just has three terms:

$$\delta(x,y,\alpha,\beta) = c_1(x,y)\alpha + c_2(x,y)\beta + c_3(x,y), \tag{2.9}$$

where

$c_1(x,y)$=( 4*b2*a3-4*b3*a2+4*a1*b3-4*a1*b2-4*b1*a3+4*b1*a2)*x+(2*b1*b3^2-2*b1*r3^2-2*b1*a2^2+2*b3*b2^2+2*b1^2*b2+2*r1^2*b3+2*b1*a3^2+2*a1^2*b2-2*a1^2*b3-2*b3*r2^2-2*b2*a3^2-2*r1^2*b2-2*b1*b2^2-2*b1^2*b3-2*b2*b3^2+2*b2*r3^2+2*b3*a2^2+2*b1*r2^2-4*b3*r2*r+4*b1*r2*r-4*b1*r3*r+4*b2*r3*r+4*r1*r*b3-4*r1*r*b2);

$c_2(x,y)$=( 4*b2*a3-4*b3*a2+4*a1*b3-4*a1*b2-4*b1*a3+4*b1*a2)*y+(2*a2*b3^2+2*a1*r3^2-2*b2^2*a3+2*a2*a3^2+2*a1*b2^2+2*r1^2*a2-4*a2*r3*r+2*b1^2*a3+4*a1*r3*r-2*a2^2*a3-2*r1^2*a3-2* a1*a3^2+4*r2*r*a3-2*a2*r3^2+4*r1*r*a2-4*a1*r2*r-4*r1*r*a3+2*a1*a2^2-2*a1*b3^2+2*a1^2*a3-2*a1*r2^2-2*a1^2*a2-2*b1^2*a2+2*r2^2*a3);

$c_3(x,y)$=(2*b1*b3^2-2*b1*r3^2-2*b1*a2^2+2*b3*b2^2+2*b1^2*b2+2*r1^2*b3+2*b1*a3^2+2*a1^2*b2-2*a1^2*b3-2*b3*r2^2-2*b2*a3^2-2*r1^2*b2-2*b1*b2^2-2*b1^2*b3-2*b2*b3^2+2*b2*r3^2+2*b3*a2^2+2*b1*r2^2-4*b3*r2*r+4*b1*r2*r-4*b1*r3*r+4*b2*r3*r+4*r1*r*b3-4*r1*r*b2)*x+(2*a2*b3^2+2*a1*r3^2-2*b2^2*a3+2*a2*a3^2+2*a1*b2^2+2*r1^2*a2-4*a2*r3*r+2*b1^2*a3+4*a1*r3*r-2*a2^2*a3-2*r1^2*a3-2*a1*a3^2+4*r2*r*a3-2*a2*r3^2+4*r1*r*a2-4*a1*r2*r-4*r1*r*a3+2*a1*a2^2-2*a1*b3^2+2*a1^2*a3-2*a1*r2^2-2*a1^2*a2-2*b1^2*a2+2*r2^2*a3)*y+(-8*b1*a3*r2*r+8*b1*a2*r3*r-4*b1*a2*b3^2-4*b1*a2*a3^2-4*a1*b2*r^2+4*a1*b2*a3^2+4*a1*b2*b3^2-4*a1*b3*b2^2+4*a1*b3*r^2+4*a1*b3*r2^2-8*a1*b2*r3*r+8*a1*b3*r2*r-4*r1^2*a2*b3+4*r1^2*a3*b2+4*b1^2*a2*b3-4*b1^2*a3*b2+4*r^2*a3*b2+8*r1*r*a3*b2+4*a1^2*a2*b3-4*a1^2*a3*b2-4*a1*b3*a2^2+4*b1*a3*a2^2+4*b1*a3*b2^2-4*b1*a3*r^2-4*b1*a3*r2^2+4*b1*a2*r^2+4*b1*a2*r3^2-8*r1*r*a2*b3-4*a1*b2*r3^2-4*r^2*a2*b3).

**Step 3** Calculate the Dixon resultant det($D$) or it's sub-resultant of the sub-matrix of $D$ with maximal rank (If D is Singular, Without lose of the generality , again denote the sub-resultant by det($D$)).

The Dixon resultant det($D$) has 2022 terms which is the function of the radius $r$ of the solution circle. The vanishing of the det($D$) is the necessary condition for the solution.

**Step 4** Find the necessary condition for the solution.

By solving the equation det($D$)=0, we got the constrain to the radius $r$. In the expression of the radius $r$, there are total 5697 terms (omitted here).

**Step 5** Get the solution

Substituting the solution of the radius $r$ to the equations $c_1(x,y)=0$ , $c_2(x,y)=0$ and $c_3(x,y)=0$ above, we can finally get the coordinate of the center of the solution circle.

A special example is given hare. Given three circles as follows:

$$\boldsymbol{C}_1\textbf{:}\ (x-0)^2+(y-0)^2=1,\quad \boldsymbol{C}_2\textbf{:}\ (x-0)^2+(y-5)^2=9,\quad \boldsymbol{C}_3\textbf{:}\ (x-5)^2+(y-0)^2=4. \tag{2.10}$$

Substituting the coordinates and radii into the expressions from (1.5) to (1.8) we can get the corresponding Dixon polynomial:

$$\delta(x,y,\ \alpha,\beta)=(-220+20*r+100*x)\alpha+(-170+40*r+100*y)\beta+$$
$$(100+100*r\text{^}2+200*r)+(-220+20*r)x+(-170+40*r)y.$$

Hence, the Dixon derived system is

$$\begin{cases} f_1\ (x,y)=100*x-220+20*r \\ f_2\ (x,y)=100*y-170+40*r \\ f_3\ (x,y)=(-220+20*r)*x+(-170+40*r)*y+(100+100*r\text{^}2+200*r). \end{cases} \tag{2.11}$$

So, the necessary condition for the linear equation system has non-zero solution is that the determinant of the coefficient matrix vanishes, which gives constrain to the radius $r$. By calculating, we get that

$$r=1.27874,\quad x=1.944252,\quad y=1.188504.$$

The rest seven cases can be solved similarly.

**3. The Combinatorial Matrix Approach**

From the review of the Dixon resultant method above, we found that if the Dixon matrix becomes singular the Dixon resultant det($D$) vanishes identically and gives us no further information. We have to find a non-singular sub-resultant by means of other techniques. On the other hand, the degree of the coefficients in Dixon resultant pertaining to the coefficients of the original polynomials is $(n+1)\ n!\ \Pi_{i=1}^{n}\ d_i$, where $n$ is the quantity of the variable and $d_i$ is the maximal degree of the variable $x_i$ among the $n$+1 polynomials ( $i$ =1,2,…, $n$). The following sections will present the Combinatorial Matrix approach with which one can decrease the coefficient degree in calculating the so-called Combinatorial Resultant for the derived polynomial system. The Combinatorial Matrix approach tries to find several mutual independent derived polynomials that have equal or less degree pertaining to the given polynomials by using of the combination method. After adding them to the original polynomial system, they form an independent square like polynomial system. Then we can get the necessary condition for the derived polynomial system has common zero. The necessary condition is that the determinant of the coefficient matrix vanishes.

We first consider the following three generic 2-degree symbolic coefficient polynomial equations in two variables:

$$PS:\quad \begin{cases} p_1(x,y) \equiv a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 \\ p_2(x,y) \equiv b_1x^2 + b_2xy + b_3y^2 + b_4x + b_5y + b_6 \\ p_3(x,y) \equiv c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 \end{cases} \tag{3.1}$$

Our objective is to find the necessary condition for the existence of a common zero for the given polynomial system (3.1) by means of the so called Combinatorial Matrix Approach.

Let $p_{ij}(x,y)$ denote the monomial which is in the $i^{th}$ polynomial and $j^{th}$ term in (3.1) ( $i$=1,2,3; $j$=1,2,…,6). And let $F_i(j_1, j_2,..., j_t) = \sum_{s=1}^{t} p_{i,j_s}$ be the polynomial formed by picking up the monomials $p_{ij_1}(x, y)$, $p_{ij_2}(x, y)$,…, $p_{ij_t}(x, y)$, where $i$=1,2,3 and $j_t \in (1,2,…,6)$. We separate the polynomial $p_i(x, y)$ into three parts $F_i(j_1,j_2,…,j_t)$, $F_i(k_1,k_2,…,k_l)$ and $F_i(v_1,v_2,…,v_e)$ such that $p_i(x, y) = F_i(j_1,j_2,…,j_t) + F_i(k_1,k_2,…,k_l) + F_i(v_1,v_2,…,v_e)$, where $j$'s, $k$'s and $v$'s are different , $i$=1,2,3 and $t+l+e$=6. We then construct a determinant with polynomial entries as follows:

$$P(L_1,L_2,L_3) = \begin{vmatrix} \mathrm{F}_1(j_1,j_2,...,j_t) & \mathrm{F}_1(k_1,k_2,...,k_l) & \mathrm{F}_1(v_1,v_2,...,v_e) \\ \mathrm{F}_2(j_1,j_2,...,j_t) & \mathrm{F}_2(k_1,k_2,...,k_l) & \mathrm{F}_2(v_1,v_2,...,v_e) \\ \mathrm{F}_3(j_1,j_2,...,j_t) & \mathrm{F}_3(k_1,k_2,...,k_l) & \mathrm{F}_3(v_1,v_2,...,v_e) \end{vmatrix} \tag{3.2}$$

Where $L_1 = [j_1,j_2,…,j_t]$, $L_2 = [k_1,k_2,…,k_l]$ and $L_3 = [v_1,v_2,…,v_e]$. The $P(L_1, L_2, L_3)$ represents any one of the combinations. For example,

$$P([1,2],[5],[3,4,6]) = \begin{vmatrix} \mathrm{F}_1(1,2) & \mathrm{F}_1(5) & \mathrm{F}_1(3,4,6) \\ \mathrm{F}_2(1,2) & \mathrm{F}_2(5) & \mathrm{F}_2(3,4,6) \\ \mathrm{F}_3(1,2) & \mathrm{F}_3(5) & \mathrm{F}_3(3,4,6) \end{vmatrix} = \begin{vmatrix} a_1x^2 + a_2xy & a_5y & a_3y^2 + a_4x + a_6 \\ b_1x^2 + b_2xy & b_5y & b_3y^2 + b_4x + b_6 \\ c_1x^2 + c_2xy & c_5y & c_3y^2 + c_4x + c_6 \end{vmatrix}.$$

It is obvious that if there is a common zero for the polynomial system (3.1), then it must be zero of the polynomial (3.2), since

$$P(L_1,L_2,L_3)=\begin{vmatrix} F_1(j_1,j_2,\dots,j_t) & F_1(k_1,k_2,\dots,k_l) & F_1(j_1,j_2,\dots,j_t)+F_1(k_1,k_2,\dots,k_l)+F_1(v_1,v_2,\dots,v_e) \\ F_2(j_1,j_2,\dots,j_t) & F_2(k_1,k_2,\dots,k_l) & F_2(j_1,j_2,\dots,j_t)+F_2(k_1,k_2,\dots,k_l)+F_2(v_1,v_2,\dots,v_e) \\ F_3(j_1,j_2,\dots,j_t) & F_3(k_1,k_2,\dots,k_l) & F_3(j_1,j_2,\dots,j_t)+F_3(k_1,k_2,\dots,k_l)+F_3(v_1,v_2,\dots,v_e) \end{vmatrix}$$

$$=\begin{vmatrix} F_1(j_1,j_2,\dots,j_t) & F_1(k_1,k_2,\dots,k_l) & p_1(x,y) \\ F_2(j_1,j_2,\dots,j_t) & F_2(k_1,k_2,\dots,k_l) & p_2(x,y) \\ F_3(j_1,j_2,\dots,j_t) & F_3(k_1,k_2,\dots,k_l) & p_3(x,y) \end{vmatrix}.$$

Let $l_i=|L_i|$, $i$=1,2,3. Define the type of the polynomial $P(L_1,L_2,L_3)$ as follows:

**Definition 3.1** A polynomial $P(L_1,L_2,L_3)$ is called type $t(l_1,l_2,l_3)$ if it has the form of (3.2) and $l_i=|L_i|$, $i$=1,2,3.

Then there are 15 different type $t$(1, 1, 4) polynomials, 60 different type $t$(1, 2, 3) polynomials and 90 different type $t$(2, 2, 2) polynomials. So there are total 165 different combinations. But the type $t$(1, 1, 4) polynomials turn out to be the constant coefficient combination of the original polynomials, which could not provide any useful information. We skip them here. For the type $t$(1, 2, 3) polynomials we can chose three of them as follows:

$$P([1],[2,3],[4,5,6])=\begin{vmatrix} a_1x^2 & a_2xy+a_3y^2 & a_4x+a_5y+a_6 \\ b_1x^2 & b_2xy+b_3y^2 & b_4x+b_5y+b_6 \\ c_1x^2 & c_2xy+c_3y^2 & c_4x+c_5y+c_6 \end{vmatrix}=x^2y\begin{vmatrix} a_1 & a_2x+a_3y & a_4x+a_5y+a_6 \\ b_1 & b_2x+b_3y & b_4x+b_5y+b_6 \\ c_1 & c_2x+c_3y & c_4x+c_5y+c_6 \end{vmatrix},$$

$$P([6],[3,5],[1,2,4])=\begin{vmatrix} a_6 & a_3y^2+a_5y & a_1x^2+a_2xy+a_4x \\ b_6 & b_3y^2+b_5y & b_1x^2+b_2xy+b_4x \\ c_6 & c_3y^2+c_5y & c_1x^2+c_2xy+c_4x \end{vmatrix}=xy\begin{vmatrix} a_6 & a_3y+a_5 & a_1x+a_2y+a_4 \\ b_6 & b_3y+b_5 & b_1x+b_2y+b_4 \\ c_6 & c_3y+c_5 & c_1x+c_2y+c_4 \end{vmatrix},$$

$$P([3],[1,2],[4,5,6])=\begin{vmatrix} a_3y^2 & a_1x^2+a_2xy & a_4x+a_5y+a_6 \\ b_3y^2 & b_1x^2+b_2xy & b_4x+b_5y+b_6 \\ c_3y^2 & c_1x^2+c_2xy & c_4x+c_5y+c_6 \end{vmatrix}=xy^2\begin{vmatrix} a_3 & a_1x+a_2y & a_4x+a_5y+a_6 \\ b_3 & b_1x+b_2y & b_4x+b_5y+b_6 \\ c_3 & c_1x+c_2y & c_4x+c_5y+c_6 \end{vmatrix}.$$

Let $p_4(x,y)$, $p_5(x,y)$ and $p_6(x,y)$ denote the determinant on the right hand above respectively. And let $\tau(i,j,k)$ denote the coefficient determinant corresponding to $i^{th}$, $j^{th}$ and $k^{th}$ columns of the coefficient matrix in (2.1) ($i\neq j\neq k$). Then $p_4(x,y)$, $p_5(x,y)$ and $p_6(x,y)$ can be written as follows

$$\begin{cases} p_4(x,y)=\tau(1,2,4)x^2+(\tau(1,2,5)+\tau(1,3,4))xy+\tau(1,3,5)y^2+\tau(1,2,6)x+\tau(1,3,6)y, \\ p_5(x,y)=\tau(1,3,6)xy+\tau(2,3,6)y^2+\tau(1,5,6)x+(-\tau(3,4,6)+\tau(2,5,6))y+\tau(4,5,6), \\ p_6(x,y)=\tau(1,3,4)x^2+(\tau(1,3,5)+\tau(2,3,4))xy+\tau(2,3,5)y^2+\tau(1,3,6)x+\tau(2,3,6)y. \end{cases} \quad (3.3)$$

If we consider the polynomials $p_i(x,y)$($i$=1,2,…,6) as the linear function of the power products of $\{x^2,xy,y^2,x,y,1\}$(denote it as $\{v_1,v_2,\dots,v_6\}$), then the system

$$DS: \qquad \{\, p_i(x,y)=0; \quad i=1,2,\dots,6 \,\} \qquad (3.4)$$

forms a homogeneous linear equation system pertaining to the variables $v_1, v_2, \ldots, v_6$.
Let $C\,M$ denote the following matrix:

$$C\,M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \tau(1,2,4) & \tau(1,2,5)+\tau(1,3,4) & \tau(1,3,5) & \tau(1,2,6) & \tau(1,3,6) & 0 \\ 0 & \tau(1,3,6) & \tau(2,3,6) & \tau(1,5,6) & -\tau(3,4,6)+\tau(2,5,6) & \tau(4,5,6) \\ \tau(1,3,4) & \tau(1,3,5)+\tau(2,3,4) & \tau(2,3,5) & \tau(1,3,6) & \tau(2,3,6) & 0 \end{pmatrix} \quad (3.5)$$

With Maple V, after 5.3 second calculation it is shown that *det*(*CM)* is non-vanish and has 21894 terms.

Then the equations in the system $\{ p_i(x, y) = 0; i=1,2,\ldots,6\}$ are independent.

**Definition 3.2** The full rank matrix *CM* is called a Combinatorial Matrix, the determinant *det*(*CM*) is called a Combinatorial resultant and the linear system $\{ p_i(x, y) = 0; i=1,2,\ldots,6\}$ is called a derived equation system for the polynomial system (3.1).

Since the common zeros of the polynomial system (3.1) is also the zeros of the derived polynomial system (3.4), then the zero set of the derived polynomial system (3.4) include all zeros of the system (3.1). That means

$$Zero(PS) \subseteq Zero(DS). \quad (3.6)$$

On the other hand, the necessary condition for the derived polynomial system (3.4) has non-trivial solution is that the determinant of the coefficient matrix *CM* vanishes. Therefore we have following theorem:

**Theorem 3.1** Suppose *PS* is a polynomial equation system of three generic 2-degree equations in two variables. Then the necessary condition for the existence of a common zero for the polynomial system *PS* is that the determinant of the combinatorial matrix vanishes.

The combinatorial matrix is not unique. The matrix (3.5) is just one of them. One can constructs any independent derived polynomial system. The key is that we must construct some derived polynomials with lower degree. Usually they must be less or equal to the degree of the given polynomials after eliminating some power products.

Comparing with other method, we tried Maple V *wsolve*($\{p_1, p_2, p_3\}$,$[a_1, x, y]$,{}) and the Characteristic set method *csolve*($\{p_1, p_2, p_3\}$,$[x,y]$), but the computer shown that "*Error: object too large*". With the Gröbner bases method, the *gsolve*($\{p_1, p_2, p_3\}$) shown no useful information. By successive Sylvester method, it also shown that "*Error: object too large*". Even with Dixon resultant method, while calculate the Dixon resultant the computer shown "*Error*, *(in minor) object too large*". Only with the Combinatorial method, after 5.3 second calculation it shown that *det*(*CM)* is non-vanish and has 21894 terms. The reason is that, with the system (3.1), the degree of the coefficients in the resultant pertaining to the coefficients of original polynomials is $4\times8=32$ for the Wu's and Sylvester's methods, $3\times8=24$ for the Dixon resultant method but just $3\times3+3=12$ for the Combinatorial matrix method! This is the reason why Combinatorial matrix method faster than others. It is also true for the case of a system with three generic 3-degree symbolic coefficient polynomial equations in two variables.

We now consider a system with three generic 3-degree symbolic coefficient polynomial equations in two variables then construct its combinatorial matrix and resultant.

Given a general polynomial equation system with three 3-degree symbolic coefficient polynomials in two variables as follows:

$$PS: \begin{cases} p_1(x,y) \equiv a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_5x^2 + a_6xy + a_7y^2 + a_8x + a_9y + a_{10} \\ p_2(x,y) \equiv b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3 + b_5x^2 + b_6xy + b_7y^2 + b_8x + b_9y + b_{10} \\ p_3(x,y) \equiv c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 + c_5x^2 + c_6xy + c_7y^2 + c_8x + c_9y + c_{10} \end{cases} \quad (3.7)$$

Our goal is to find the condition that the system (3.7) has common zero. To this end, again we first construct some derived polynomials such that by adding the original polynomials together, the coefficient matrix of the homogenous equation system forms a squared and full rank matrix. But for the system (3.7), the constructing technique is little different. Some derived polynomials are formed by two or three combinatorial matrices. This is we can not construct enough derived polynomials just by use of single matrix. In this case, the number of the power products will increase. The criterion is to find a square like homogenous equation system, in which the coefficient matrix is non-singular. Here is one set of polynomials we found that satisfies the criterion.

$$\begin{cases} p_4(x,y) \equiv \frac{1}{x^2y^2}P([1,2,5],[4,7],[3,6,8,9,10]) + \frac{1}{x^2y^2}P([1,2,5,6,8],[3],[4,7,9,10]), \\ p_5(x,y) \equiv \frac{1}{xy}P([1,2,3,5,6,8],[4,7,9],[10]), \\ p_6(x,y) \equiv \frac{1}{xy^2}P([1,2,3,5,6,8],[4,7],[9,10]), \\ p_7(x,y) \equiv \frac{1}{xy^3}P([1,2,3,5,6,8],[4],[7,9,10]), \\ p_8(x,y) \equiv \frac{1}{x^2y^3}P([1,2,5],[4],[3,6,7,8,9,10]), \\ p_9(x,y) \equiv \frac{1}{x^3y^2}P([1],[4,7],[2,3,5,6,8,9,10]) + \frac{1}{x^3y^2}P([1,2,5],[3],[4,6,7,8,9,10]), \\ p_{10}(x,y) \equiv \frac{1}{x^4y}P([1],[3,6],[2,4,5,7,8,9,10]) + \frac{1}{x^3y^2}P([1,2,5],[3],[4,6,7,8,9,10]), \\ p_{11}(x,y) \equiv \frac{1}{x^2y}P([1,2,5],[4,7,9],[3,6,8,10]) + \frac{1}{x^2y}P([1,2,5,8],[3,6],[4,7,9,10]), \\ p_{12}(x,y) \equiv \frac{1}{x^3y}P([1],[4,7,9],[2,3,5,6,8,10]) + \frac{1}{x^3y}P([1,3,5,6,8],[2],[4,7,9,10]) \\ \qquad + \frac{1}{x^3y}P([1,2,5],[3,6],[4,7,8,9,10]), \\ p_{13}(x,y) \equiv \frac{1}{x^3y^2}P([1,2],[3],[4,5,6,7,8,9,10]). \end{cases}$$

These polynomials all are the linear combination of the 13 power products $\{y^4, xy^3, x^2y^2, x^3, x^2y,$

$xy^2, y^3, x^2, xy, y^2, x, y, 1\}$. Then

$$DS: \quad \{ p_i(x, y) = 0; \quad i=1,2,\ldots,13 \} \tag{3.8}$$

forms a order 13 homogenous equation system. We denote the coefficient matrix as *CM*.
To see if the matrix *CM* is full rank, we use Maple V to calculate the rank of the order 13 matrix with symbolic entries. Unfortunately, after calculating 700 seconds, the computer shows "Error, (in expand / big proud) Object too Large". This is because the degree of the coefficients pertaining to the coefficients in the original polynomials is $3\times 10+3=33$. Though it is reduced from 54 with successive Sylvester resultant method or Characteristic set method to 33, it is still intractable for Maple V. So we have to try another method to determine if it is full rank. Evaluate special values to the parameters in the coefficient matrix of the system (3.7) as follows:

$$\{a_1, -3, 1, 4, 1, 19, 7, 13, 17, 5\},$$

$$\{9, b_2, 23, 1, 27, 47, 43, 5, 31, 1\},$$

$$\{5, 53, c_3, 1, 13, 3, 11, 33, 29, 7\}.$$

After substitute the parameters in (3.8) with the values above, we got a special case of the coefficient matrix *CM*, denote it as matrix *C*. If rank(*C*)=13, then we can assert that the matrix *CM* is also full rank. After calculating rank(*C*) for 188 seconds, the computer shows that the rank(*C*)=13. So, the matrix *CM* is a full rank matrix. Then we have

**Theorem 3.2** Suppose *PS* is a polynomial equation system of three generic 3-degree equations in two variables. Then the necessary condition for the existence of a common zero for the polynomial system *PS* is that the determinant of the combinatorial matrix vanishes.

From the constructing of the derived system *DS*, we know that the derived polynomial system is not unique. One can constructs different equivalent derived system. The less the dimension of the combinatorial matrix is, the better. The heuristic algorithm of the combinatorial matrix for three generic 3-degree equations in two variables can be summarized as follows:

**Algorithm** *CombMatrx*

***Input*** : ***PS***, a polynomial set; ***X***, the power product set in ***PS***; ***CX***, the cofactor sequence.

***Output:*** ***DS***, the combinatorial derived polynomial set.

***Initiaizationl***: ***PS***=$\{p_1, p_2, p_3\}$, ***DS***= ***PS***, ***X***=$\{x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1\}$, ***CX***=$\{xy, x^2y, xy^2, x^3y, x^2y^2, xy^3, x^4y, x^3y^2, x^2y^3, xy^4, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5\}$.

***Step 1:*** Construct a polynomial $p_4$ by combinatorial matrix method with ***PS,*** such that after eliminating the cofactor ***CX***(1)= $xy$, $p_4$ has lower as possible degree of power product and $\{p_1, p_2, p_3, p_4\}$ is independent. Denote power product set as $\boldsymbol{Y}_4$. Let ***X***= $\boldsymbol{X}\cup\boldsymbol{Y}_4$ and ***DS***= $\boldsymbol{DS}\cup\{p_4\}$. Let $l$(***DS***) denote the number of polynomials in ***DS***, $l$(***X***) denote the number of the power products in ***X***. If $l$(***DS***) = $l$(***X***) then go to ***Step 4***, otherwise do ***Step 2***.

***Step 2***: For $i$ =5 do the following loop:

While ($l$(***DS***) $\neq$ $l$(***X***)) do

Construct a polynomial $p_i$ by combinatorial matrix method with ***PS*,** such that after eliminating the cofactor ***CX***($i$), $p_i$ has lower as possible degree of power product and ***DS*** $\bigcup \{p_i\}$ is independent. Denote power product set as $\boldsymbol{Y}_i$.

If $\boldsymbol{Y}_i \subseteq \boldsymbol{X}$ then ***DS*** =***DS*** $\bigcup \{p_i\}$. If $l$(***DS***) = $l$(***X***) then go to ***Step 4***, otherwise $i$=$i$+$1$ and go on ***Step 2***. If $\boldsymbol{Y}_i \not\subset \boldsymbol{X}$, go to ***Step 3***.

If we have exhausted ***CX*** and still have $l$(***DS***)<$l$(***X***), then the combinatorial matrix is singular and go to ***Step 4***.

***Step 3***: Denote the new power products in $\boldsymbol{Y}_i$ by $\{e_1, e_2, \ldots, e_s\}$. Assume that the coefficients corresponding to $e_1, e_2, \ldots, e_s$ in $p_i$ are $\tau_1(i_1, j_1, k_1)$, $\tau_2(i_2, j_2, k_2)$ ,…, $\tau_s(i_s, j_s, k_s)$. Let $l_1 = \bigcup_{t=1}^{s} i_t$, $l_2 = \bigcup_{t=1}^{s} j_t$, $l_3 = \bigcup_{t=1}^{s} k_t$. Then construct a polynomial determinant from $\{p_1, p_2, p_3\}$ such that the first column includes the columns with column index in $l_1$, the second column includes the columns with column index in $l_3$ and the third column includes the columns with column index in $l_2$. Finally the cofactor must be ***CX***($i$). After eliminating the cofactor ***CX***($i$), denote the result polynomial by $q_i$. From the construction of the polynomial $q_i$, we know that the coefficients corresponding to the $e_1, e_2, \ldots, e_s$ in $q_i$ are $-\tau_1(i_1, j_1, k_1)$, $-\tau_2(i_2, j_2, k_2)$ ,…, $-\tau_s(i_s, j_s, k_s)$. So the polynomial $p_i + q_i$ must eliminate the terms corresponding to the power products $e_1$, $e_2, \ldots, e_s$.

If the set of power products of the polynomial $(p_i + q_i) \subseteq \boldsymbol{X}$ and the polynomial $(p_i + q_i)$ is independent to ***DS***, then ***DS*** = ***DS*** $\bigcup \{p_i + q_i\}$. If $l$(***DS***) = $l$(***X***) then go to ***Step 4***, otherwise $i$=$i$+$1$ and go on ***Step 2***.

If the set of power products of the polynomial $(p_i + q_i) \not\subset \boldsymbol{X}$, then go back to ***Step 3*** to eliminate new power products. The process will be end after several iterations. If the sum polynomial obtained this way is independent to ***DS***, then it can be added to ***DS***. If $l$(***DS***) = $l$(***X***) then go to ***Step 4***, otherwise $i$=$i$+$1$ and go on ***Step 2***.

***Step 4:*** Output ***DS***, terminate the process. ▌

Here we show some steps of the ***CM*** algorithm.

For the cofactor ***CX***($1$)=$xy$ , construct a polynomial determinant

$P$([1,2,3,5,6,8],[4,7,9],[10])

$$= \begin{vmatrix} a_1x^3 + a_2x^2y + a_3xy^2 + a_5x^2 + a_6xy + a_8x & a_4y^3 + a_7y^2 + a_9y & a_{10} \\ b_1x^3 + b_2x^2y + b_3xy^2 + b_5x^2 + b_6xy + b_8x & b_4y^3 + b_7y^2 + b_9y & b_{10} \\ c_1x^3 + c_2x^2y + c_3xy^2 + c_5x^2 + c_6xy + c_8x & c_4y^3 + c_7y^2 + c_9y & c_{10} \end{vmatrix}$$

$=(xy)\, p_4$, where

$$p_4 = \begin{vmatrix} a_1x^2 + a_2xy + a_3y^2 + a_5x + a_6y + a_8 & a_4y^2 + a_7y + a_9 & a_{10} \\ b_1x^2 + b_2xy + b_3y^2 + b_5x + b_6y + b_8 & b_4y^2 + b_7y + b_9 & b_{10} \\ c_1x^2 + c_2xy + c_3y^2 + c_5x + c_6y + c_8 & c_4y^2 + c_7y + c_9 & c_{10} \end{vmatrix},$$

$\boldsymbol{Y}_4=\{x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1\} \cup \{xy^3, x^2y^2, y^4\}$

$= \{xy^3, x^2y^2, y^4, x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1\}$.

$\boldsymbol{DS} = \boldsymbol{PS} \bigcup \{p_4\} = \{p_1, p_2, p_3, p_4\}$.

Then for the cofactor $\boldsymbol{CX}(2)=x^2y$, construct a polynomial determinant $P$([1,2,5],[4,7,9],[3,6,8,10])

$$= \begin{vmatrix} a_1x^3 + a_2x^2y + a_5x^2 & a_4y^3 + a_7y^2 + a_9y & a_3xy^2 + a_6xy + a_8x + a_{10} \\ b_1x^3 + b_2x^2y + b_5x^2 & b_4y^3 + b_7y^2 + b_9y & b_3xy^2 + b_6xy + b_8x + b_{10} \\ c_1x^3 + c_2x^2y + c_5x^2 & c_4y^3 + c_7y^2 + c_9y & c_3xy^2 + c_6xy + c_8x + c_{10} \end{vmatrix}$$

$$=(x^2y) \begin{vmatrix} a_1x + a_2y + a_5 & a_4y^2 + a_7y + a_9 & a_3xy^2 + a_6xy + a_8x + a_{10} \\ b_1x + b_2y + b_5 & b_4y^2 + b_7y + b_9 & b_3xy^2 + b_6xy + b_8x + b_{10} \\ c_1x + c_2y + c_5 & c_4y^2 + c_7y + c_9 & c_3xy^2 + c_6xy + c_8x + c_{10} \end{vmatrix}$$

$=(x^2y)\, p_5$.

Since there are new power products $\{xy^5, xy^4, x^2y^3, x^2y^4\}$ in $p_5$, we should construct another polynomial determinant to eliminate them. To this end, we first consider the coefficients corresponding to new power products $\{xy^5, xy^4, x^2y^3, x^2y^4\}$ in $p_5$. They are $\tau_1$(2, 4, 3), ($\tau_2$(5, 4, 3), $\tau_3$(2, 4, 6), $\tau_4$(2, 7, 3)), ($\tau_5$(1, 4, 6), $\tau_6$(1, 7, 3)), $\tau_7$(1, 4, 3). In this case $l_1$={1,2,5}, $l_2$={4,7}, $l_3$={3,6}. Then construct a $3\times3$ polynomial determinant such that its first column includes the terms with column index 1,2 and 5 , the second column includes in the terms with column index 3 and 6, the third column includes the terms with column index 4 and 7 of the original system (3.7) as follows:

$$\begin{vmatrix} a_1x^3 + a_2x^2y + a_5x^2 & a_3xy^2 + a_6xy & a_4y^3 + a_7y^2 + a_8x + a_9y + a_{10} \\ b_1x^3 + b_2x^2y + b_5x^2 & b_3xy^2 + b_6xy & b_4y^3 + b_7y^2 + b_8x + b_9y + b_{10} \\ c_1x^3 + c_2x^2y + c_5x^2 & c_3xy^2 + c_6xy & c_4y^3 + c_7y^2 + c_8x + c_9y + c_{10} \end{vmatrix}$$

$$= (x^2y) \begin{vmatrix} a_1x + a_2y + a_5 & a_3xy + a_6x & a_4y^3 + a_7y^2 + a_8x + a_9y + a_{10} \\ b_1x + b_2y + b_5 & b_3xy + b_6x & b_4y^3 + b_7y^2 + b_8x + b_9y + b_{10} \\ c_1x + c_2y + c_5 & c_3xy + c_6x & c_4y^3 + c_7y^2 + c_8x + c_9y + c_{10} \end{vmatrix}$$

$= (x^2y)\, q_5$.

Obviously, the polynomial $p_5^{'}=(p_5 + q_5)$ will eliminate the terms with the power products $xy^5$, $xy^4$, $x^2y^3$, $x^2y^4$. The calculation also shows that $p_5^{'}$ is independent to $\{p_1, p_2, p_3, p_4\}$. Therefore,

$\boldsymbol{DS} = \boldsymbol{DS} \bigcup \{p_5^{'}\} = \{p_1, p_2, p_3, p_4, p_5^{'}\}$.

In this way, the process can be carried on until a square shape independent combinatorial matrix is found.

We now show that the Dixon derived system (2.4) can be obtained by the Combinatorial Matrix Method. Transpose the determinant (2.2) and do some column permutations we can get an equivalent polynomial determinant below:

$$\begin{vmatrix} p_1(\alpha_1,\alpha_2,\ldots,\alpha_n) & p_1(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_n) & \cdots & p_1(\alpha_1,x_2,\ldots,x_{n-1},x_n) & p_1(x_1,x_2,\ldots,x_n) \\ p_2(\alpha_1,\alpha_2,\ldots,\alpha_n) & p_2(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_n) & \cdots & p_2(\alpha_1,x_2,\ldots,x_{n-1},x_n) & p_2(x_1,x_2,\ldots,x_n) \\ p_3(\alpha_1,\alpha_2,\ldots,\alpha_n) & p_3(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_n) & \cdots & p_3(\alpha_1,x_2,\ldots,x_{n-1},x_n) & p_3(x_1,x_2,\ldots,x_n) \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ p_{n+1}(\alpha_1,\alpha_2,\ldots,\alpha_n) & p_{n+1}(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_n) & \cdots & p_{n+1}(\alpha_1,x_2,\ldots,x_{n-1},x_n) & p_{n+1}(x_1,x_2,\ldots,x_n) \end{vmatrix}$$

For the determinant above, beginning from left to right, let every column subtract right hand column. Then after eliminating factor $\Pi_{i=1}^{n}(x_i-\alpha_i)$ we can get an equivalent polynomial determinant:

$$\begin{vmatrix} q_{11}(\alpha_1,\alpha_2,\ldots,\alpha_n,x_n) & q_{12}(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_{n-1},x_n) & \cdots\cdots & p_1(x_1,x_2,\ldots,x_n) \\ q_{21}(\alpha_1,\alpha_2,\ldots,\alpha_n,x_n) & q_{22}(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_{n-1},x_n) & \cdots\cdots & p_2(x_1,x_2,\ldots,x_n) \\ q_{31}(\alpha_1,\alpha_2,\ldots,\alpha_n,x_n) & q_{32}(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_{n-1},x_n) & \cdots\cdots & p_3(x_1,x_2,\ldots,x_n) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ q_{n+1,1}(\alpha_1,\alpha_2,\ldots,\alpha_n,x_n) & q_{n+1,2}(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x_{n-1},x_n) & \cdots\cdots & p_{n+1}(x_1,x_2,\ldots,x_n) \end{vmatrix} \tag{3.9}$$

For any $q_{ij}(\alpha_1,\alpha_2,\ldots,\alpha_{n-j+1},x_{n-j+1},x_{n-j+2},\ldots,x_n)$ (*i=1,2,…,n+1; j=1,2,…,n*), we can regard them as the polynomials of the power products of $\alpha_1,\alpha_2,\ldots,\alpha_n$. For a fixed index *j*, it is not difficult to find that the polynomial $q_{ij}(\alpha_1,\alpha_2,\ldots,\alpha_{n-j+1},x_{n-j+1},x_{n-j+2},\ldots,x_n)$ is the polynomial pertaining to the power products of $\alpha_1,\alpha_2,\ldots,\alpha_{n-j+1}$ and the coefficients are formed by the sum of the monomials of the variables $x_{n-j+1},x_{n-j+2},\ldots,x_n$ that corresponding to some terms in the original ***PS***. Furthermore the coefficient of the power product $\alpha_{i_1}^{m_1}\alpha_{i_2}^{m_2}\ldots\alpha_{i_t}^{m_t}$ in the (3.9) is formed by the sum of all the coefficient polynomial determinants corresponding to that power product $\alpha_{i_1}^{m_1}\alpha_{i_2}^{m_2}\ldots\alpha_{i_t}^{m_t}$. And all the coefficient polynomial determinants are corresponding to some kind of separation of the original ***PS***.

For example, in the Dixon polynomial for the system (3.7), the coefficient polynomial determinants corresponding to the variable $\alpha$ is the sum of two polynomial determinants:

$$\begin{vmatrix} a_1x+a_2y+a_5 & a_7y+a_9 & p_1(x,y) \\ b_1x+b_2y+b_5 & b_7y+b_9 & p_2(x,y) \\ c_1x+c_2y+c_5 & c_7y+c_9 & p_3(x,y) \end{vmatrix}$$

$$+\begin{vmatrix} a_1x^2+a_2xy+a_3y^2+a_5x+a_6y+a_8 & a_3y+a_6 & p_1(x,y) \\ b_1x^2+b_2xy+b_3y^2+b_5x+b_6y+b_8 & b_3y+b_6 & p_2(x,y) \\ c_1x^2+c_2xy+c_3y^2+c_5x+c_6y+c_8 & b_3y+b_6 & p_3(x,y) \end{vmatrix}.$$

Since

$$\begin{vmatrix} a_1x+a_2y+a_5 & a_7y+a_9 & p_1(x,y) \\ b_1x+b_2y+b_5 & b_7y+b_9 & p_2(x,y) \\ c_1x+c_2y+c_5 & c_7y+c_9 & p_3(x,y) \end{vmatrix} = \begin{vmatrix} a_1x+a_2y+a_5 & a_7y+a_9 & a_3xy^2+a_4y^3+a_6xy+a_{10} \\ b_1x+b_2y+b_5 & b_7y+b_9 & b_3xy^2+b_4y^3+b_6xy+b_{10} \\ c_1x+c_2y+c_5 & c_7y+c_9 & c_3xy^2+c_4y^3+c_6xy+c_{10} \end{vmatrix}$$

and

$$\begin{vmatrix} a_1x^2+a_2xy+a_3y^2+a_5x+a_6y+a_8 & a_3y+a_6 & p_1(x,y) \\ b_1x^2+b_2xy+b_3y^2+b_5x+b_6y+b_8 & b_3y+b_6 & p_2(x,y) \\ c_1x^2+c_2xy+c_3y^2+c_5x+c_6y+c_8 & b_3y+b_6 & p_3(x,y) \end{vmatrix}$$

$$= \begin{vmatrix} a_1x^2+a_2xy+a_5x+a_8 & a_3y+a_6 & a_4y^3+a_7y^2+a_9y+a_{10} \\ b_1x^2+b_2xy+b_5x+b_8 & b_3y+b_6 & b_4y^3+b_7y^2+b_9y+b_{10} \\ c_1x^2+c_2xy+c_5x+c_8 & b_3y+b_6 & c_4y^3+c_7y^2+c_9y+c_{10} \end{vmatrix},$$

we can easily find out that the right side of the expressions are corresponding to two different separations of the system (3.7).

So, every coefficient of the Dixon polynomial can be obtained by separating the original polynomial system and constructing the polynomial determinant. On the other hand the Dixon resultant method is completely mechanical and the combinatorial method can construct polynomial determinants arbitrarily. Therefore there exist some polynomials that can be constructed by the combinatorial method but cannot by Dixon resultant method. From the discussion above we can draw the conclusion as follows:

**Theorem 3.3** The derived polynomial system by Dixon resultant method can be obtained by the combinatorial matrix approach. Furthermore, there exist some polynomials that can be derived by combinatorial matrix method which cannot be derived by Dixon resultant method.

## 4. The Application of the Combinatorial Matrix Approach

The combinatorial matrix method can be used to solve many geometrical and automated theory proving problems faster than other methods. However, some times a combinatorial derived system for a polynomial equation system may be singular. In those cases, we can choose a part of equations to form a non-singular sub-derived system and find the solution.

**Example 1** (*Apollonius problem*) Given three circles on the plane, find other circles tangent to all three. Equation (2.6) gives the relationship between the solution circle and the given circles. Theoretically there are eight solutions corresponding to the choice of the signs in (2.6).

Generally, equation (2.6) has the form below:

$$PS: \quad \begin{cases} p_1(x,y) \equiv x^2+y^2+a_3x+a_4y+a_5=0, \\ p_2(x,y) \equiv x^2+y^2+b_3x+b_4y+b_5=0, \\ p_3(x,y) \equiv x^2+y^2+c_3x+c_4y+c_5=0. \end{cases}$$

We can construct two polynomials as follows:

$$p_4(x,y) \equiv \tfrac{1}{xy} P([1,5],[3],[2,4]) = \tau(1,3,4)x^2 + \tau(2,3,5)y + \tau(3,4,5)=0,$$
$$p_5(x,y) \equiv \tfrac{1}{xy} P([1,3],[2,4],[5]) = \tau(1,4,5)x - \tau(1,3,5)y + \tau(3,4,5)=0,$$

Since the combinatorial matrix $CM$ is not singular ($\det(CM) \neq 0$ and has 306 terms), then $\{p_1(x,y),$ $p_2(x,y),\ p_3(x,y),\ p_4(x,y),\ p_5(x,y)\}$ forms the derived system for $PS$ and it has solution.

To give a concrete example, take three circles center at (0,0) with radius 1, center at (0,5) with radius 3 and center at (5,0) with radius 2 respectively. We first calculate the solution circle that all three circles are outside it.

By substituting $a_3 = 0$, $a_4 = 0$, $a_5 = -r^2 - 2*r - 1$, $b_3 = 0$, $b_4 = -10$, $b_5 = -r^2 - 2*r - 9$, $c_3 = -10$, $c_4 = 0$, $c_5 = -r^2 - 2*r - 4$ to det($CM$)=0, we can get a equation pertaining to the radius $r$:

$$10000*(16-r^2-6*r)^2+10000*(21-r^2-4*r)^2-1000000+20000*(-r^2-2*r-1)^2$$
$$-1000000*r^2-2000000*r-20000*(16-r^2-6*r)*(-r^2-2*r-1)$$
$$-20000*(21-r^2-4*r)*(-r^2-2*r-1)=0.$$

Solving it we get the radius

$$r = -(53/20)+(21/20)*\sqrt{14} = 1.278740256. \tag{4.1}$$

By solving the derived system we get the center of the solution circle is

$$(x=1.944251949,\ y=1.188503898).$$

Other solutions are listed below:

$$r=2.003, \quad (x=1.799400000,\ y=0.8988000000), \tag{4.2}$$
$$r=4.342793268, \quad (x=1.331441347,\ y=-0.0371173070), \tag{4.3}$$
$$r=3.068813079, \quad (x=1.586237384,\ y=0.4724747680), \tag{4.4}$$
$$r=4.568813079, \quad (x=1.286237384,\ y=-0.1275252320), \tag{4.5}$$
$$r=4.842793268, \quad (x=1.231441347,\ y=-0.2371173070), \tag{4.6}$$
$$r=-2.003, \quad \text{(no real solution for this case)} \tag{4.7}$$
$$r=6.578740256, \quad (x=0.8842519490,\ y=-0.9314961020), \tag{4.8}$$

It took less than one second to finish the calculation.

**Example 2** (*Heymann problem* [5,7] ) Let $ABC$ be a triangle, $a$, $b$ and $c$ the length of the sides $BC$, $AC$ and $AB$, $a_i$ and $a_e$ the length of internal and external bisectors of angle $A$, and $b_e$ the length of the external angle bisector of B. The objective is to express the side length $a$ in terms of $a_i$, $a_e$ and $b_e$. Furthermore determine if, given general value of the three angle bisectors, can one draw the triangle just using a compass and a ruler (if the expression involving $a$, $a_i$, $a_e$ and $b_e$ is of degree $2^m$ in $a$ for some integral $m$).

At first, we express $a_i$, $a_e$ and $b_e$ in terms of the lengths $a$, $b$ and $c$. It is easily be done by Euclidean geometry. The expressions are below:

$$\begin{cases} a_i^2 = \dfrac{cb(c+b-a)(c+b+a)}{(b+c)^2}, \\ a_e^2 = \dfrac{cb(-c+b+a)(c-b+a)}{(c-b)^2}, \\ b_e^2 = \dfrac{ac(-c+b+a)(c+b-a)}{(c-a)^2}. \end{cases}$$

Regarding $a$, $b$ as independent variables and $c$ , $a_i$, $a_e$ , $b_e$ as dependent variables we rewrite the expression as follows:

$$PS: \begin{cases} p_1 = a_i^2(b+c)^2 - cb(c+b-a)(c+b+a) \quad , \\ p_2 = a_e^2(c-b)^2 - cb(-c+b+a)(c-b+a) \; , \\ p_3 = b_e^2(c-a)^2 - ca(-c+b+a)(c+b-a) \cdot \end{cases}$$

After calculating the 13 derived equations from $PS$ by the combinatorial matrix method, we found the combinatorial matrix method is singular. Fortunately we can pick up 10 independent equations from the 13 derived polynomials after doing the fraction-free *Gauss* elimination to form a triangle like equation system $DS = \{ q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10} \}$. If let $T=(t_{ij})$ be the coefficient matrix for $DS$, then the last factor in $T=(t_{ij})$ is

$D = t_{10,10} = -c^{20}(4c^2 - a_i^2 - a_e^2)^2\,(-288\,c^{10}\,a_e^{12}\,b_e^{8} - 512\,c^{14}\,a_e^{10}\,b_e^{6} + \ldots + 20488\,c^{6}\,a_i^{14}\,a_e^{8}\,b_e^{2})$. (326 terms)

*Let* $d = (4c^2 - a_i^2 - a_e^2)^2\,(-288\,c^{10}\,a_e^{12}\,b_e^{8} - 512\,c^{14}\,a_e^{10}\,b_e^{6} + \ldots + 20488\,c^{6}\,a_i^{14}\,a_e^{8}\,b_e^{2})$,

$P_0 = P_0(c, a_i, a_e, b_e) = t_{9,9} = 174\,a_i^{16}\,c^{25} b_e^{4} + \ldots - 7680\,a_i^{14}\,c^{25}\,a_e^{6}$. (174 terms)

$P_1 = P_1(c, a_i, a_e, b_e) = t_{9,10} = 4448\,c^{30} a_e^{12}\,b_e^{4} + \ldots + 256\,c^{22}\,a_i^{16}\,a_e^{8}$. (209 terms)

$Q_0 = Q_0(c, a_i, a_e, b_e) = t_{8,8} = -22\,c^{22}\,a_e^{14}\,b_e^{2} + \ldots + 40\,c^{20} a_i^{2}\,a_e^{12}\,b_e^{4}$. (96 terms)

$Q_1 = Q_1(c, a_i, a_e, b_e) = t_{8,9} = 48\,c^{22}\,a_e^{14}\,b_e^{2} + \ldots - 48 c^{20}\,a_i^{2}\,a_e^{12}\,b_e^{4}$. (93 terms)

$Q_2 = Q_2(c, a_i, a_e, b_e) = t_{8,10} = -680\,c^{23}\,a_i^{12}\,a_e^{2}\,b_e^{2} + \ldots + 3056 c^{27}\,a_i^{8}\,a_e^{4}$. (108 terms)

If $a_i$ ,$a_e$ ,$b_i$ and $c$ satisfy $d=0$, then $c$ can be determined from $d=0$. Suppose $c = F(a_i, a_e, b_i)$, we can get an ascending sub-list from $DS$ as follows:

$$P_0\, b + P_1 = 0,$$

$$Q_0\, a + Q_1\, b + Q_2 = 0.$$

From the expressions above we can get $a$ and $b$:

$$b = -P_1 / P_0, \quad a = -(Q_1(-P_1/P_0) + Q_2)/\, Q_0 .$$

In this way, the parameters $a$, $b$ and $c$ are presented as the function of the variable $a_i$, $a_e$ and $b_i$. The total time spent was 11 seconds.

Since the expression $D = -c^{20}(4c^2 - a_i^2 - a_e^2)^2\,(-288\,c^{10}\,a_e^{12}\,b_e^{8} - 512\,c^{14}\,a_e^{10}\,b_e^{6} + \ldots + 20488\,c^{6}\,a_i^{14}\,a_e^{8}\,b_e^{2})$ is of degree 20 in the parameter $a$, instead of degree $2^m$ in $a$, then the triangle cannot be constructed just by using of a compass and a ruler.

**Example 3** Expression for the distance of the intersection of two general conics from the origin.

$$PS: \begin{cases} p_1(x, y) \equiv a_1\, x^2 + a_2\, xy + a_3 y^2 + a_4\, x + a_5 y + a_6 = 0, \\ p_2(x, y) \equiv b_1\, x^2 + b_2\, xy + b_3 y^2 + b_4\, x + b_5 y + b_6 = 0, \\ p_3(x, y) \equiv x^2 + y^2 - T = 0. \end{cases}$$

The derived system by combinatorial matrix method is constructed as follows:

$$DS:\begin{cases} a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6=0,\\ b_1x^2+b_2xy+b_3y^2+b_4x+b_5y+b_6=0,\\ x^2+y^2-T=0,\\ \tau(1,2,4)x^2+(\tau(1,2,5)+\tau(1,3,4))xy+\tau(1,3,5)y^2+\tau(1,2,6)x+\tau(1,3,6)y=0,\\ \tau(1,3,6)xy+\tau(2,3,6)y^2+\tau(1,5,6)x+(\tau(2,5,6)-\tau(3,4,6))y+\tau(4,5,6)=0,\\ \tau(1,3,4)x^2+(\tau(1,3,5)+\tau(2,3,4))xy+\tau(2,3,5)y^2+\tau(1,3,6)x+\tau(2,3,6)y=0.\end{cases}$$

Let $C$ denote the coefficient matrix of the $DS$. Then

$$det(C)=6a_3^2b_1^2a_6^2b_6^2+\ldots-2a_2^2b_3b_4^3a_4a_6T. \qquad \text{(2424 terms)}$$

From $det(C)=0$, we can solve the parameter $T=T(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6)$.

Then do the fraction-free *Gauss* elimination to get a triangle like coefficient matrix $M$.

Let

$$D_0=M_{5,5}=b_2^3a_5a_6^2-a_6b_4^3a_3^2+\ldots-2a_1^2b_2b_5b_1a_6T, \qquad \text{(348 terms)}$$

$$D_1=M_{5,6}=b_2b_6^2a_3^2a_6+\ldots-2a_1^3b_5b_3b_4T^2, \qquad \text{(348 terms)}$$

$$Q_0=M_{4,4}=2a_4^2b_1b_3-b_2^2a_6a_3+\ldots-b_2^2a_4^2, \qquad \text{(36 terms)}$$

$$Q_1=M_{4,5}=b_2a_3^2b_6+b_2a_1^2b_6+\ldots+a_5b_1a_1b_4, \qquad \text{(40terms)}$$

$$Q_2=M_{4,6}=-b_6a_3^2b_4-a_2^2b_6b_4+\ldots-b_1a_5a_2b_3T. \qquad \text{(46 terms)}$$

Then, $D_0y+D_1=0$ and $Q_0x+Q_1y+Q_2=0$. From the expressions we can solve $x$ and $y$:

$$y=-D_1/D_0,\quad x=-(Q_1(-D_1/D_0)+Q_2)/Q_0.$$

The time to obtain the condition for the common zero was 1.6 seconds.

**Example 4** Find the conditions for perpendicular intersection of a general conic and a general ellipse.

$$PS:\begin{cases} p_1(x,y)\equiv a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6=0,\\ p_2(x,y)\equiv b_1x^2+b_3y^2+b_4x+b_5y+b_6=0,\\ p_3(x,y)\equiv \dfrac{\partial p_1}{\partial x}\dfrac{\partial p_2}{\partial x}+\dfrac{\partial p_1}{\partial y}\dfrac{\partial p_2}{\partial y}=0.\end{cases}$$

The derived system by combinatorial matrix method is constructed as follows:

$$DS:\begin{cases} a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6=0,\\ b_1x^2+b_3y^2+b_4x+b_5y+b_6=0,\\ 4a_1b_1x^2+(2a_2b_1+2a_2b_3)xy+4a_3b_3y^2+(2a_4+2a_1b_4+a_2b_5)x\\ \qquad +(a_2b_4+2a_5b_3+2a_3b_5)y+a_4b_4+a_5b_5=0,\\ \tau(1,2,4)x^2+(\tau(1,2,5)+\tau(1,3,4))xy+\tau(1,3,5)y^2+\tau(1,2,6)x+\tau(1,3,6)y=0,\\ \tau(1,3,6)xy+\tau(2,3,6)y^2+\tau(1,5,6)x+(\tau(2,5,6)-\tau(3,4,6))y+\tau(4,5,6)=0,\\ \tau(1,3,4)x^2+(\tau(1,3,5)+\tau(2,3,4))xy+\tau(2,3,5)y^2+\tau(1,3,6)x+\tau(2,3,6)y=0.\end{cases}$$

Let $C$ denote the coefficient matrix of the $DS$. Then

$$det(C)=-448a_1b_1^4a_2^2b_3^2a_6^2a_4a_5b_5a_3b_4+\ldots+16a_3^4b_1^6a_4^4b_6^2. \qquad \text{(8465 terms)}$$

So, $det(C)=0$, is the necessary condition for $PS$ has common zero. If the coefficients $a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_3, b_4, b_5, b_6$ make $det(C)=0$, then the common zeros can be found by *Gauss Elimination* as above. It took 6.4 seconds to calculate the necessary condition.

## 5. Experiment results and comparisons

After comparing the combinatorial matrix method with other methods, we found the combinatorial matrix method has less calculating time in deriving the necessary condition for existing the common zeros. The calculating comparison for example 1 to 4 and for general cases (3.1) is listed in the table below:

| Example # | Singular | Terms | Combinatorial Matrix | Dixon | Gröbner | Sylvester |
|---|---|---|---|---|---|---|
| 1 | S | 306 | 0.1 *s* | 0.66 *s* | * | 5.1 *s* |
| 2 | S | 326 | 4.2 *s* | 6 *s* | * | 750 *s* |
| 3 | S | 2424 | 1.4 *s* | 6.7 *s* | * | * |
| 4 | N | 8465 | 6.2 *s* | 8.5 *s* | * | * |
| (3.1) | N | 21894 | 5.3 *s* | * | * | * |

Table 1: Comparison of different Methods

In the table above, a (*) indicate that either the computer shown "*Error: object too large* " or gives no useful information. All examples were done on an AMD Duron 800 128M microcomputer.

## 6. Conclusion

The Combinatorial Matrix Approach is presented and discussed. It is shown that the method in the cases of the polynomial systems with bi-variable in two degrees or with bi-variable in three degrees is more efficient than other methods. By applying the method one can eliminate several variables simultaneously. On the methodology of the deriving polynomials, it has more freedom. With the method one can reduce the degree of the coefficients in the finding of the necessary condition for the existence of the common zero and deduce the time complexity of the equation solving for the polynomial equation system with symbolic coefficients. The algorithm given here is just to show an idea and the outline of the Combinatorial Matrix Method.

Many work need to be done to expand the method to the situation of any symbolic polynomial system with higher degree and more variables. Besides, the singular cases need to be treated in detail. By the way, it is unclear how the Dixon resultant can derive independent polynomials. It is still a black box that needs to be investigated further.

**Acknowledgements:** I would like to thank Professor Jingzhong Zhang for his guidance.

## References

[1] Robert H. Lewis, Solving Large Polynomial Systems with the Dixon Resultant, http://www.bway.net/~lewis/.

[2] Robert H. Lewis, Conic Tangency Equations and Apollonius Problems in Biochemistry and Pharmacology, Preprint submitted to Elsevier Science May 2002

[3] Chou. SC, Proving elementary geometry theorem using Wu's Algorithm. " Automated Theorem Proving" after 25 years. MAS. Contemporary Mathematics, 1984,29:243

[4] Chou. SC, Mechanical Geometry Theorem Proving. D. Reidel Publishing Company, 1988

[5] D. Kapur, T. Saxena & L. Yang, Algebraic and geometric reasoning using Dixon resultants, Proceedings of ISSAC'94, ACM Press, 1994

[6] L. Yang & X.R. Hou, Gather-and-sift: a symbolic method for solving polynomial systems, Proc. ATCM'95, Singapore, Dec,1995

[7] Heymann ,W. Problem der Winkehalbierenden , Ztschr . f . Math . and Phys . No .35 ,1890.