CONTS

Exploring Mathematics with the TI-89 Titanium® and Voyage 200®

Wade Ellis
West Valley Community College
San Jose, CA
wade_ellis@wvmccd.cc.ca.us

Carl Leinbach
Gettysburg College
Gettysburg, PA
leinbach@gettysburg.edu

Purpose

In this "hands-on" workshop we will explore some of the possibilities for using two exciting entries in the Texas Instruments Handheld Computer Algebra System (CAS) line, the TI-89 Titanium® or the Voyage 200®, to learn mathematics. We will concentrate on exploring applications of mathematics in several different courses. For the most part we will deal with the CAS capabilities of these hand-helds. Obviously, 90 minutes is a short period of time to cover all of the possibilities of these exciting new learning and teaching tools. We hope to give you some of the flavor of the TI-89 Titanium and Voyage 200 and to motivate you to learn more about them and their potentials as learning tools.

A Very Brief Overview of the two Hand-Helds

When you turn on either of these hand-helds you will see that there are definite changes in the user interface. The calculator screen has several icons each icon represents a different application that is available on that particular hand-held. Note: There are certain applications that are available on every one of hand-helds. You can customize your particular version by downloading additional applications from the TI web site or third party developers web sites. Many of the applications are supplied without cost. Others usually have a very reasonable cost. The expanded RAM and flash memory on these hand-helds allows for several applications to be loaded on each machine.

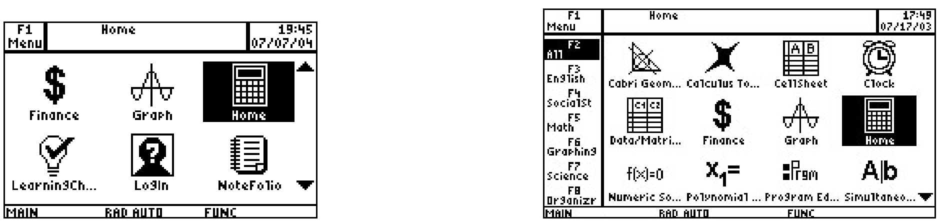


Figure 1: *Examples of the Welcome Screen on the TI-89Tiatanium (left) and the Voyage 200 (right)*

To navigate through the applications on your individual hand-held use the four arrow keys to highlight the various Apps (applications). When you have an application that you wish to use, simply press the enter key and the application is started. This is the preferred user interface for these machines. It is possible to set the interface to the older pull down menu. Our bet is that you will choose to use the icons to select your Apps. The look and feel is very much that of a PC or Mac.

A Brief Introduction to CAS and the Basic Functionality of the TI Handheld CAS

We will start at the very beginning. Highlight the icon showing a handheld with the word “Home” underneath it. Press a key with the word “Enter” written on it. A screen similar to one one of those shown in Figure 2 should appear.

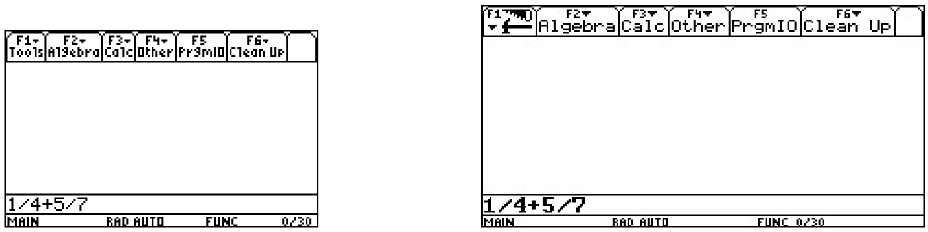


Figure 2: *The “Home” Screen with an expression in the command line*

Type “1”, “÷”, “4”, “+”, “5”, “÷”, “7”. Only type what appears inside the quotes. Do not include the quotes. Your screen should now look exactly like Figure 2. Press “Enter” and your screen should look like one of the screen shots in Figure 3.

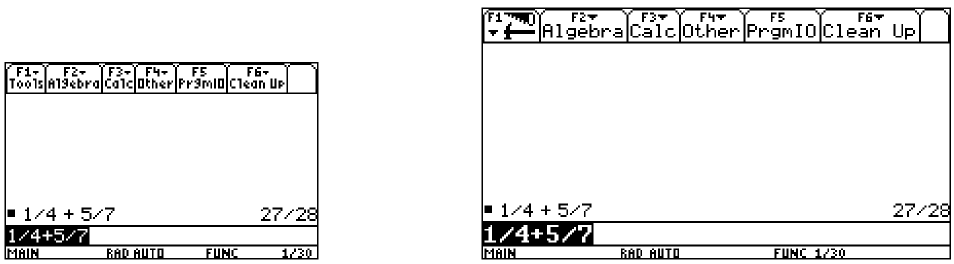


Figure 3: *The result of pressing the “Enter” key*

Note that the CAS displays the expression that you typed in the command line on the left of the display and the result in rational form on the right. If you press the key with a diamond and then the “Enter” key the result .964286 appears on the right hand side of the display. This is the approximate value of the result evaluated to the number of decimal places that you specify in the “Display Digits” option of the “Mode”. You can achieve the same approximate result by entering one of the operands as a decimal number, i.e. enter the 7 as 7. When the CAS reads the decimal point, it is assumed that a decimal answer is preferred by the user.

What about doing some algebra? First clear your screen by pressing the [F1] key on the top row followed by pressing the numeric keypad key, “8”. Also press the “Clear” key.

Now press the [F2] key. The “Algebra” menu is pulled down. Highlight option number 2 or press “2” on the numeric keypad. This types the “factor(” on the command line. Now type, “x^3+x^2–2x–2)”. Press “Enter” and you will see the top line of the display in Figure 4. Next press the right arrow key at the upper right hand side of the handheld CAS and backspace over the “)” and type “,x)”. Press enter and the second line of the display in Figure 4 appears.

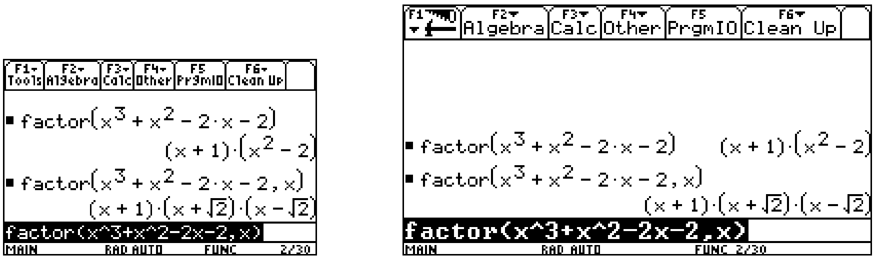


Figure 4: *Algebraic factoring over the rationals and the reals.*

Notice that the factor option without the second argument gives the irreducible factors over the rationals. With the second argument, “,x”, the result is all of the linear factors over the reals.

Press [F1], option “8” to clear the display. Do not press “Clear”. Press the right arrow key, but do not press the backspace key. Using the left arrow key and backspace change the argument of factor to “x^3+x^2+2x+2”. Press “Enter”. Note that the first line of the display in Figure 5 shows the irreducible factors over the reals. To see all of the linear factors we must use the “cfactor(” or complex factor operator. Press the left arrow key and put a “c” in front of “factor” and press the “Enter” key.

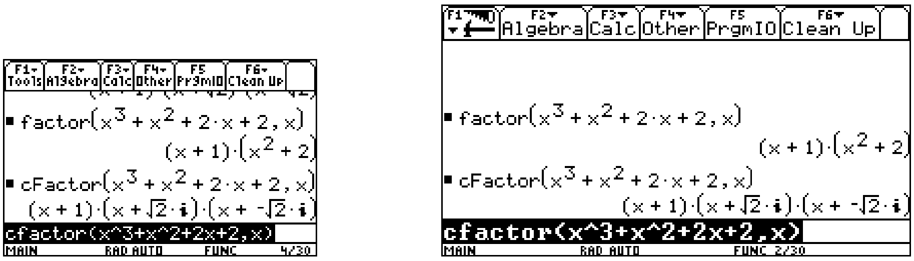


Figure 5: *Factoring a polynomial with complex roots*

If we wish to graph our two polynomial examples we press the key with the diamond and [F1] on the Titanium or green key with diamond and “w” on the Voyage 200. Alternatively, return to the main window by pressing the “APPS” key and choosing the icon with “Y=” and pressing “Enter”. Type “x^3+x^2–2x–2” and press “Enter”. Then type “x^3+x^2+2x+2” and press enter. Your display should look like the left hand side of Figure 6. The Titanium display is on the top and the Voyage 200 display is on the bottom in this figure. To see the graphs of the two functions return to the main window and press the icon with a grqph on it that is labeled “Graph”. This will produce a display similar to the right hand side of Figure 6. Can you tell which graph corresponds to which function? You can adjust the size of your window by pressing [F2] to choose the "Zoom" option. You can also adjust it manually by pressing the green key with the diamond and "E" or by returning to the main window and choose the "Window Editor" icon.

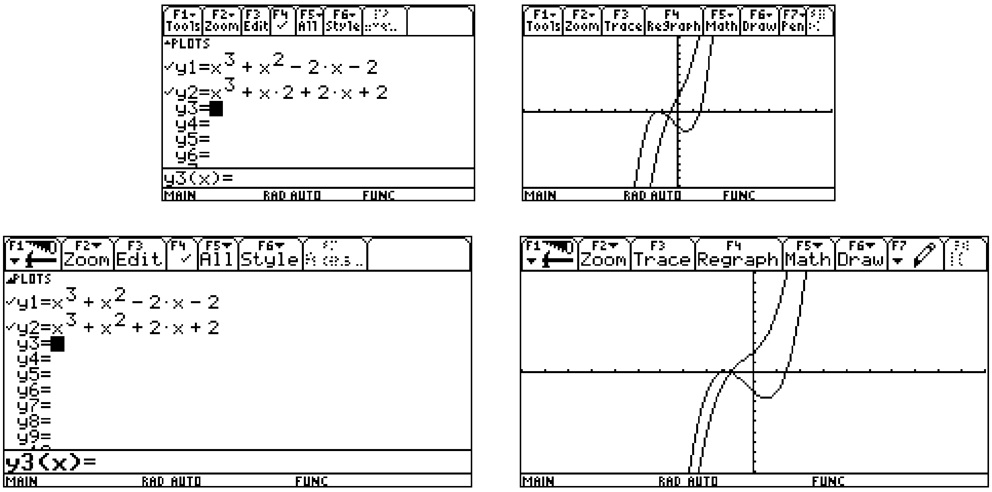


Figure 6: *Entering functions (left) and displaying their graphs (right)*

We return to the home screen for a calculus exercise. Press [F1] option "8" and the "Clear" to have a clean work space. We will integrate x/(x^3 + 1) with respect to x. Press the blue key labeled 2ND followed by "7". This places "∫(" on the command line. Type "x÷(x^3+1)", press "Enter", and the resulting display is that shown in Figure 7.

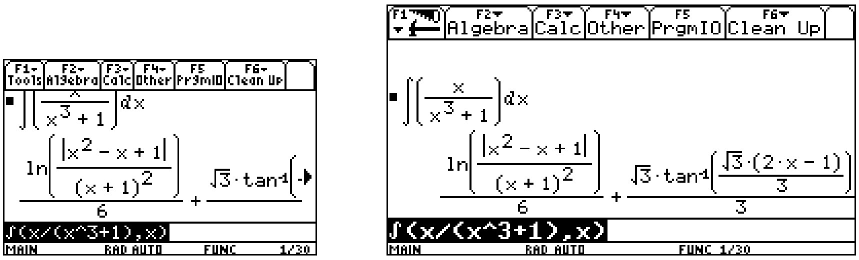


Figure 7: *An example calculus problem*

Note that to read the entire answer on the Titanium, we need to arrow up and scroll through the answer. Because of the smaller screen size of the Titanium, we will use the Voyage 200 to generate most of the future screen shots. It is a price we pay for having so much power and versatility in a machine that can easily fit in our pants pocket.

These examples illustrate a lot of manipulative power. Our job is to determine how we can use it to enhance learning and not as a substitute for thinking. That is what we hope to illustrate in the next sections.

An Excursion into Algebra

In this section we will use fundamental properties of polynomials and exponentials to solve an equation that can not be solved using the built in “solve” operator. *Acknowledgement*: Professor Marvin Brubaker of Messiah College in Grantham, PA suggested this example.

Consider the equation:

$$(x^2 + x - 1)^{x^2 - 2x - 3} = 1$$

Find all real values of x that satisfy the equation.

Our first thought is to use the powerful “solve” operator. Figure 8 shows the status of the calculator after about 7 – 8 minutes of waiting. Pressing the “ON” key terminates the computation.

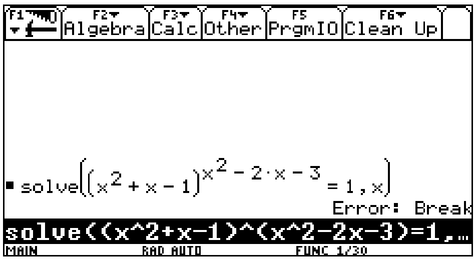


Figure 8: *Attempting to Solve an Unusual Equation Using the solve operator*

We will have to use our heads. However, the hand-helds can still help us. Go to the “Y=” editor and clear out the any previous equations that may be there. Enter the three equations as shown on the left in Figure 9. Note that y3(x) depends on y1(x), the base of the exponential and y2(x), the exponent. We can attempt to graph y3(x) together with the line y = 1 as shown on the right of Figure 9. The graph is rather messy and does not really help us find the solutions. It has “false lines” as it jumps from pixel to pixel on the screen. Computer graphics is a useful tool, but not for all problems.

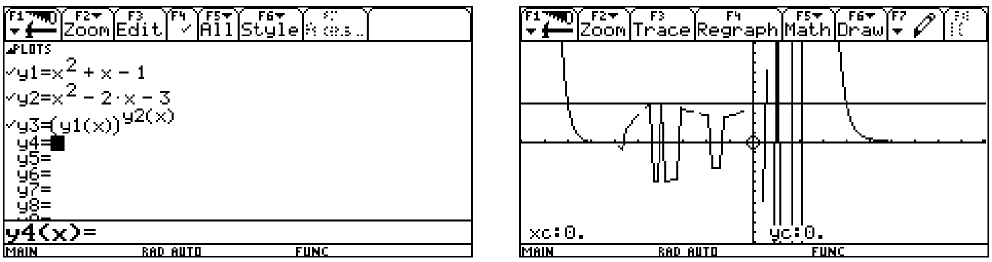
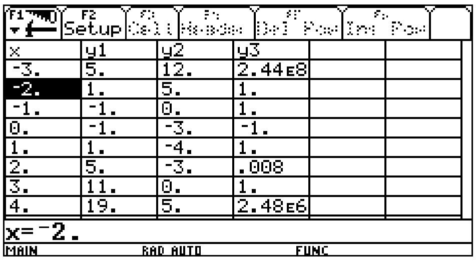


Figure 9: *Defining the Right Hand Side of the Equation and Attempting a Graphical Solution*

A better approach is to “look at the numbers” by using the “Table” APP. Enter this APP from the keyboard (green key with diamond followed by “Y”) or the main window. Figure 10 shows the table.



x	y1	y2	y3
-3.	5.	12.	2.44E8
-2.	1.	5.	1.
-1.	-1.	0.	1.
0.	-1.	-3.	-1.
1.	1.	-4.	1.
2.	5.	-3.	.008
3.	11.	0.	1.
4.	19.	5.	2.48E6

x=⁻2.

MAIN RAD AUTO FUNC

Figure 10: *Displaying Some Numerical Values for y1, y2, and y3*

The table shows that y3(x) = 1 for x = -2, -1, 1, and 3. This example was concocted to have nice solutions. What if the solutions would not have been integers, or worse still, not rational? We use this example as the basis for a more general solution that will work in all cases.

Exercises:

1. The table shows four solutions to the original problem. Are there any more? How do you know? What property of exponentials did you use to make your conclusion?

2. What characterizes a solution to an equation of the form $p(x)^{q(x)} = 1$ where $p(x)$ and $q(x)$ are polynomials? Write your conclusion as a theorem.

3. Have you considered all possibilities in your "theorem"? Consider the problem $(x^2 - x - 2)^{(x^2+x-6)} = 1$. Does your "theorem" hold true in this case? What types of exceptions, if any, must you consider before you have a universally true statement?

4. Construct an equation of the form $p(x)^{q(x)} = 1$ that has solutions at $x = -4, -1$, 3, and 5.

5. Construct an equation of the form $p(x)^{q(x)} = 1$ that has five solutions. Let p be a cubic polynomial. You choose the solutions to the problem. Use your hand held CAS to expand the polynomials p and q.

Cracking the Code – Constructing a Public Key Code

We use this exercise in our general liberal arts computing course at Gettysburg College. It is presented in the Computer Security section of the course. The students find the exercise challenging, but most of them are able to construct a code and decode the sentence that they are assigned.

Imagine a spies' "phone book" that lists all of the spies in my network (using pseudonyms, of course) and for each spy two numbers. If we want to send a message to one of our spies, say Jane, we simply look up her numbers and use them to encode the message. The amazing thing is that even if someone steals our "phone book" and intercepts the message to Jane, they will not be able to decode it. On the other hand Jane can decode the message easily using her hand-held CAS. We are using a Public Key Code that was invented by three mathematicians/computer scientists; Ronald Rivest, Adi Shamir, and Leonard Adleman. This type of code is known as an RSA code in honor of these three individuals. Their method has its roots in a result by the famous French mathematician, Fermat.

Public key codes rely on the fact that some mathematical operations are easily done, while reversing the operation is very difficult. For example, if we have two 300 digit numbers, it is relatively easy to multiply them and find their product. On the other hand, if we are given the product, it is a difficult task to find the two numbers that made up the product. This is especially true if we do not know *a priori* that both of the numbers consisted of 300 digits each. It appears that finding the factors of an extremely large number is intractable. It is possible to write an algorithm to find the factors that will do the job, but it can take an extremely long time to complete the task. In fact, it can take so long that the answer no longer has any value to the person who intercepted the message.

Here is how it works. Suppose Jane has a public key of $n = 7$ and $r = 35$. She also has a secret number, $m = 31$ which no one else knows. So how does it work? Suppose that we want to send a message. Suppose that we want to tell Jane to use sector 17 on a map. We will send her a simple message of 17. It is encoded in this way: compute the remainder of the exponential 17^7 when it is divided by 35. That is, send

$$a^n \bmod r, \text{ where } a = \text{the numerical value of the message.}$$

The first line of Figure 11 has the computation.

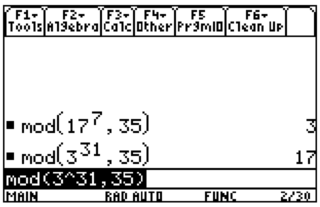


Figure 11: *Encoding and Decoding a Simple Message on the TI-89 Titanium*

How does Jane decode the message? Remember she has a secret number, m. This number is chosen so that it reverses the process.

$$c^m \bmod r, \text{ where } c = \text{the numerical value of the coded message.}$$

This is the second line in the display of Figure 11.

OK, what is under the hood? First of all Jane choose her public key, n and r, as well as her private, secret, key, m. She first chose two primes, p and q, and made $r = p\cdot q$ (Note: $35 = 5\cdot 7$). In practice, p and q would most likely be at least 300 digit long numbers, i.e. greater than 10^{300}. Now that Jane has r, she must get m and n. A theorem of Fermat guarantees that n and m will encode and decode the message if we choose them so that

$$mn = k(p-1)(q-1)+1 \text{ for some positive integer } k.$$

There are several possibilities for k, m, and n. In our example, we arbitrarily choose $k = 9$ which yields $mn = 217$. We are now sent to send any numerical message, a, as long as $a < 35 = r$.

Exercises

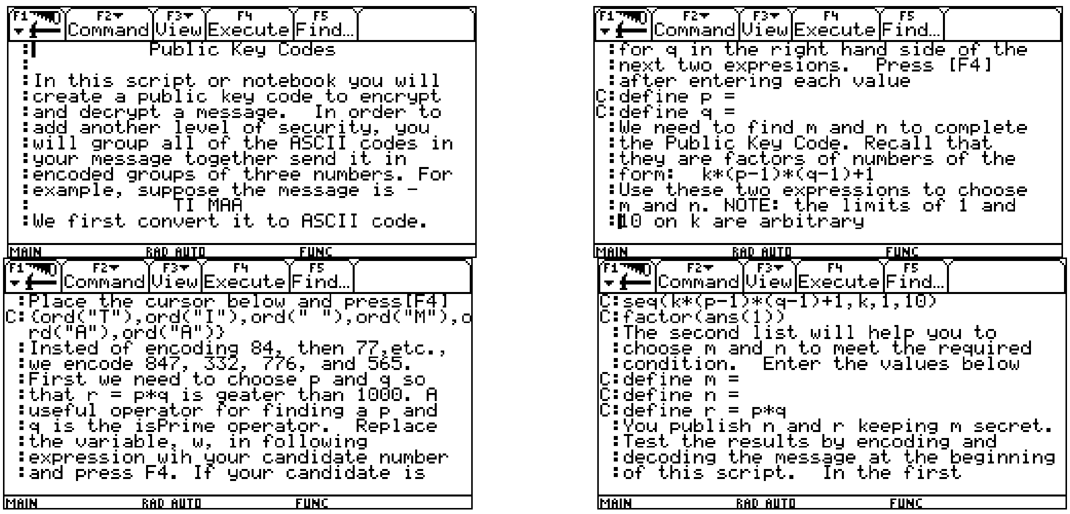
1. Show that Jane could have chosen $k = 2$ to send her message using the same p and q.
2. Enter the following on the command line:

   seq(char(k), k, 65, 90)

   These values of k are the ASCII code for the alphabet. Write down the ASCII code for your initials.
3. Let $p = 7$ and $q = 13$. Choose m, n, and r. Encode each of your initials using your public key. Check your work by decoding your coded initials.

The material shown as Figure 12 consists of screen captures from the “Text Editor” APP and should have been loaded on your machine with the Montreal folder. It is called “encrypt”. Open the “Text Editor” and at the welcome prompt choose: “Open” and then choose the Folder, “Montreal” and Variable, “encrypt”. Figure 12 is arranged to be read as two-column text. Within the hand-held, you can scroll through it using your arrow keys. A recommended practice is to press F3 and choose Option 1, Script View. Lines that are preceded with the letter “C” are commands and can be executed by pressing the F4 key. By choosing the Script View you can see the results of executing the command. In short, the text editor acts like the notebook feature in some other CAS’s.

This script is similar to the material that is given to the Liberal Arts Computing class at Gettysburg College. Credit for that material goes to Professor Jim Fink of Gettysburg College.



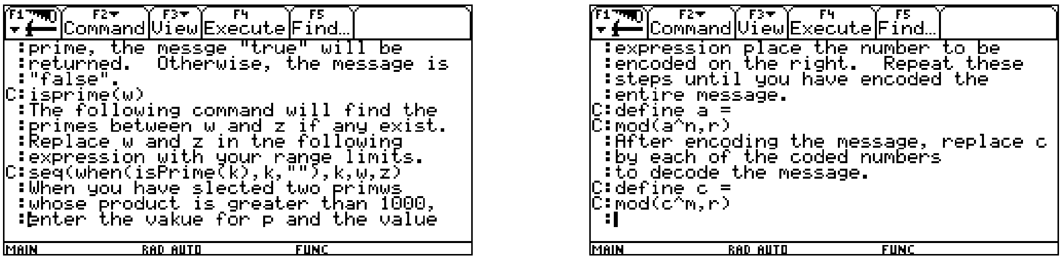


Figure 12: *A Script for Creating a Public Key Code*

Symbolic Math Guide

One of the wonderful things about the TI-89 Titanium and Voyage 200 is that they have room for adding applications. The Texas Instruments web site http://education.ti.com has an on line "store" for distributing these applications. Many of them are free. Others are very reasonably priced. We will look at the *Symbolic Math Guide*.

One of the complaints made about Computer Algebra is that it does too much for the students. The Symbolic Math Guide is an attempt to allow students control the solution process. For example, a student can direct the process for simplifying certain expressions, solving certain types of equations, and implementing the rules of differentiation. We will use the SMG in the next section to shed light on finding the exact roots of a cubic equation. In this section we will look at a general problem.

How does one find the slope from knowing only one point on the graph? As teachers, we need to motivate the answers to these questions. One way is to show that the tangent line is the "limiting position" of secant lines. Thus, it makes sense that the slope of the tangent line is the "limiting value" of the slopes of the secant lines. This means examining the slopes of the secant lines or looking at the "difference quotients" and simplifying them. We use SMG to look at the formula for the difference quotients for $\sqrt{(x)}$ at $x = 1$.

First we highlight the Flash APP labeled "Symbolic Math Guide" and press ENTER. We are greeted with the welcome screen shown in Figure 13

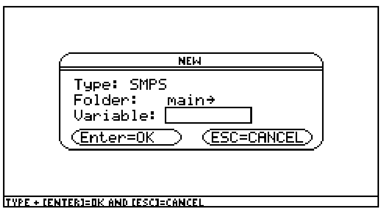


Figure 13: *SMG Welcome Screen*

Type the name "diffq" in the "Variable" box and press ENTER twice. That will bring up the screen shown on the left in Figure 14.

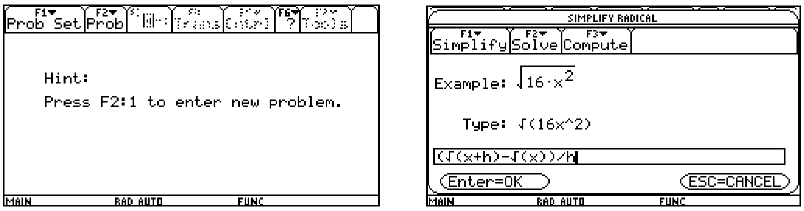


Figure 14: *Choosing an Expression with Radicals to Simplify*

We choose [F2] Option 1: New Problem and then press [F1] "Simplify" and choose Option 3: Radical. We type the expression for the difference quotient and press ENTER. The resulting screen is shown on the left in Figure 15.

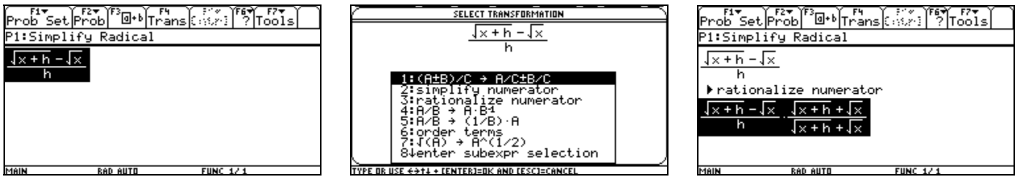


Figure 15: *The First Steps Towards Simplifying the Difference Quotient*

Our first step might be (We use might because there are other options available) to press [F4] "Transform". This brings up the expression on which we are working and the menu of possible actions. We chose Option 3: rationalize numerator. The screen on the left shows the result of this choice. Note that the option that was chosen in printed in the workspace and the result is show. Also note, SMG does not do to much for you. You need to direct the process. The file "dquote" contains a method for simplifying this expression. You may choose different steps. A usesful key is the [F3] key that allows you to pick out sub expressions to simplify. You can see the use of this operation in the file "dquote".

There are several ways that SMG can be used in class to involve the students in the solution of a problem. One is to go around the class having different students do a next step. Another is to divide the class into small groups and see which groups can come up with a solution involving the fewest number of steps. Another is to see which group can come up with a solution involving the greatest number of non circular steps.

Exercises

1. Create a new file called "integral" and choose [F2] > new problem > compute > indefinite integral and enter $\int((x + 1)/x, x)$ as the integral. Simplify and evaluate this integral
2. Press [F2] >new problem and enter $\int(x/(x^2 + 1), x)$ as the integral. Simplify and evaluate this integral

The Cell Sheet on the Handheld CAS – A Numeric and Symbolic Spreadsheet

From the very beginning TI CAS devices (TI 92, TI 92 plus, and the TI 89) have had a “Data/Matrix Editor”. This APP had many of the properties of a Spreadsheet and also allowed for symbolic manipulation. However, one could only refer to results in other columns and required the use of the “seq” operator to define entire columns. Yes, many of the functionalities of a spreadsheet were emulated, but it certainly did not have the versatility of a spreadsheet. The situation changed with the development of the Cell Sheet for TI Handhelds that are included with the purchase of a Titanium TI-89 or a Voyage 200.

The CAS Cell Sheet as a Numeric Spreadsheet

The Cell Sheet has the same functionality as any of the spread sheets with which you may be familiar. Alphabetical information is filled in highlighting the desired cell, pressing ENTER key and then typing a ″ (2ND “L”) followed by the desired information and then pressing the ENTER key again. Numbers are entered by pressing the ENTER key, typing the number and then pressing ENTER again. Formulas are typed in the same way they are preceded by an = sign. Figure 16 illustrates the process of building a Cell Sheet. The highlighted cell is entered as a result of a formula.

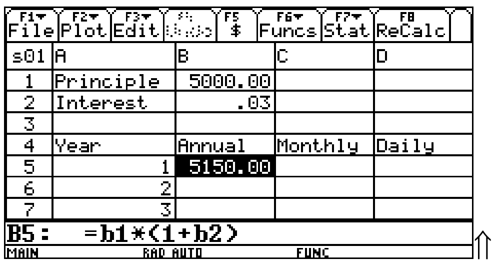


Figure 16: *The First Steps in Building a Savings Account Spreadsheet*

The Cell Sheet that we are developing is to show the differences in the results of depositing a particular amount of money at a given interest rate when the interest is compounded annually, monthly, and daily. The user of the Cell Sheet will have the ability to change the initial principle and the yearly interest rate. In order to enter a general formula into a range of cells, first highlight the cells by holding down the shift key, ⇑, and moving the along the range with the arrow keys. Next press [F3] and choose Option 3: Fill Range. A dialogue box appears with the range filled in. The user types in the formula as is illustrated in Figure 17. Note that absolute addresses are preceded with a dollar sign, $. This is similar to the practice in most of the common spreadsheets.

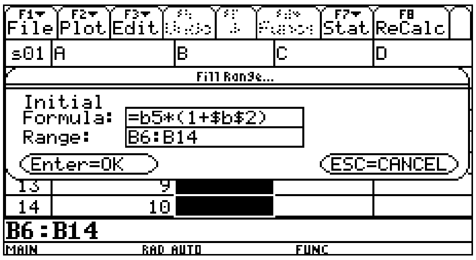


Figure 17: *Filling a Range of Cells with a Common Formula*

This is typical of the types of operations that you will use when working with the Cell Sheet. It remains to fill in the other columns of the Cell Sheet. Figure 18 shows this process.

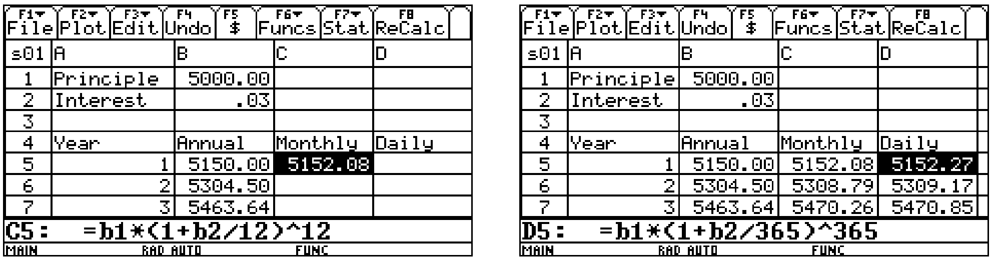


Figure 18: *Entering the Formulas for Monthly and Daily Compounding*

It is interesting to note that the big difference occurs between the Annual compounding and the Monthly compounding. Going from Monthly to Daily compounding nets less than $3.00 over the ten year period. Of course, if the initial principle is larger or the interest rate is higher, this difference can be significant.

The Symbolic Cell Sheet on a CAS

This is a really attractive feature on the CAS version of the cell sheet. It can be a useful way to have students explore and generate conjectures. Some possibilities are to study the effect of changing some of the parameters of a polynomial on the roots of the polynomial. Another possibility in a calculus course would be to search for a general pattern for the effects of increasing n in the derivatives or integral of functions such as:

$$\frac{x}{x^n+1} \quad \text{or} \quad \cos(x^n).$$

The Cell Sheet makes setting up such investigations a rather easy task.

Our task will be to find the equation of a circle through three points that are entered into the Cell Sheet by the user. The general equation for a circle with center at (c_1, c_2) and radius, r, is: $(x-c_1)^2+(y-c_2)^2=r^2$. We have three values for (x, y). That will give us enough material to write three equations in three unknowns. The surprising fact is that these equations will turn out to be linear in c_1, c_2 and a third variable we will call d. We can rewrite the equation as:

$$2xc_1 + 2yc_2 + d = x^2 + y^2 \quad (1)$$

where

$$d = r^2 - (c_1^{\,2} + c_2^{\,2})$$

Now we are ready to set up the Cell Sheet. Figure 19 shows the sheet. The highlighted cell is equation (1) above where the references to the cells are the particular values for x and y entered by the user of the Cell Sheet. Column D lists the values of the three parameters in equation (1), c_1, c_2 and d. These values are found as result of the Computer Algebra statement in cell C6.

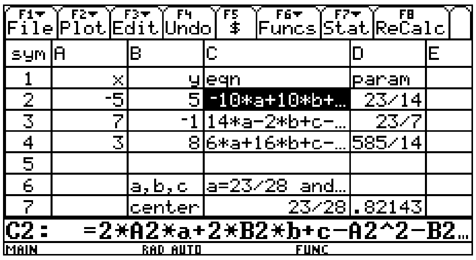


Figure 19: *An Example of a Symbolic Cell Sheet*

On the left side of Figure 20 cell C6 is highlighted. Note that the command line shows the formula for the cell. It is a command from the CAS's Algebra commands. It is the solve operator applied to a linear system of three equations in three unknowns. The right hand side of the figure shows the result of [F3] Option 9: Pretty Print. Note that this displays the contents of the cell and not the formula that created the cell. The option to show the formulas for the cell on the command line is [F1] Option 9: Format. If you desire, you can show the values for the cells on the command line. I prefer to have the look and feel of a spreadsheet.

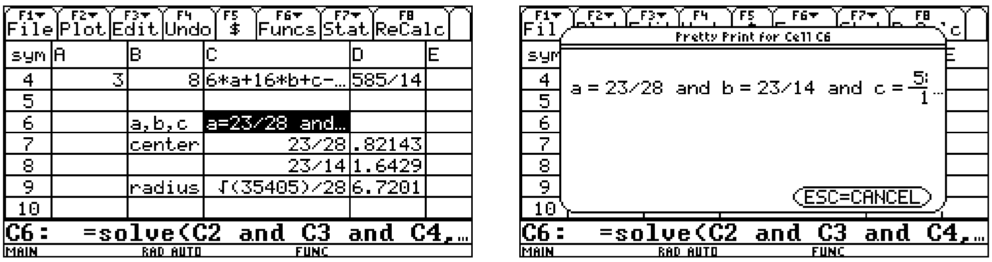


Figure 20: *Displaying the Result of an Algebraic Cell Sheet Computation*

A Heuristic that Works – Motivating Regression

In this section we will use a common sense approach to finding a best fitting curve. We will not prove any formal theorems, but will rely on method that seems to make sense based on physical evidence. What happens is that a least square best fit is developed as a result of our heuristic approach. We do not prove this, but hope that it gives some insight to the many regression techniques that appear on today's handhelds. In fact we do not even mention the term regression and leave it to the instructor to use or ignore.

Motivation from the playground

Suppose that a 150 pound person and a 100 pound child wish to ride a teeter-totter. If they wish to make the work load equal, they will place the pivot point in a position where the teeter-totter balances. This means that the pivot point will be off center. Suppose the teeter-totter is 10 feet long. Figure 21 shows the arrangement .

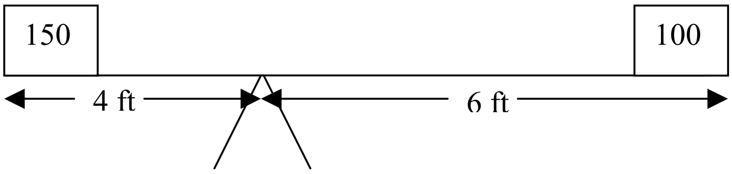


Figure 21: *A Teeter-Totter Balancing with Unequal Weights*

Note that the pivot point for the Teeter-Totter needs to be moved towards the heavier person so that the product of the person's weight times the distance is equal to the same product for the other person. In our example both products are equal to 600.

Where to put the warehouse?

A department store chain has six stores in a particular district. Top management of the store chain decides to place a warehouse in the district to supply these stores. Ideally, the management would like to place the ware house in a position that is equal distance from all six stores. This would mean finding a circle that would pass through all six of the stores. They look at a map and find that this is impossible. Hoping for the best, they put a grid over the map and assign coordinates to each of the stores. The coordinates of the stores are at (5, 5), (7, -1), (3, 8), (0, 2), (-2, 0), and (-1, -2). Figure 22 shows these locations

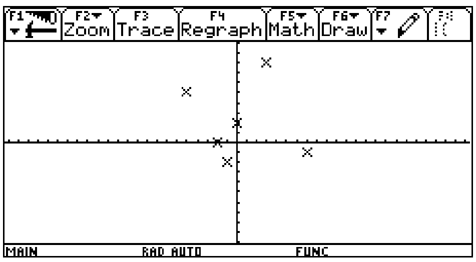


Figure 22: *The Store Locations Laid Out on a Grid – Obviously Not a Circle*

We can try to proceed as in the previous example. We enter the points and substitute in the equation for the circle as is shown on the left in Figure 23. However, this gives us 6 equations in 3 unknowns. This is an over determined system and generally has no solution. We could pick three of the equations and solve them for the center and the radius, but our answer would depend upon which three points we choose. We could solve every subset of three equations from the original set of six and average the answers for the center coordinates and radius. This would give us 20 sets of equations to solve. Furthermore, our final answer would not necessarily yield a good choice. One final idea is shown on the right side of Figure 23. That idea is to average all of the equations. The CAS will add coefficients of like terms when we sum the six equations. This is the start of a good idea, but it only gives us one equation in three unknowns, an under determined system.

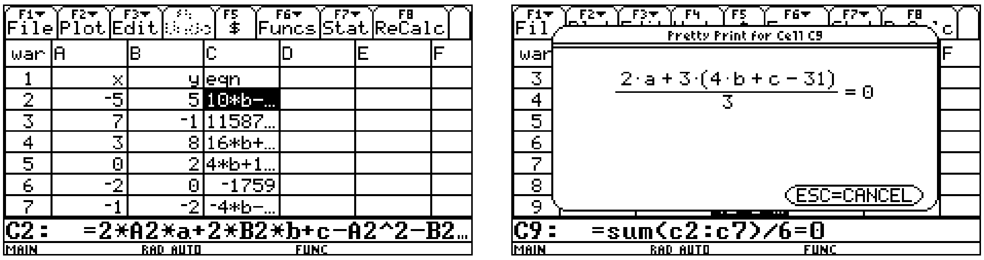


Figure 23: *Entering the points, the related equations and averaging the equations*

Actually, the last idea shown on the right of Figure 23 is not a bad idea. It just doesn't give us enough equations. Let's combine it with our playground experience. If we weight the equations in column C by the *x*-coordinates of the points, we then would have the effect of pulling the center of the circle to the right or left. If we weight the equations by the *y*-coordinates, we will pull the center up or down. These averages would also affect the radius to adjust to the new position of the center of the circle. We implemented this strategy as shown in Figure 24.

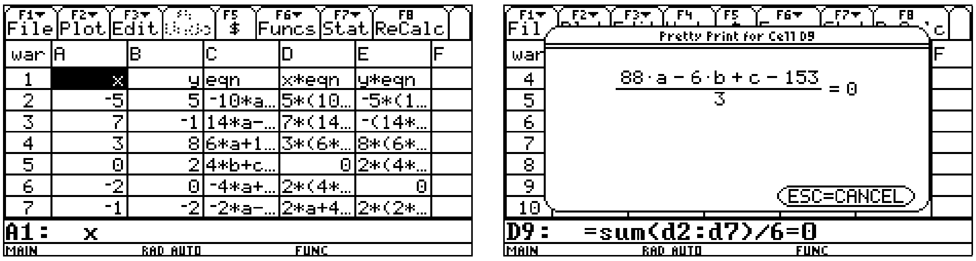


Figure 24: *Adding the Equations weighted by x and y and averaging them*

We have reduced our original set of six equations averaging the equations and by weighting them using the *x* – and *y*-coordinates of the points that generated the equations and then taking the averages of these equations. We now have three linear equations in three unknowns. Given the way in which these equations were generated, they should be independent and have a unique solution. Finding this solution is shown in Figure 25 together with the graph of the points and the resulting circle. This circle actually turns out to be the circle which best fits the points. The proof of this statement requires a bit more theory.

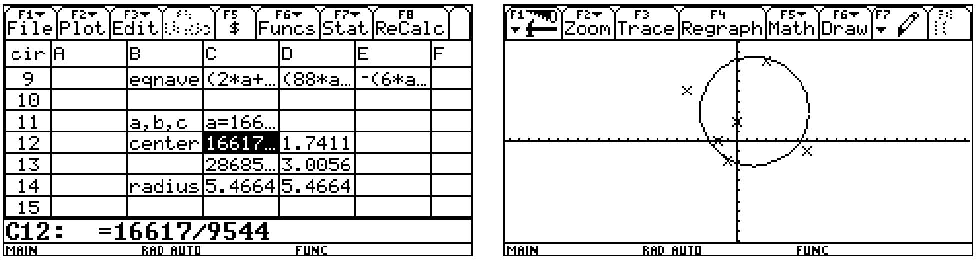


Figure 25:*The Solution to the Problem and the Graph of the Points and the Circle*