RANDOM WALKS, RANDOM SHOTS AND DISTRIBUTIONS OF SAMPLES WITH CABRI 2 PLUS

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Abstract
We will show different applications of the random function of Cabri to illustrate statistic problems:
1/Simulations of random walks: examples on a line, in the plane and into the space.
2/ Random shots: examples of estimation of ares in the plane ans volume into space.
3/ Curves of random functions to illustrate distributions of samples and modelize their variations in a dynamic way.

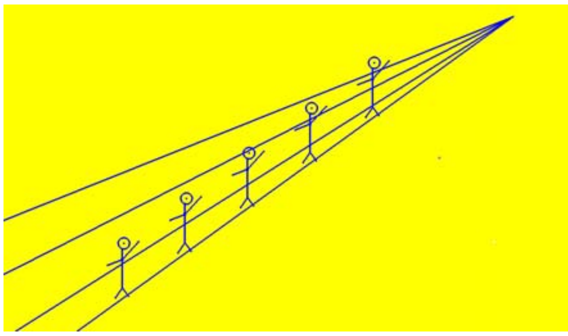
1. THE DIFFERENT WAYS TO RECEIVE AND TO USE A FIGURE

According to Duval there are four different ways to receive and to use a figure to solve a problem of geometry. Here below are these four ways that Duval calls in French “les 4 appréhensions d’une figure”..Each way is illustrated with a figure conceived in the Cabri environment

1.1. The perceptive way

This way is the one we use when we receive the figure with our eyes and when this one is automatically treated with our brain; this automatic interpretation lets us understand the figure as a 2D figure or a 3D figure

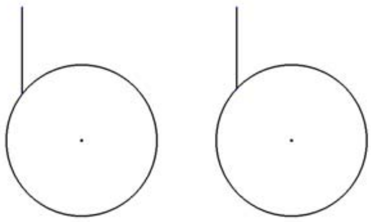
The figure on the right is received as a 3D figure where the different characters look bigger and bigger as they go backwards



Perceptive.fig

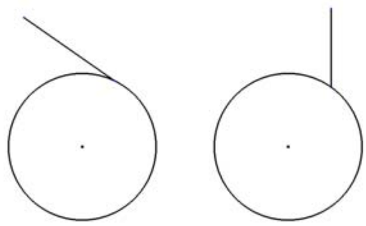
1.2. The sequential way

These 2 figures seem to be similar:



sequential1.fig

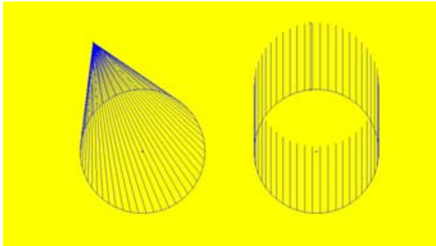
But if we drag the point of the circle belonging to the segment, the consequences are different:



sequential2.fig

The reason is located in the way each figure has been built. The step by step construction allows to plan what really could happen when we ask for the locus of the segment (when its second extremity moves on the circle)

sequential3.fig ⟶

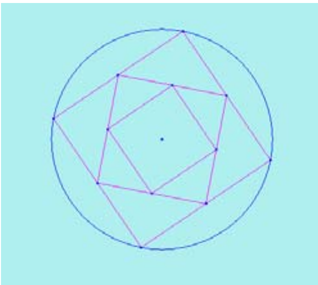


So the sequential way is the one we use when the figure is understood with the step by step process of its construction

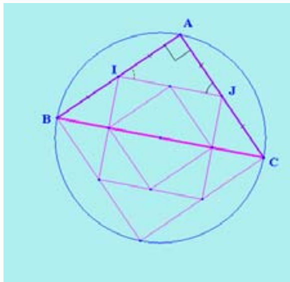
1.3. The discursive way

The figure on the left is given with conditions: the bigger polygon is a square inscribed in a circle and all the other points are middle points of other segments.
The figure on the right shows how it is caught in the discursive way: there is **a mental logic reasoning** that permits us to receive the second polygon as a square, not because we perceive a square but, because with logic reasoning using known maths theorems, **we have proven this fact.**

Discursive1.fig



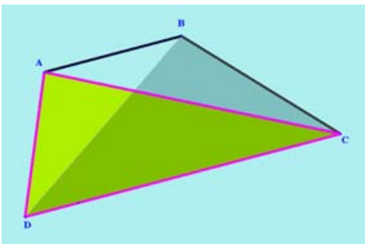
Discursive2.fig



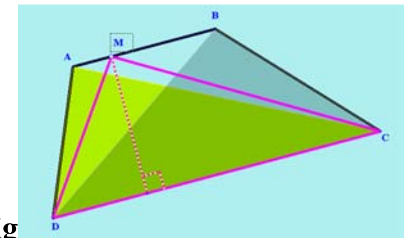
1.4. The operational way

First example: The problem to solve here is: let us prove that the two triangles ADC and BDC have the same area; a dynamical way to approach the solution is to imagine a deformation of ADC to BDC using the triangles MDC where M is sliding from D to B.
It is what we have done to the righthand figure: this allows the experimentator to realise that all triangles have the same basis and also the same height and so the same area.

operational.fig

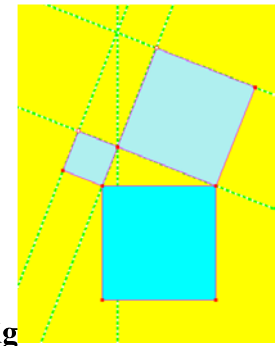


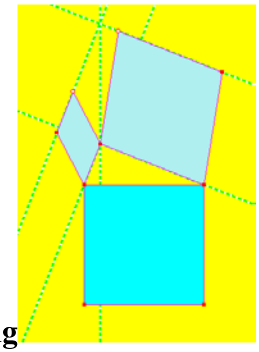
operational.fig

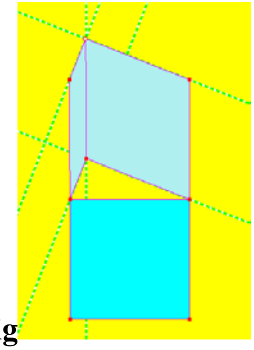


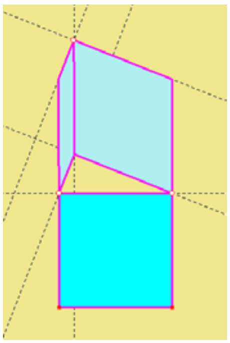
That is one of the operational ways

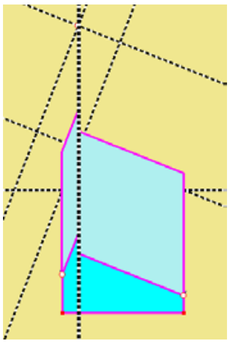
Second example: The example we give below is what we used to call a demonstration of the pythagorean theorem without any words

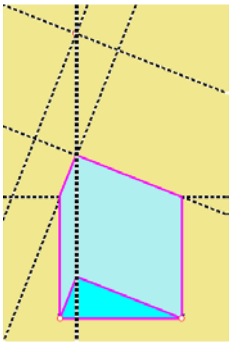
pyth1.fig


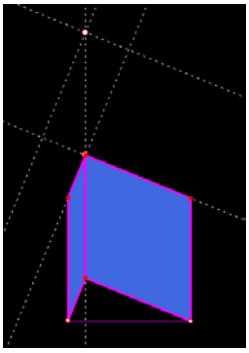
pyth1.fig


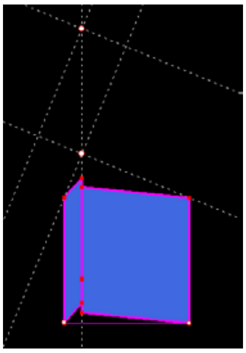
pyth1.fig


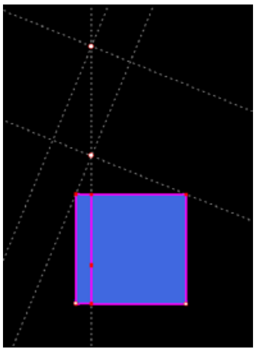
pyth2.fig


pyth2.fig


pyth2.fig


pyth3.fig


pyth3.fig


pyth3.fig


1.5. The roles of these ways in the heuristic processes of solving geometric problems

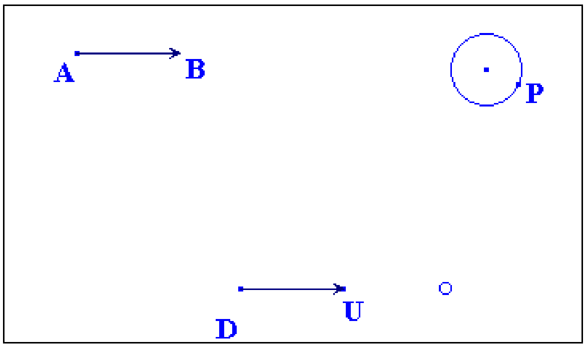
In his works Raymond Duval has shown that the ways that enhance the chances to reach the solution are **the perceptive and the operational ways**. So there are the more heuristic processes in the the way of solving geometric problems.
In our lecture we will give several examples showing how the use of Cabri can be heuristic because it allows very often to use the perceptive and the operational ways and espescially with the special tool "locus". We will show also that the sequential way cannot be always considered as non heuristic in the Cabri environment.

2. SIMULATIONS OF RANDOM WALKS

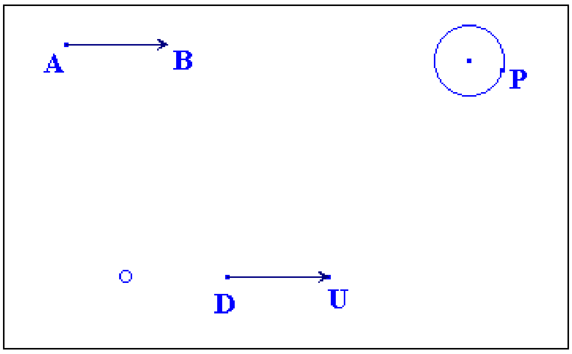
2.1. ON A LINE

A/ Simulation of a random jump

We want to simulate a random jump from D to another point (the circled one) located either at the distance 2AB in the direction of vector AB or at the distance AB in the opposite direction of vector AB. We want also that the choice of the circled point changes in relation with the location of pilot point P.

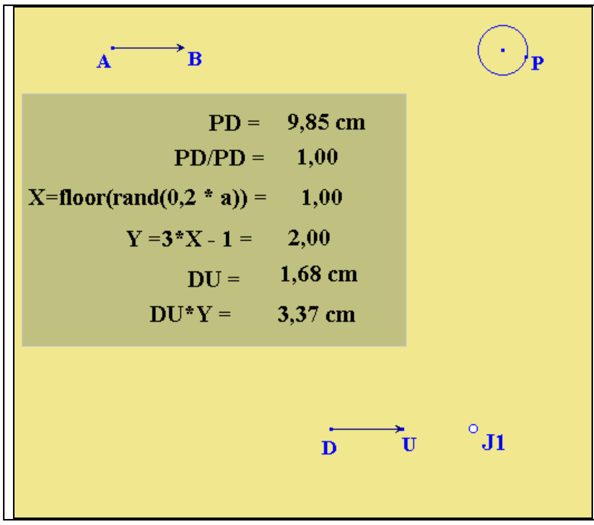


A randline.fig



A randline.fig

B/ The Cabri file to experiment (A randline.fig)



X is a random number equal to 2 or –1 in relation with the location of pilot point P.
Y depends from X:
Y= 2 when X=1
Y=-1 when X= 0
So Y is the abscissa of J1 in the axis (D, $\overrightarrow{DU}$).
J1 has been constructed as the measurement transfer of Y on vector $\overrightarrow{DU}$.
J& is the point got after our random jump from D.
Remark: J1 is the translated point of D in the translation with respect to vector $\overrightarrow{AB}$.

When this file is over, draggring point P generates randomly two different positions for J1.

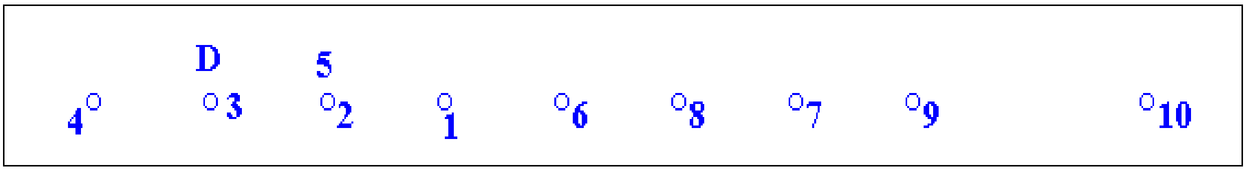
C/ Recording of this random jump algorithm

We create a new tool called a macro construction untitled “**2 or –1.mac**” having
As initial objects: Pilot point P, vector AB and starting point D
As final object: the random point J1

D/ Recording of a random walk algorithm

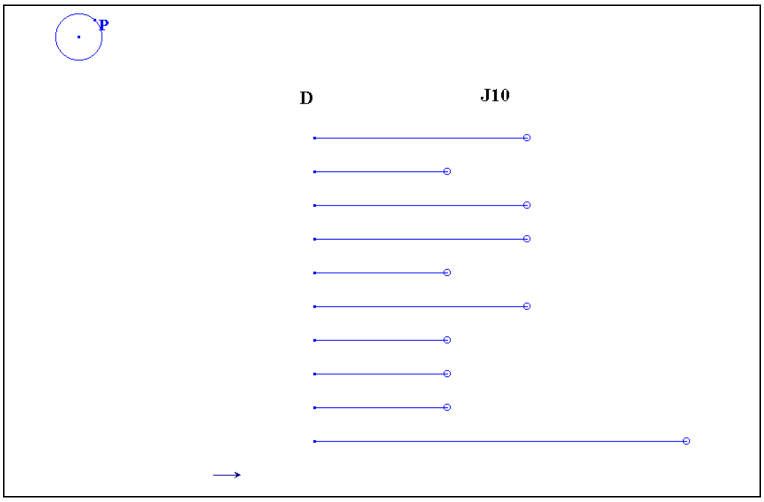
We apply this macro to J1 to get J2 and to J2 to get J3 until we reach J10.
We create a new macro construction untitled "2 or –1 ten times" having:

As initial objects: Pilot point P, vector $\overrightarrow{AB}$ and starting point D
As final object: the random point J10



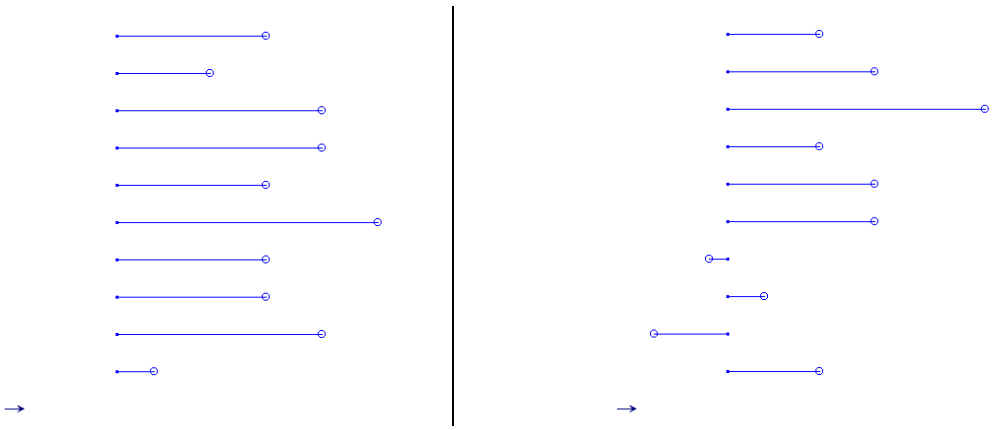
E/ Experimentation with the macro "2 or –1 ten times"

We have applied this macro to 10 different points and linked D to J10 with a segment

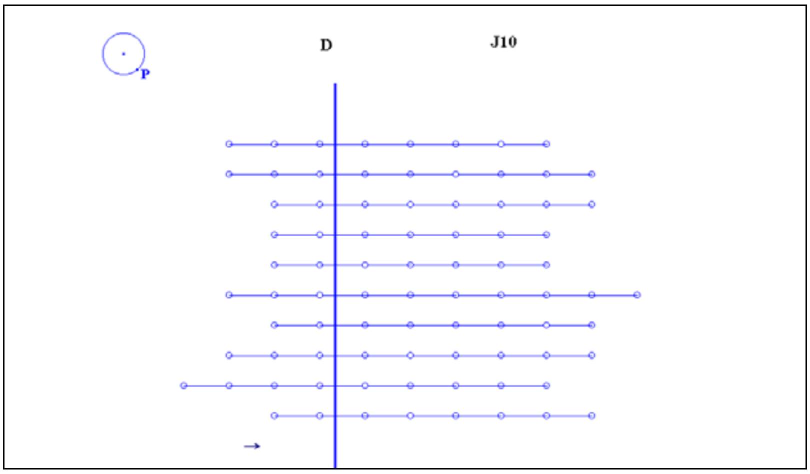


B randlinesamples.fig

Here we have dragged P to get new random walks of 10 jumps each:



We can notice that, it seems we have more chance to go right than left. We could estimate this conjecture when we start an animation of P after putting the traces of those segments ON.

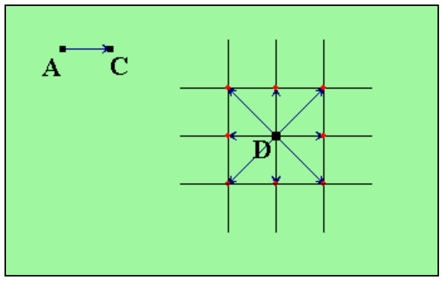


This conjecture is more plausible now because there are more points at the right of points D.

2.2. IN THE PLANE

2.2.1. Simulation of a random jump

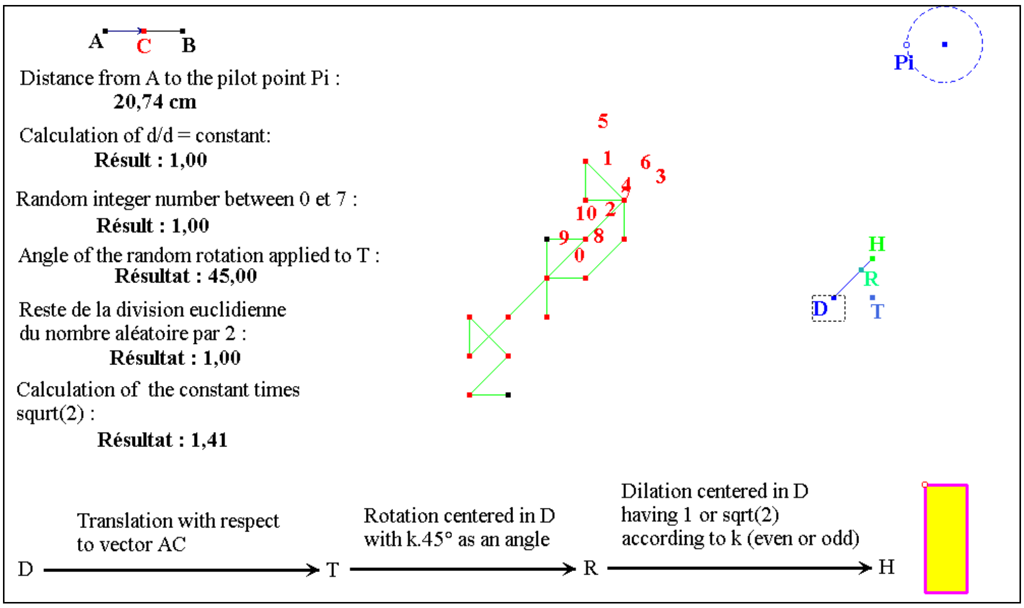
Starting from a point D of the plane, a point must jump randomly to one of the 8 points surrounding D on the grid. These points are defined with D and vector $\overrightarrow{AC}$. A sequence of that sort of jumps is called a random walk on a grid of the plane.



C randwalkM.fig

A/ The Cabri file to experiment (C randwalkM2.fig)

After the construction we can obtain :



C randwalkM2.fig

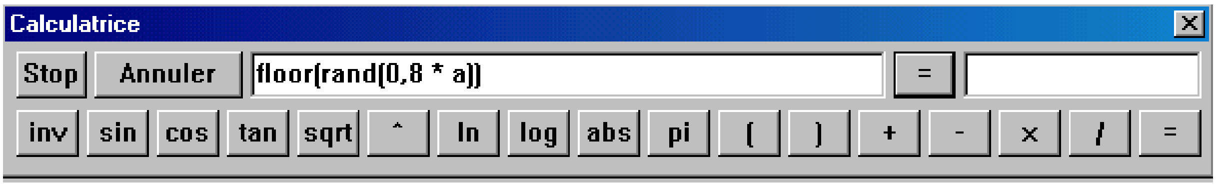
B/ What we have done to prepare the random jumping

♦A ray [AB) is created with 2 points of the grid (there are located on a parallel line to the system of axis)

♦A given dotted circle is constructed in the right part of the screen. With “Point on object” we have created a leader point called P on this circle. Un point appelé point pilote est créé sur ce cercle en tant que « point sur objet ». Distance d between this leader point and A must never be 0. This distance displayed here as **20,74cm is the first displayed numeric value.**

♦Ratio d/d is evaluated with the tool « Calculate » of Cabri and displayed as 1,00. this number depends from the location of the leader point but also from the location of A. Each new position of the leader point will generate a new calculus of ratio d/d. **This result has always the value 1 and will be the second displayed numeric value**

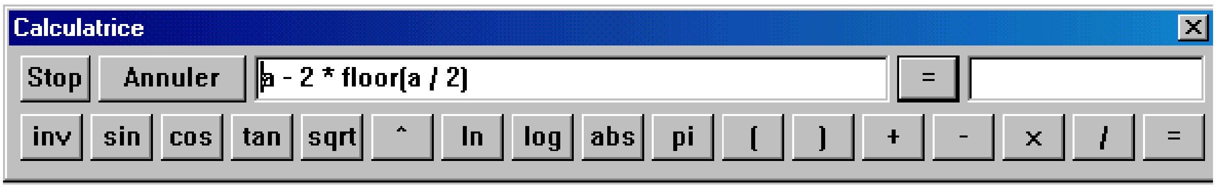
♦Then thanks also to the tool “Calculate” we create un random integer number between 0 and 7 using the formula: Integer of rand(0, 8.d/d) and edited like below :



where *a* is the previous d/d. Le numbet we get is the third displayed numerical value (here 7). Now this number will be called k.

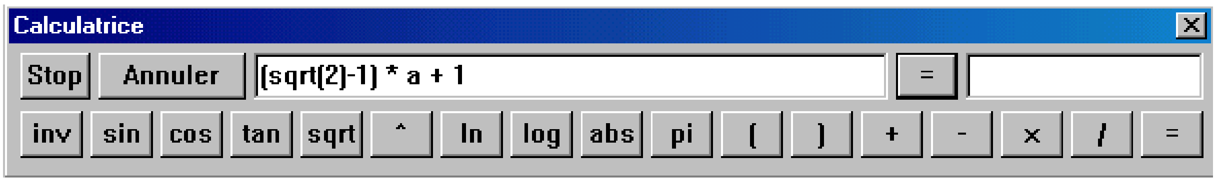
♦We calculate with the tool « Calculate » the result of k.45 to obtain one of the 8 angles necessaty to know the direction of the jump.. **This number will be the fouirth displayed value (here 45).**

♦Then we calculate and display the rest of the euclidian division of k with 2 which will give 0 or 1On fait ensuite calculer (toujours avec la calculatrice) le reste de la division euclidienne de k par 2 qui donnera 1 ou 0 using the formula, k – 2.Integer of a/2, edited like below. Number a represents k.



This number will be the fifth displayed value (here 1) The rest is 0 when the angle (k.45°) is in relation with an horizontal or vertical direction et is 1 when the angle is in relation with a direction having a slope of 45° or-45°.

♦Calculation of the image of the previous number using the linear function y = ($\sqrt{2}$ -1).x + 1. This image is equal to $\sqrt{2}$ when x equals 1 and 1 when x equals 0. This calculation is edited like below:



This result will be the sixth displayed value (here 1,41 which is the displayof $\sqrt{2}$).

C/ The sequence of transformations to realise a random jump

♦First value is transfered as C on the ray [AB) and then we create vector $\overrightarrow{AC}$.

♦We create a free point D in the plane (the starting point of our random walk) which is transformed in T withh respect to the translation of vector $\overrightarrow{AC}$.

♦We transform this point T with respect to the rotation centered in D and having as an random angle k.45 which is the fourth value displayed to get point R. This point R lies on the grid only if k is an even number, so the displayed fifth result is 0. When k is an odd numberR is located at the distance of 1 vector from D instead of $\sqrt{2}$; it is why :

♦We tranform R with respect to the dilation centered D and having as coefficient the sixth displayed value to get at last point H. H is the point we get after the random jump on the grid. H and R are superposed when k is an even number.

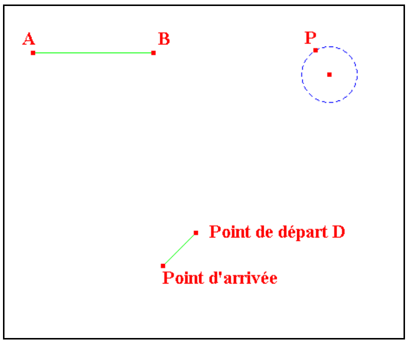
D/ Validation of this algorithm

If we drag the leader point, the positions of the mages of D are recalculated, so these images change randomly. T is always at the same place.

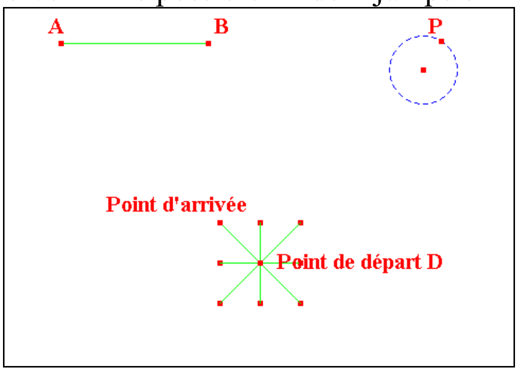
E/ Recording of this random jump algorithm

The technic we have shown is long so in order to realise a sequence of random jumping instead of one, we will record our construction in a macro construction. A macro construction is a tool which will permit us to generate the random point got after a random jump from one given point D and a leader point.

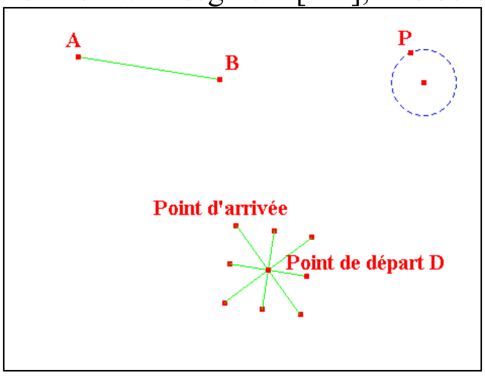
The macro « **Promenade aléatoire 1.mac** » admits as **initial objects**:
A segment which defines the grid where the random jumping will take place
A point which will be the leader point (here P), then
A point which will be the starting point of the random walk (here D)



This macro admits as final objects:
The point got after one random jumping from our starting point (here : « Point d'arrivée »)
And the segment linking the starting point and the point got after the random jump.
If we put the trace of this segment ON and if we start an animation of the pilot point, we can observe all the possible random jumps on the grid in relation with segment [AB], like below :



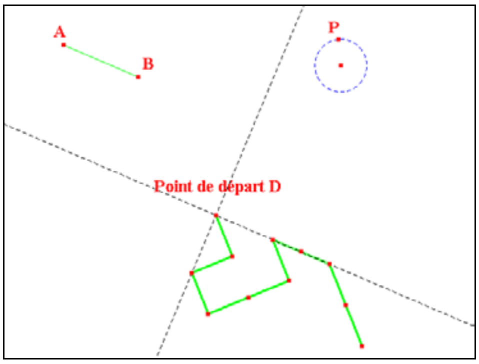
E randwalkMvalidation.fig



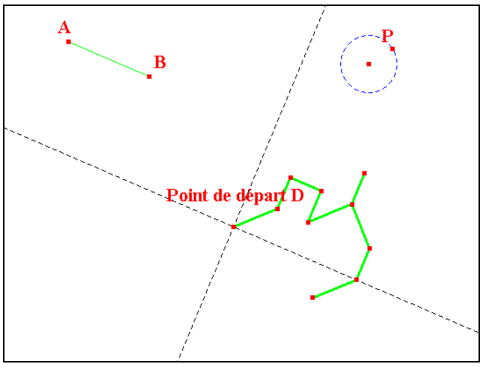
E randwalkMvalidationbis.fig

F/ Recording of the random walk algorithm

If we apply our macro on each point we have reached after a random jump, we get a sequence of random jumps. When we repeat 10 times this opération, we can visualise a random walk of 10 jumps that can be recorded as a new macro construction called « Promenade aléatoire 10.mac » which will be useful further. This macro admits the same initial objects than the previous one but now final objects are all the segments and points we have got at each jump.
If we drag the leader point, each position of this point generates a new walk. Below, two samples are displayed :



E randwalkM1validation.fig

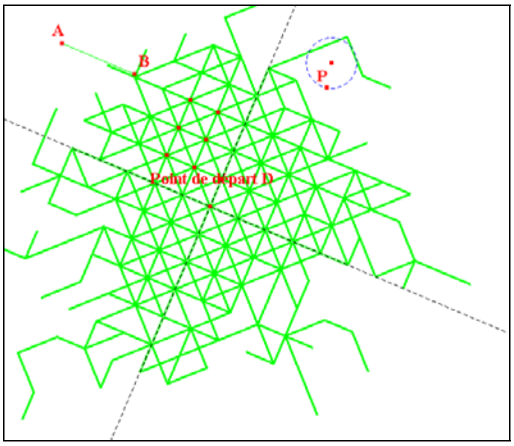


E randwalkM1validation.fig

G/ Validation of the random walk algorithm

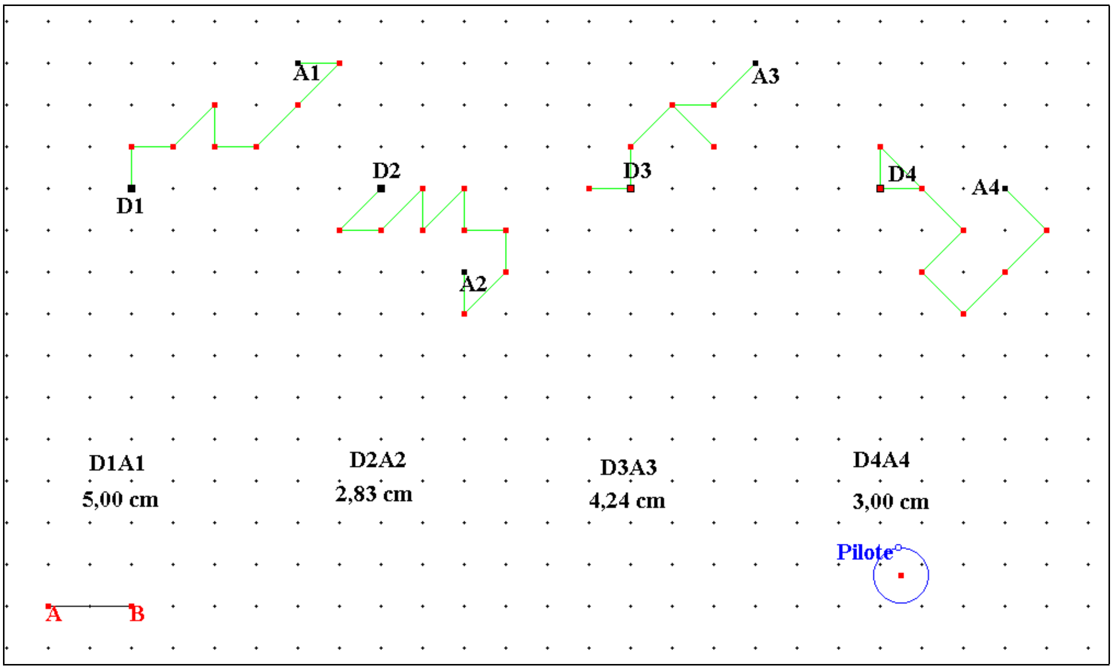
If we put the the traces of all segments of a walk ON and if we start an animation of the leader point, different random walks (having 10 jumps each) appear on the screen..

E randwalkM2validation.fig ⟶



2.2.2. The use of random walks for solving statistic problems

We can use the second macro to realise samples of random walk in order to study the distance between the starting point and the arrival point. We will have only to measure these distances and to store them in the data table of Cabri. If we start an animation of the leader point we can store all these random distances. All theses datas can be pasted in an excel sheet to be statistically treated.



D randwalkMsamples.fig

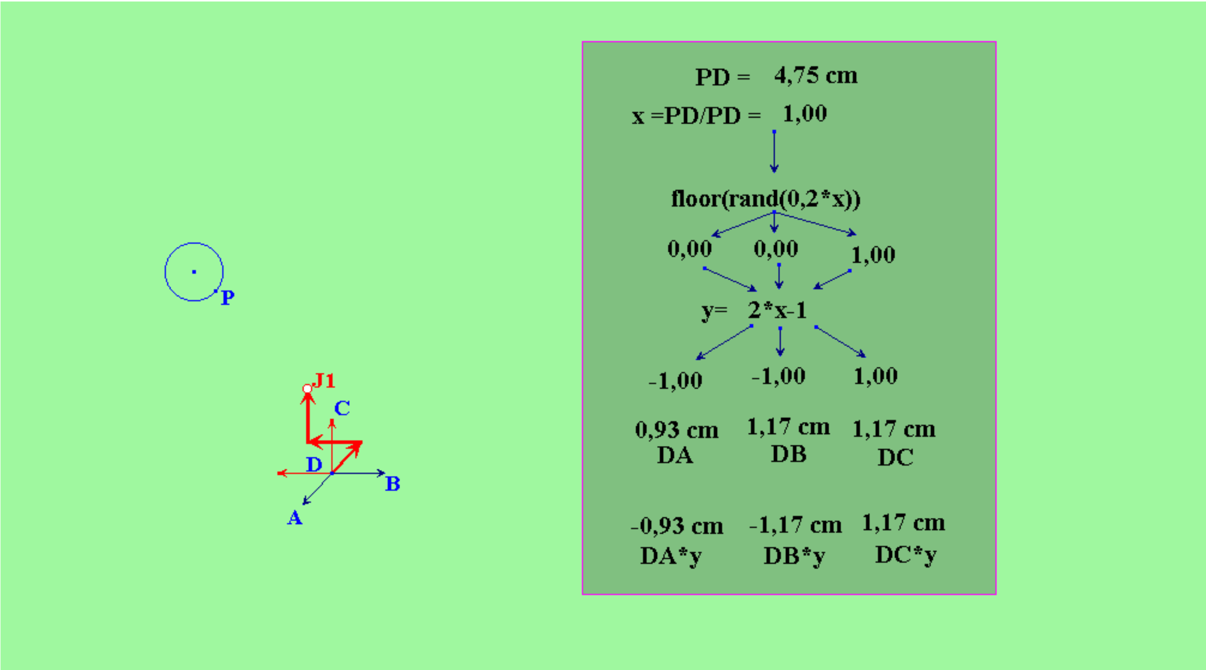
In the previous Cabri screen we have applied the second macro to points D1, D2, D3 and D4 to obtain the random pathes that reach points A1, A2, A3 et A4. Then, we have measured distances DiAi. If we catch these measurements in the data table of Cabri and if we start an animation of the leader point, 4 columns will be filled with the random distances of each random walk. We can imagine a distribution of samples with a big number of points Di (the size of each sample can be a number less than 999)

2.2.3. Conclusion

It is possible to tackle statistics and especially simulations with Cabri if we use solid knowledge about transformations.
We can imagine to give this file with the macros to the students so they can concentrate their attention to the problems of distributions of samples.
We can imagine also to construct this file with the students guided step by step. We will do with them some putting up like in Physics when they have to prepare an experimentation. This part is very important because they work both on the improvement of information technics and the improvement of their knowledges

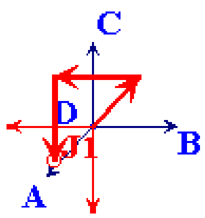
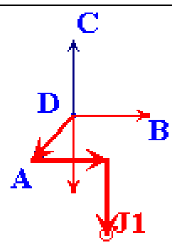
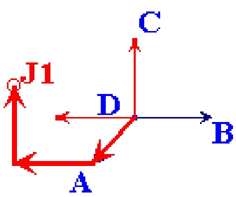
2.4. IN SPACE

Using the same ideas of the previous paragraphs, we can create the next file allowing us to jump randomly from a starting point D of space to another one J1 after 3 jups of 1 à-1 in the directions of the 3 vectors of a system of axis.
We can also create a macro constructing J1 starting drom D and from the 3 vectors of the system of axis (randspace and randspacebis).



F randspace1.fig

If we drag the pilot point P we get other random jumps

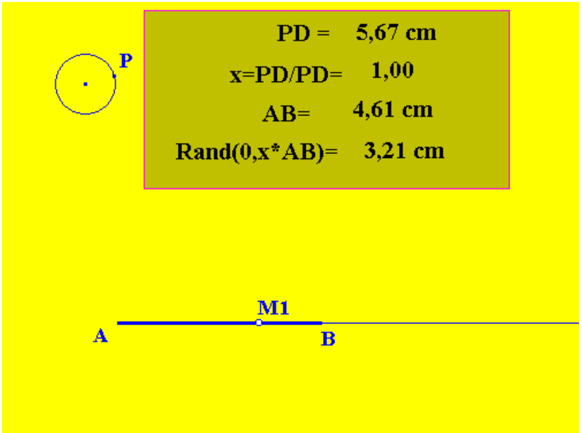
 **F randspace2.fig**	 **F randspace2.fig**	 **F randspace2.fig**

3. RANDOM SHOTS

3.2.IN THE PLANE TO ESTIMATE AREAS

3.2.1. Macro generating a random point on a segment

In the right file M1 is the point got by transferring number rand(0,x*AB) on the ray [AB). This number is more than0 and less than AB. So M1 will always lie on segment [AB] and change randomly when the pilot point P is dragged.
We have create a macro constructing M1 as a final objects when P and [AB] are given as initial objects ("randsegment")

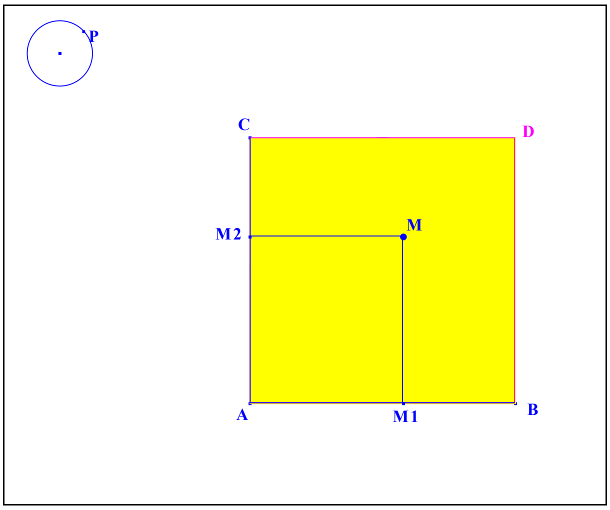


G shotsegment.fig

3.2.2. Random shots into a square (Monte Carlo method)

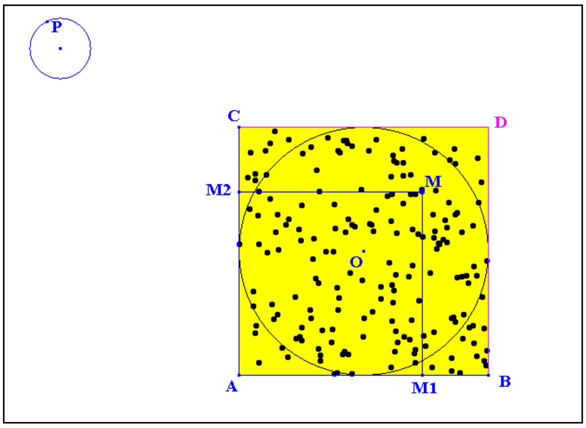
To get such a point inside of the yellow square, we apply the previous macro to segments [AB] and [AC] to get the random points M1 and M2. The random point inside of the square is constructed like in the right screen.

H monte carlo1.fig ⟶



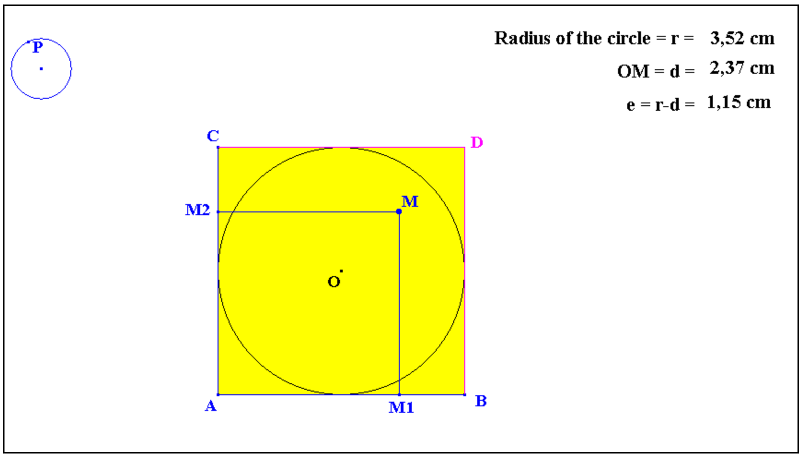
If we pu the trace of M ON and if we start an animation of point P, we will machine-gun the square to get the right screen.

H monte carlo2.fig ⟶



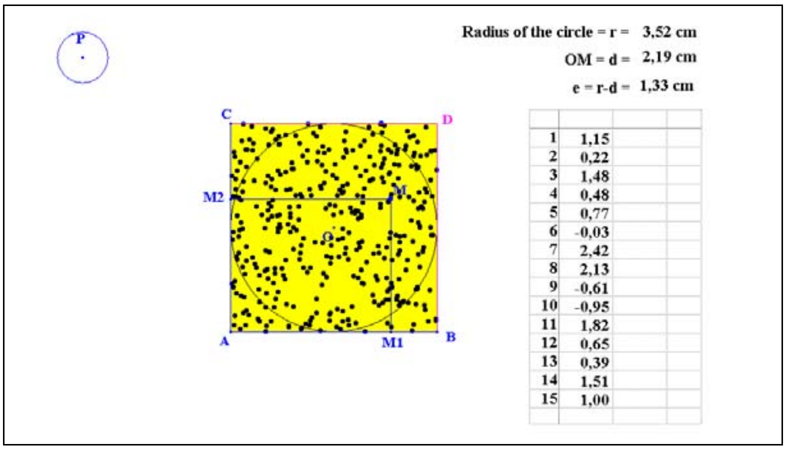
Here we have evaluated e = r-d; this number is positive if M is inside the circle and negative outside.

H monte carlo3.fig ⟶



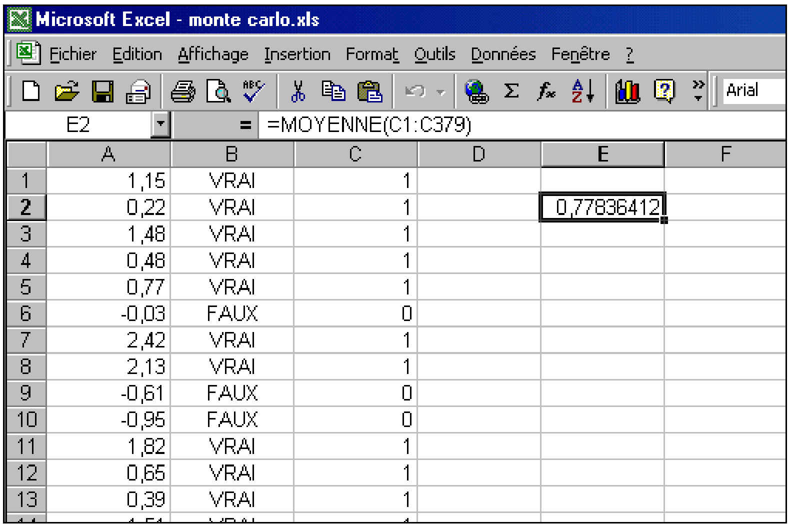
Here, during an animation of P we have stored the numbers e generated into the data table of Cabri. We have copy and paste this table on an excel sheet to be treated

H monte carlo4.fig ⟶



The datas are pasted on the first column.
On the second column a test gives us true if e is positive or equal to 0.
The third column contains 1 e is positive or equal to 0 and 0 if e is negative.
In the colomn E we have évaluated the mean of the 379 values of the third colum of this sheet to get 0.778

monte carlo.xls ⟶



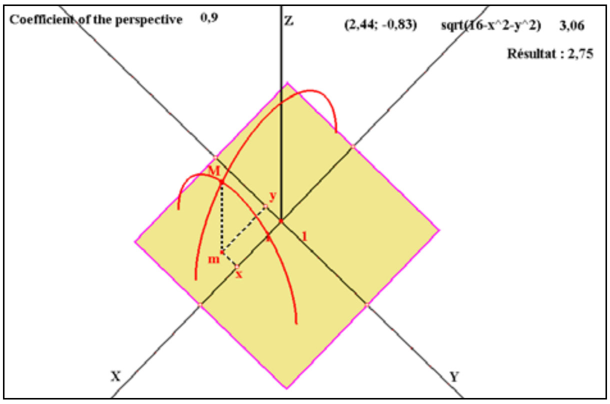
	A	B	C	D	E	F
1	1,15	VRAI	1			
2	0,22	VRAI	1		0,77836412	
3	1,48	VRAI	1			
4	0,48	VRAI	1			
5	0,77	VRAI	1			
6	-0,03	FAUX	0			
7	2,42	VRAI	1			
8	2,13	VRAI	1			
9	-0,61	FAUX	0			
10	-0,95	FAUX	0			
11	1,82	VRAI	1			
12	0,65	VRAI	1			
13	0,39	VRAI	1			

This last number is an estimation of the ratio between the area of the disk and the area of a square which is $\pi/4$. So our experimentation allows us to give the apprproximation of π equal to: 0.778*4= 3.112. This result, according to the size of our sample is a very good result (a lucky result)

3.3. IN SPACE TO ESTIMATE VOLUMES

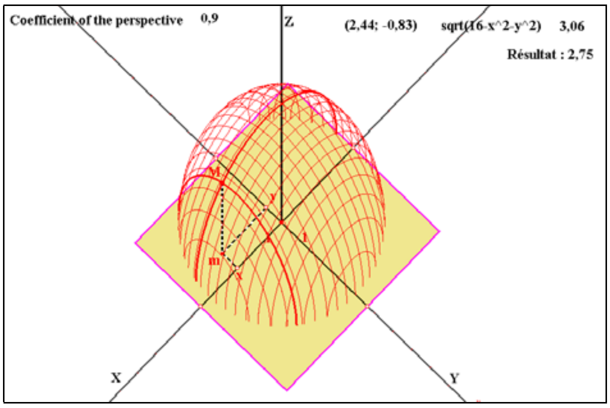
3.3.1.Curves of functions of 2 variables:

The example choosen is the representation of half a sphere in military perspective with a coefficient that can be changed. So we use the expression $\sqrt{16-x^2-y^2}$ edited in the Cabri file, sqrt(16-x²-y²) with the tool “Expression”. This expression is evaluated for the coordinates x and y of point m (this point depend of x and y that belongs to segments of the x and y axis)



I funcspacesphere1.fig

M has been constructed such as:
mM= f(x;y) = $\sqrt{16-x^2-y^2}$.
The 2 red curves are the locus L1of M when x moves and the locus L2 of M when y moves. They represent the sections of the surface with planes parallel to xOz and yOz planes.



I funcspacesphere2.fig

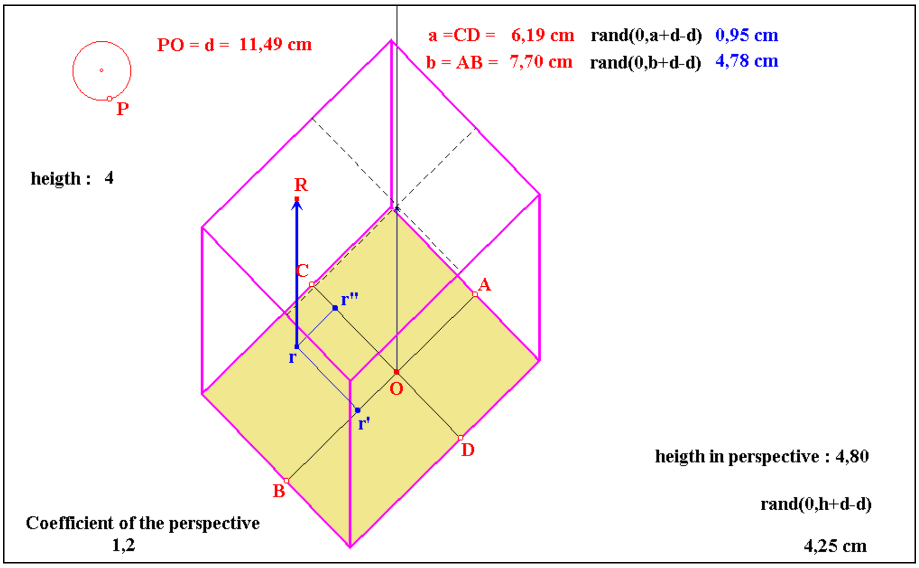
The frame of the surface is obtained with 2 loci:
The first is the locus of L1 when y moves.
The second is the locus of L2 when x moves.
The number of points in a locus has been fixed before: here we have choosen “20”.

3.3..2. Random function of 2 variables

Here is one way to generate a random point in a box represented below:

Creation of the random point r’ on the segment [AB]	Ar’= random number between 0 and AB
Creation of the random point r” on the segment [CD]	Cr”= random number between 0 and CD
Creation of the random point R on the vertical segment going from r to the top of the box	rR = random number between 0 and h (h is the heigth of the box

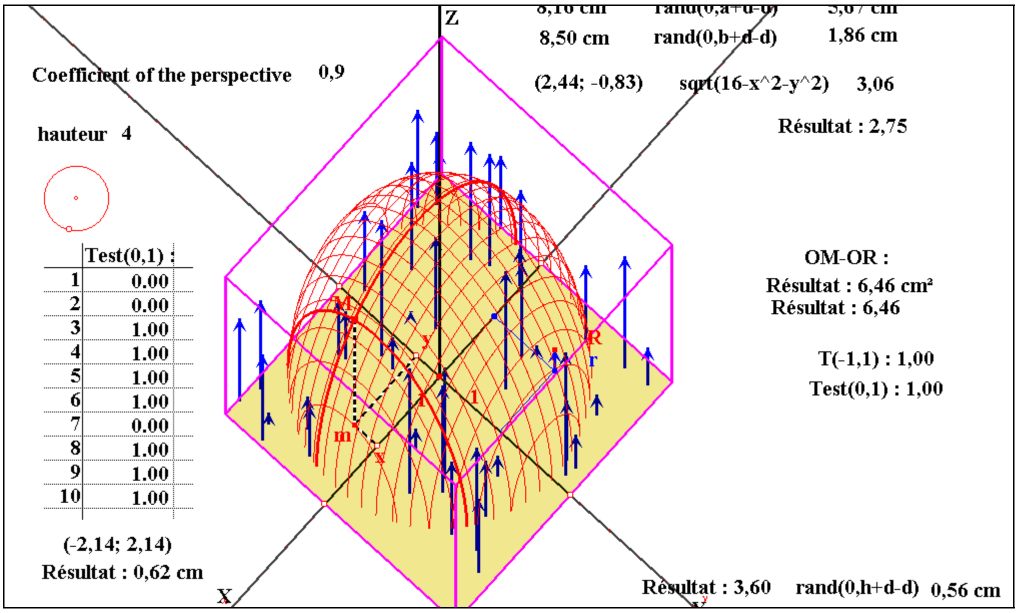
P is a point that can move on a circle. This point command all the random numbers calculated: to get this result we have use the distance d = PO in each formula using the “rand” tool



K funcspacesphereran1.fig

3.3.3.Evaluation of the volume under a surface with a random shot

If we create in a same page what we have created before in 5.2. and 5.3.
If we create a number equal to 1 when the random point lies inside the sphere and 0 if not.
If we catch in the table of Cabri these numbers during an animation.
If the trace of the random vector is activated, we obtain the screen captured below



K funcspacesphereran2.fig

We copy now this table in an excel page in order to calculate the mean of it. This mean k is the ratio of random points of the box lying inside the sphere. For a sample of size 70 we have got a mean of 0.514
So we can get an approximate value of the volume of half of this sphere with the formula k*v where v is the volume of the box, here: 8.16cm*8.50cm*4cm = 277.44cm². So:
k.v = 0.514*277.44 = 142.60cm².

Or the true volume is given by the formula $\frac{2}{3}\pi.(radius)^3 = \frac{2}{3}\pi.(4)^3 = \frac{128\pi}{3} = 134.04.$

This can be considered as a good result considering the size of the sample.

4. CURVES OF RANDOM "FUNCTIONS" AND STATISTICS

The experiment will use a dice; we will create samples having 10 or 100 as a size.
The variable is always the abscissa x of a point belonging to a segment included in the x abscissa. We have created 2 random functions:

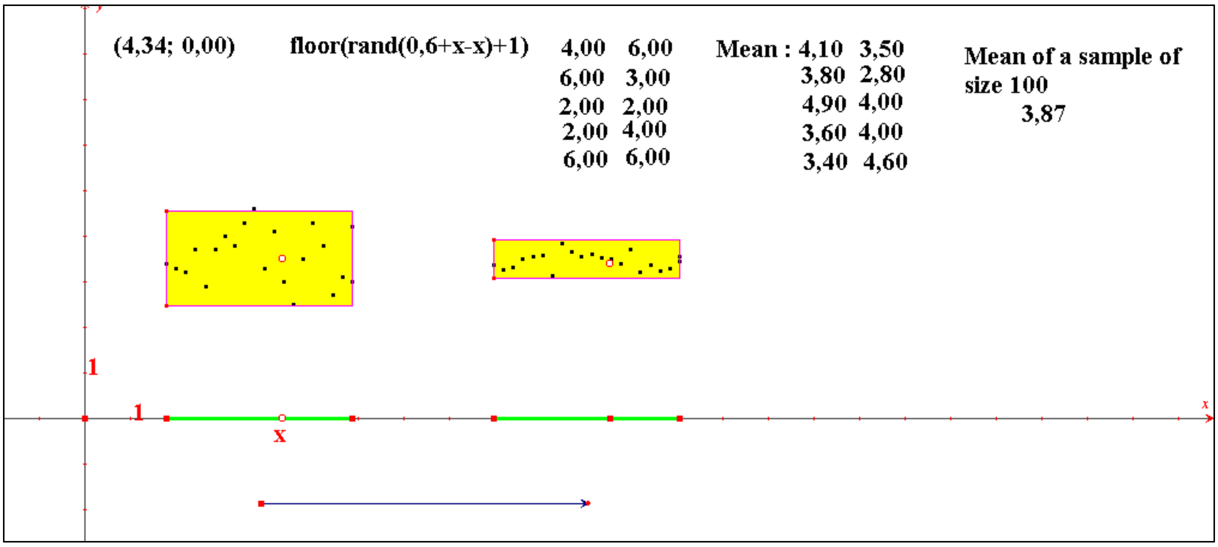
The first one: x ⟶ Mean of a sample having a size of 10

We first create the function x ⟶ Int(rand(0,6+x-x)+1) which generates a random integer number between 1 and 6. We record the function in a macro M1. We apply this macro 10 times to the same number x to obtain a sample of size 10. We calculate the mean of this sample; the calculus of the mean of 10 numbers is recorded in a second macro M2. At last we create a third macro M3 giving the mean of this sample of size 10

The second one: x ⟶ Mean of a sample having a size of 100

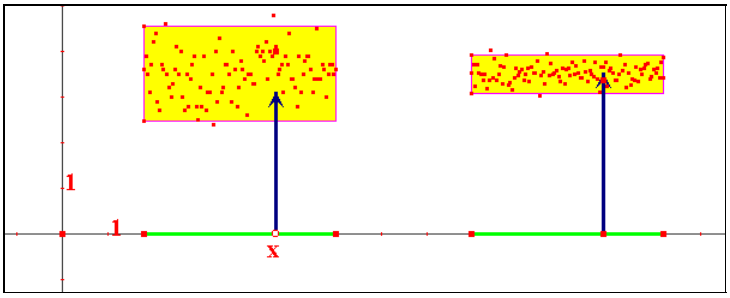
We apply the third macro M3 to x, ten times. We apply the second macro M2 to these 10 numbers to get the mean of a sample of size 100.

We build below one point of each curve of these 2 functions. We can obtain the 2 statistic graphics after choosing 20 points not linked in the options of loci. We obtain them in asking Cabri for giving the loci of these generic points when x moves.



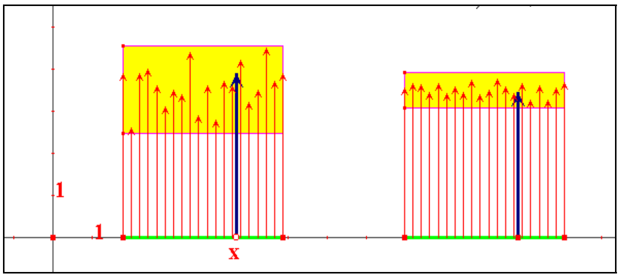
N funcrand1.fig

It is possible to visualise that the width of these curves depend of the size of the sample and not of the number of samples. Rightside we have increased the size of the samples and the width of the curve does not change.

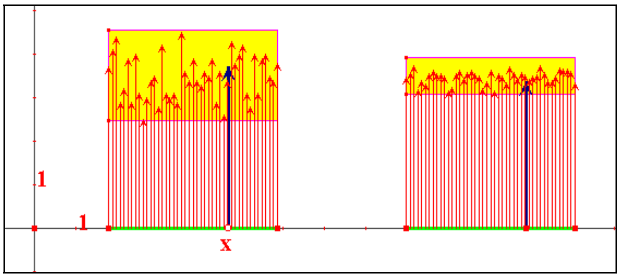


N funcrand2.fig

The 2 next screens represent the same phenomenom with loci of vectors in place of loci of points



M funcrand1.fig



M funcrand2.fig

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