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CONTS

Ten Years of International *Derive* Conferences

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ABSTRACT

Everything is going so fast. Mathematical computers programs are so powerful that we may think that it has always been the case. This lecture focuses on how technology has changed the way I teach mathematics, through specific examples.

A souvenir from Plymouth, U.K. in 1994: in version 3 of *Derive* for DOS , implicit 2D plots became available. Could you imagine investigating multiple variable calculus without implicit plots? Many ideas for using *Derive* as a teaching tool originated from this conference.

In Bonn, Germany, 1996, Bert Waits said, about the TI-92: "this machine will change the way we teach mathematics for the next fifteen years!". This handheld technology allowed computer algebra to be used in the classroom by teachers and students as never before, changing forever the opinion of many mathematicians about graphic calculators.

From Gettysburg, USA, 1998, I will recall two independent souvenirs. First, TI-89 and TI-92 Plus Module with Flash technology, giving the possibility to upgrade to future software versions without buying a new calculator. Second: the collaboration between David Parker (Acrospin, Cyclone, DPGraph) and the authors of *Derive* announced upcoming interesting results.

In Liverpool, U.K. in 2000, *Derive* 5 and its new interface became a reality and spectacular 3D plots capabilities were added. New programming capabilities were introduced, "that have revolutionised the way that programs can be written and displayed in the *Derive* mathematical environment" as Terence Etchells said.

Finally, Vienna, Austria, 2002. I asked the following question: if we use both systems, *Derive* and the symbolic TI, can the two devices communicate? Some Derivers were concerned with another question: are you organizing the next conference? Well, *Derive* 6 now offers the link I was looking for (and even much more, for example, the ability to show the steps of many simplifications). And without the collaboration of my colleagues Gilles Picard and Kathleen Pineau, this TIME-2004 symposium would not have been possible.

1. Introduction

Using three examples, we will see how *Derive* has changed. It became more sophisticated while it stayed easy to use —and some of us will say easier. *Derive for Windows* ("*Derive* 4") confirmed the passage from the DOS operating system to the windows operating system. If you are a teacher of multiple variable calculus, you will probably say that the most dramatic changes came from the new 3D plots capabilities in version 5. With version 6, teachers and students have access to a display step button. This new feature tells the users which methods *Derive* used for the simplification performed. And, in *Derive* 6, we can now understand how polynomial systems are solved, something quite complicated for the non mathematician: the new Groebner_basis function gives a better taste of what is going on (and Eugenio Roanes-Lozano's keynote lecture is a very good explanation of that).

Our first example is a typical multiple variable calculus problem at the age of technology. It was impossible to give such examples to students 15 years ago. The example will show the advantage of using *Derive* : simplicity and power. Our second example will use the "step mode" in simplifications of integrals: this is amazing to see what *Derive* is doing! Our third and last example will consist on another look to our first example by linking to the Voyage 200: another new feature of *Derive* 6.

2. The first example: the power of visualisation!

Let us take a look at a multiple variable calculus problem found in a textbook. We have to find the critical points of the function

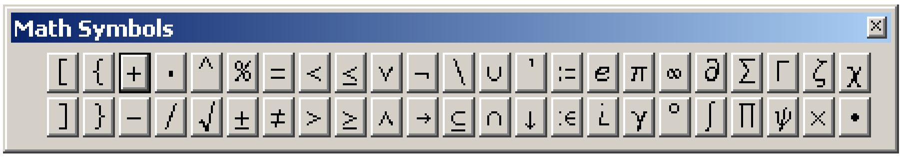
$$f(x, y) = \left(x^2 + xy + 5y^2 + x - y\right)e^{-\left(x^2+y^2\right)}.$$

First, you just don't ask this question without technology! Second, even with technology, there is still a lot to be done. Define this function by founding the good symbol for the exponential ..., plot the graph of this function in a correct 3D box, plot some level curves, find the partial derivatives and the critical points ... All of this can be done so easily with *Derive*, no complicated commands to know or to use. Let's do this.

With recent versions of *Derive* (5 or 6), we can rapidly observe 5 critical points because, after plotting, we just have to rotate the box. Version 6, using Gröbner basis methods (after simplifying the exponential factor in order to use Gröbner basis) can even find, in numerical mode, the critical points. Of course, with version 4 or earlier, one needs to use Newton's method 5 times in order to find the critical points. Let's show this, using *Derive* 6.01.

#1: $$f(x, y) := (x^2 + x\cdot y + 5\cdot y^2 + x - y)\cdot e^{-x^2 - y^2}$$

A comment about this (new) symbol for base *e*. Exactly the same as of the TI and as found in textbooks. And, as a teacher, take a look, with your students, to special mathematical constants or symbols that can be found on the algebra window. This is another important feature of *Derive* (it started with *Derive for Windows*):



Using the auto scale, we get easily this picture:

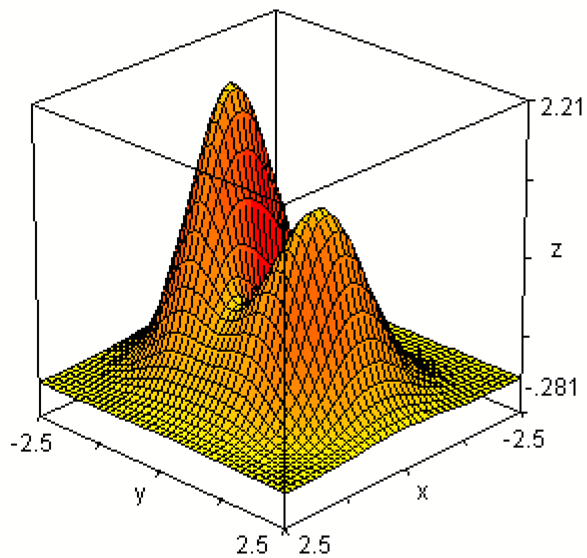


Figure 1

This figure 1 shows (and, of course, it would have been the same with another CAS) the *power of visualisation*! Without using pencil and paper, we actually see (at least) 2 local maximums. And, using the (new) feature of rotating the box with the mouse (or, as we used to do in *Derive* 5, with the arrow keys), we can, in fact, observe 3 more critical points, for a total of 5 critical points. And if you want to observe some level curves, you can now use the (new) slide bar, by intersecting the former surface with an horizontal plane, $z = c$, with c ranging from -1 to 2: here is a picture when $c = 0.5$.

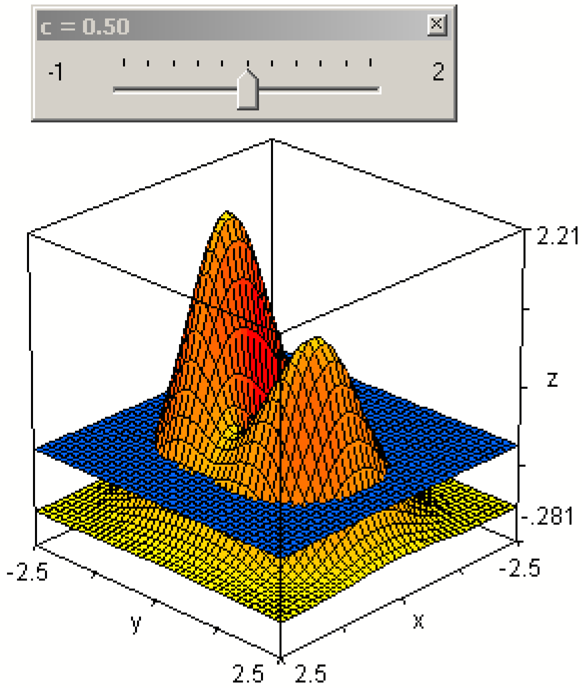


Figure 2

As a teacher, I want my students to recall that a critical point of a function of 2 variables (which is everywhere differentiable) is a solution of the simultaneous system of equations

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$$

Don't overestimate your students! Ask them to plot each curve defined by the above equations. And, here, with f defined earlier by #1, they will see how implicit plotting works if they zoom out on the graphs: of course, removing the exponential factors won't produce the same effect:

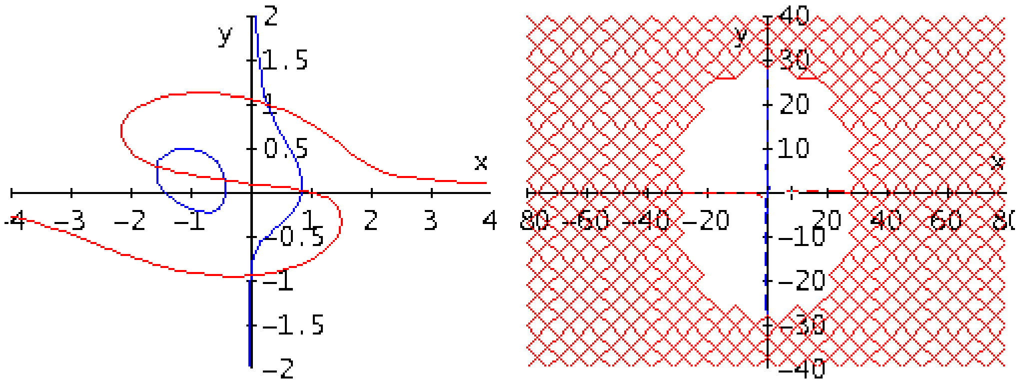


Figure 3

Let us recall that *Derive* plots a curve defined by an equation $u(x, y) = 0$ without having to use a command like "implicit 2D plot". This is again a great advantage. Of course, students will probably note that an explicit plot is faster and more accurate: another opportunity for the teacher to talk about how a CAS can produce implicit 2D plots.

Now, let us obtain the 5 critical points by using a *solve* command or a *solutions* command: here, Gröber basis method will lead to a fifth degree polynomial that can not be solved exactly. So, we obtain this by using *simplify* and, after, *approximate*:

$$\#2: \quad \text{SOLUTIONS}\left(\left[\frac{d}{dx} f(x, y), \frac{d}{dy} f(x, y)\right], [x, y]\right)$$

$$\#3: \quad [\,]$$

$$\#4: \quad \begin{bmatrix} -0.4213654636 & 0.1329768087 \\ -1.580824344 & 0.2926858331 \\ 0.2750570473 & 1.001318218 \\ 0.8584345393 & 0.02078520632 \\ 0.004921131562 & -0.9555060044 \end{bmatrix}$$

The last matrix (#4) contains the critical points (points in 2D). Ask your students to show that they are located at the intersection of the two curves in figure 3 ... or ask them to plot some level curves and these 5 points:

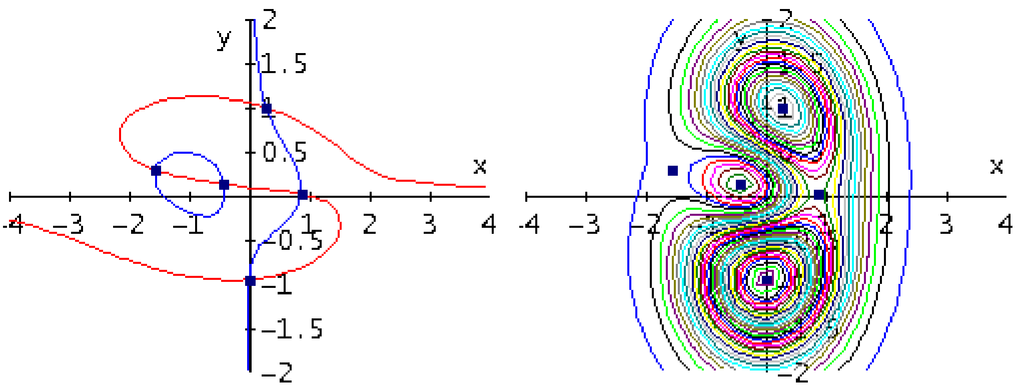


Figure 4

Now, what about an image, in 3D, of these points? This is a new feature of *Derive* 6, the ability of plotting points in space, with different size. And, using again the mouse, we can actually see that there are 2 saddle points, one global minimum, one local maximum and one global maximum. Of course, students will show this using some calculus techniques (Hessian matrix):

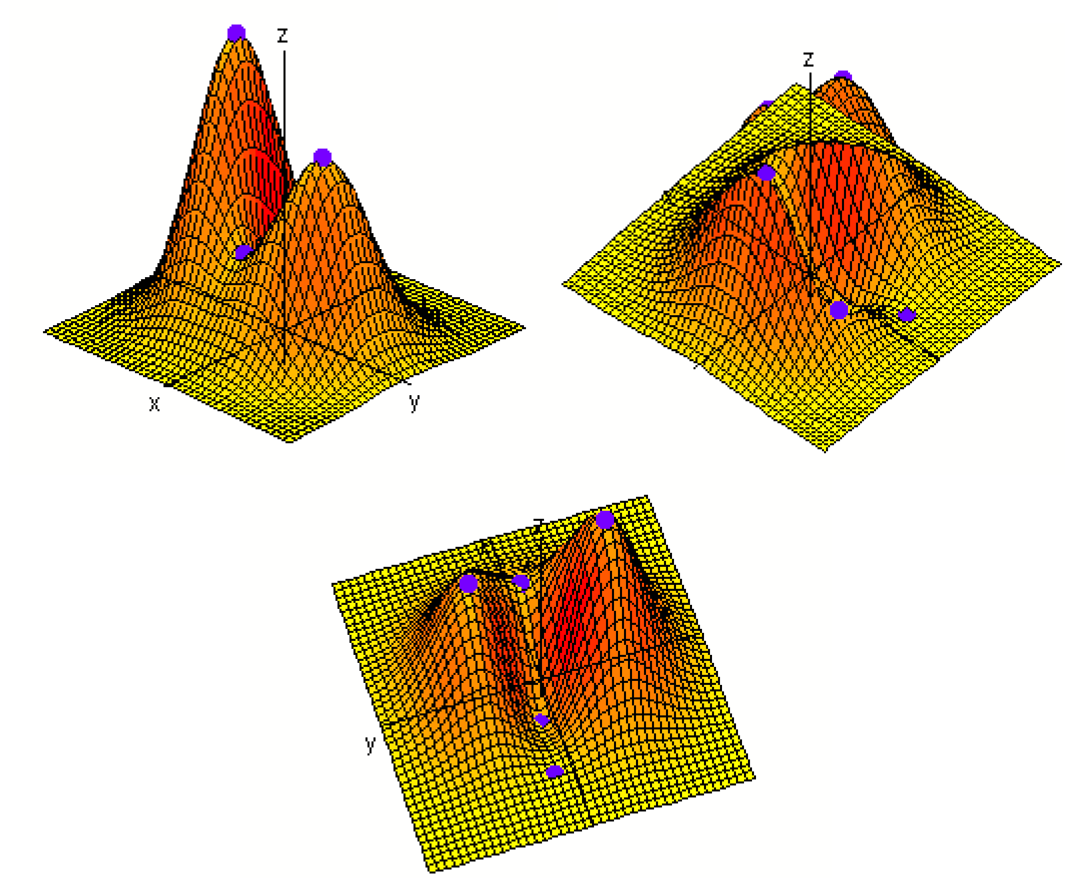


Figure 5

Finally, one can ask the following question: how can we be sure that there are only 5 critical points? Well, removing the exponential factors that arise from #4, we can see that the critical points are solutions of the following polynomial system: $f_1 = 0$ and $f_2 = 0$ where

#5: $f1 := 2 \cdot x^3 + 2 \cdot x^2 \cdot (y + 1) + 2 \cdot x \cdot (5 \cdot y^2 - y - 1) - y - 1$

#6: $f2 := 2 \cdot x^2 \cdot y + x \cdot (2 \cdot y^2 + 2 \cdot y - 1) + 10 \cdot y^3 - 2 \cdot y^2 - 10 \cdot y + 1$

Here, it is possible to solve $f_1 = 0$ for y and, then, substitute into f_2. Sometimes it is impossible to do so. How can we be sure that there are only 5 critical points? If we could reduce the system to a system of 2 equations but with one univariate equation, that will be possible. This is exactly what the (new) *groebner_basis* function does:

#7: GROEBNER_BASIS([f1, f2], [x, y])

#8: $\left[1292 \cdot y^5 - 636 \cdot y^4 - 1148 \cdot y^3 + 548 \cdot y^2 - 59 \cdot y + 1,\ 1855 \cdot x + 23902 \cdot y^4 - 27270 \cdot y^3 - 19743 \cdot y^2 + 24699 \cdot y - 2097\right]$

In #8, the first expression is a filth order polynomial and one can show that there are 5 real roots. Substituting in the next expression, we find, for each y, exactly one x because the second expression is linear in x!

3. The second example: show me the steps!

Let us take a look at some integrals and integration methods. We want to find $\int x(2x+3)^{10}\,dx$. This is a typical exercise we give our students, hoping they will think to make a change of variable ... and not to expand! Well, a change of variable should not be so good if, instead of the factor x, we would have a power of x. This is (probably) the reason why the symbolic TI expands (as well as Maple) :

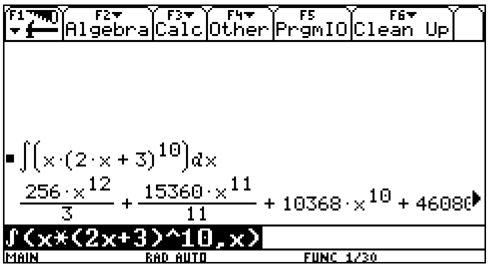


Figure 6

Derive 6 (in fact, *Derive for Dos* 3.05 or maybe earlier versions) uses a very nice reduction formula that works for powers of x times a power of a linear expression in x. This is confirmed by applying the (new) *Display Step* :

#9: $$\int x\cdot(2\cdot x+3)^{10}\,dx$$

#10: $$\frac{(2\cdot x+3)^{11}\cdot(22\cdot x-3)}{528}$$

$$\int x^{m}\cdot(a+b\cdot x)^{p}\,dx \Rightarrow \frac{x^{m}\cdot(a+b\cdot x)^{p+1}}{b\cdot(p+m+1)} - \frac{a\cdot m\cdot\int x^{m-1}\cdot(a+b\cdot x)^{p}\,dx}{b\cdot(p+m+1)}$$

#11: $$\frac{x\cdot(2\cdot x+3)^{11}}{24} - \frac{\int (2\cdot x+3)^{10}\,dx}{8}$$

By the way, this Display Step can be considered, at least for integrals, as an *Integral Tables* book. Some colleagues were not happy when CAS became available, saying that students won't need to learn integration techniques or to look up at a formula from a book. Again, we have to face the

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following question: what and how do we want our students to learn? Well, we won't answer this question here but, let us say that there is no reason at all, today, to study integration techniques without using a CAS. And, with *Derive* 6, you face many additional techniques of integration ... By the way, teachers should ask the students to *prove* the above formula used by *Derive* in order to perform the integral. One easy way is to differentiate both sides or to use undetermined coefficients. Of course, this formula can be found in some handbook of mathematical functions (see [3] for example).

Here is another example of how *Derive* works. Sometimes, one needs to do partial fractions expansion :

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#12:  ∫ x^2 / ((x + 4)·(x - 1)·(x - 6)) dx
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$$\#12: \quad \int \frac{x^2}{(x+4)\cdot(x-1)\cdot(x-6)}\,dx$$

$$\#13: \quad \frac{18\cdot LN(x-6)}{25} + \frac{8\cdot LN(x+4)}{25} - \frac{LN(x-1)}{25}$$

Distribute integral over terms of partial fraction expansion.

$$\#14: \quad \int \frac{18}{25\cdot(x-6)}\,dx + \int \frac{8}{25\cdot(x+4)}\,dx - \int \frac{1}{25\cdot(x-1)}\,dx$$

Here, let us be clear. Why not tell the students what the CAS did in order to find the coefficients of a partial fraction expansion? This a good opportunity of using limits. Let us recall that if $\dfrac{F(s)}{G(s)}$ is a proper rational function (deg F < deg G) with F and G having no common factors, then if a is a simple zero of G, the partial fraction expansion of $\dfrac{F(s)}{G(s)}$ contains a term of the form $\dfrac{A}{s-a}$ where $A = \lim\limits_{s\to a}\dfrac{(s-a)F(s)}{G(s)}$. And if a is a zero of order m of G, then partial fraction expansion of $\dfrac{F(s)}{G(s)}$ contains terms like $\dfrac{A_m}{(s-a)^m} + \dfrac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \dfrac{A_1}{s-a}$ where

$$A_m = \lim_{s\to a}\frac{(s-a)^m F(s)}{G(s)} \text{ and } A_k = \frac{1}{(m-k)!}\lim_{s\to a}\frac{d^{m-k}}{ds^{m-k}}\left(\frac{(s-a)^m F(s)}{G(s)}\right) \quad (k = 1, 2, ..., m-1)$$

And sometimes, partial fractions would be too long, so special formulas are used by *Derive* :

#15: $$\int \frac{x^2}{x^4 + 1}\, dx$$

#16: $$\frac{\sqrt{2}\cdot \mathrm{ATAN}(\sqrt{2}\cdot x - 1)}{4} + \frac{\sqrt{2}\cdot \mathrm{ATAN}(\sqrt{2}\cdot x + 1)}{4} + \frac{\sqrt{2}\cdot \mathrm{LN}\left(\frac{x^2 - \sqrt{2}\cdot x + 1}{x^2 + \sqrt{2}\cdot x + 1}\right)}{8}$$

If n>0 is even and 0<=m<n is an integer,

$$\int \frac{x^m}{a + b\cdot x^n}\, dx \Rightarrow \frac{1}{n\cdot (a^{1/n\ n - (m+1)})\cdot (b^{1/n\ m+1})}\cdot \sum_{k=1}^{n/2} \left(2\cdot \mathrm{ATAN}\left(\frac{\frac{x\cdot b^{1/n}}{a^{1/n}} - \mathrm{COS}\left(\frac{2\cdot k - 1}{n}\cdot \pi\right)}{\mathrm{SIN}\left(\frac{2\cdot k - 1}{n}\cdot \pi\right)}\right)\cdot \mathrm{SIN}\left(\frac{2\cdot k - 1}{n}\cdot (m + 1)\right)\right)$$

The last *Derive* show step was too long to be copied here. But, here again, this a good opportunity to the teacher to introduce complex numbers (some other systems use “RoofOf” for such integrals). What do all these examples have in common? Well, it gives the teacher a very good opportunity to talk about some specific integration techniques and to show other methods about which textbooks are not so enthusiastic...

And, last but not least, always recall simple methods; the good old change of variable used for some integrals, namely for these two:

$$\int \frac{x^3}{1+x^4}\,dx, \quad \int \frac{x}{1+x^4}\,dx.$$

With the *Display Step*, the user can see what he should have seen!

#17: $$\int \frac{x^3}{1 + x^4}\, dx$$

$$\int \frac{x^{n-1}}{a + b\cdot x^n}\, dx \Rightarrow \frac{\mathrm{LN}(a + b\cdot x^n)}{b\cdot n}$$

#18: $$\frac{\mathrm{LN}(x^4 + 1)}{4}$$

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$$\#19: \int \frac{x}{1 + x^4}\, dx$$

$$\int \frac{x^{n-1}}{a + b \cdot x^{2 \cdot n}}\, dx = \frac{\operatorname{ATAN}\left(\frac{\sqrt{b} \cdot x^n}{\sqrt{a}}\right)}{n \cdot \sqrt{a} \cdot \sqrt{b}}$$

$$\#20: \frac{\operatorname{ATAN}(x^2)}{2}$$

I remember, about 10 years ago, some teachers were using CAS ability to integrate *before* teaching integration methods to their students. Letting them discover some rules of integration. Now, with *Derive* 6, a teacher can introduce some techniques of integration by, first, taking a look at what the computer does and, second, writing formulas on the blackboard like before. The mix of "classical teaching" and use of technology has never been so possible.

4. The third example: working together (*Derive* and the symbolic TI)

A student is working on our example from section 2 and he (she) wants to continue on the symbolic TI. What can be done? Let's suppose that the *Derive* file contains the following information:

$$f(x, y) := e^{-x^2 - y^2} \cdot (x^2 + x \cdot (y + 1) + 5 \cdot y^2 - y)$$

$$\left[\frac{d}{dx} f(x, y), \frac{d}{dy} f(x, y)\right]$$

$$\operatorname{SOLUTIONS}\left(\left[\frac{d}{dx} f(x, y), \frac{d}{dy} f(x, y)\right], [x, y]\right)$$

$$\begin{bmatrix} -0.4213654636 & 0.1329768087 \\ -1.580824344 & 0.2926858331 \\ 0.2750570473 & 1.001318218 \\ 0.8584345393 & 0.02078520632 \\ 0.004921131562 & -0.9555060044 \end{bmatrix}$$

In *Derive* 6, we can export this to the symbolic TI (as a text file) and load it in the home screen:

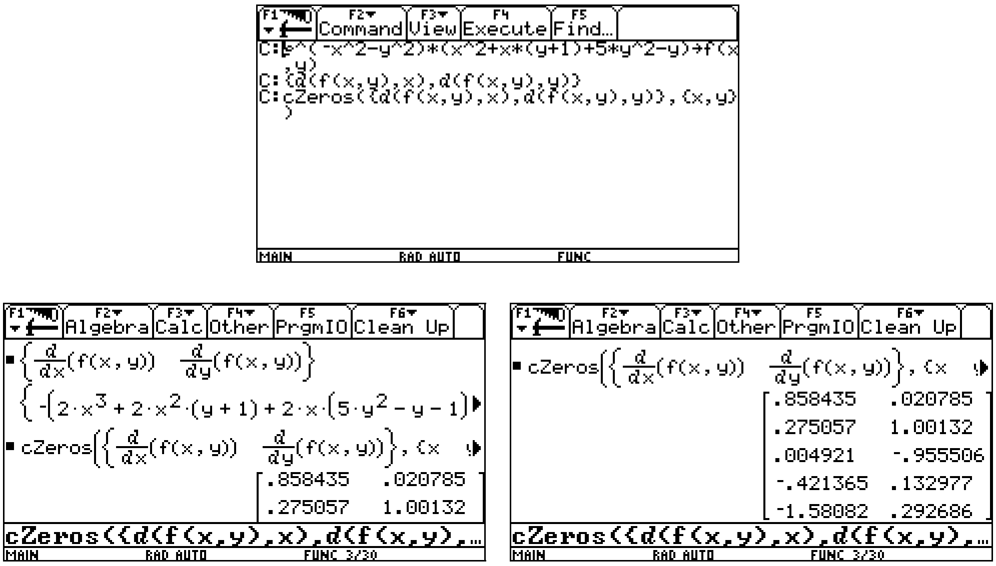

Figure 7

In this way, the symbolic TI allows the student to continue the mathematical exploration in the classroom, without having to buy a laptop (with *Derive* 6 installed on it). Of course, one needs to observe that the "solve" command of *Derive* is a "csolve" command for the TI, that lists for the TI are vectors for *Derive*, and so on. Well, this is the kind of translation we have to do often and it is quite remarquable that the symbolic TI and *Derive* 6 have now this in common.

A suggestion for Maple, Matlab and Mathematica developers: why don't you link your computer software to the symbolic TI?

5. Conclusion

Many improvements will have to be made for the connectivity between *Derive* 6 and the symbolic TI. The example we gave is a simple one, but shows that connectivity can be used by the student in order to work on the same problem, at different places, with similar but different tools. I want to thank Albert Rich and Theresa Shelby for taking some of my suggestions about *Derive* at the Visit-me conference. I also have to thank David Stoutemeyer, Gosia Brothers and Michelle Miller for accepting some of my suggestions for improving the symbolic TI calculator.

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[1] Adams, Robert, A. *Calculus of Several Variables*. Fourth Edition, Addison-Wesley, 2000.

[2] Kutzler, Bernhard & Kokol-Voljc, Vlasta. *Introduction to Derive* 6, 2003.

[3] Spiegel, Murray, R. *Formules et tables de mathématiques*. Série Schaum, McGraw-Hill, 1974.