VISUALISATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS AND SYSTEMS

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Abstract

We will first show how it is possible to obtain the curves of antiderivative functions as functions are given with their formulas with the software Cabri 2 Plus. We will show also how this file can be used by teachers or students like a program: it will be sufficent to change on the screen of Cabri the formula of the given function.
The principal part of this paper will use Euler's method; we will show how to generate the solution curves of equations like y' = f(x;y) wher F is a function whose formula can be edited in Cabri. We will see, as well, set of solutions, generation of solutions of different equations and sweeping of the plane to see better.
In this part the same technic will be use to visualize solutions of systems as x'= f(x;y)et y'= g(x;y).
We will explore classical equation and more original ones.
We will try to conclude on a new way to tackle the teaching of differential equations and also a more experimental way for teachers and students to tacle these problems.

1. CURVES OF ANTIDERIVATIVE FUNCTIONS WITH EULER'S METHOD

1.1. The basic construction leading to the first powerful macro

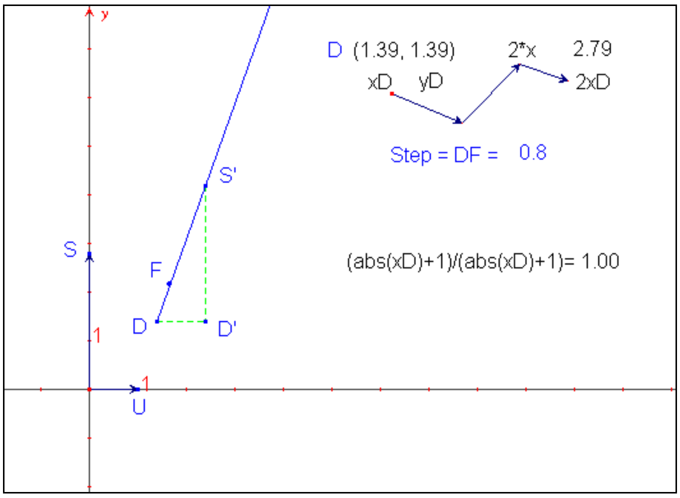
The aim of this Cabri construction is to get the point F from the point D, such as :

DF= d (here 0.8 that can be modified)

$\overrightarrow{DF}$ and $\overrightarrow{DS'}$ have the same direction.

Coordinates of vector $\overrightarrow{DS'}$ are 1 and 2xD

Anti1.fig ⟶



Remarks about the construction:

U has been built by the measurement transfer of number 1 calculated with the formula written on the screen with the tool "Calculate" of Cabri.

S has been built by the measurement transfer of number 2xD calculated by using the "Expression" 2*x edited on the screen with the tool "Apply an Expression" to the number xD which is the abscissa of point D.

D' is the translated point of D with the vector $\overrightarrow{OU}$.

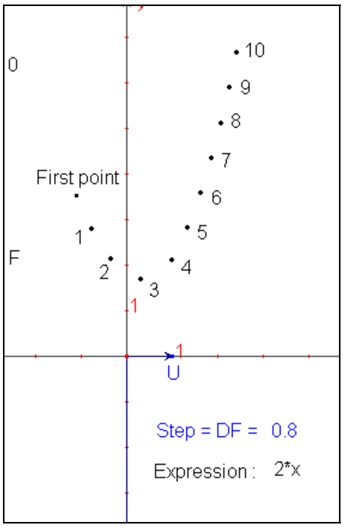
S' is the translated point of D' with the vector $\overrightarrow{OS}$

F has been built by the measurement transfer of number d (here 0,8 reprensenting the accuracy of the following construction)

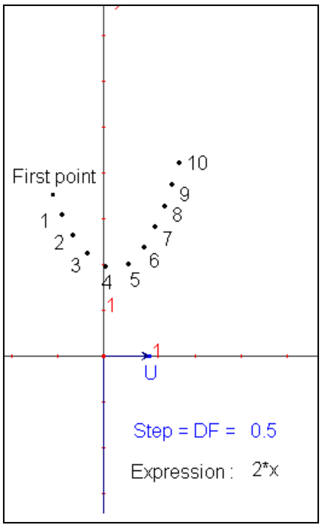
The first recorded macro "Following point 1.mac"
Initial objects : point D, a system of axis, an expression and a number d.
Final object: point F

1.2. Use of the first macro which leads to the second more powerful macro

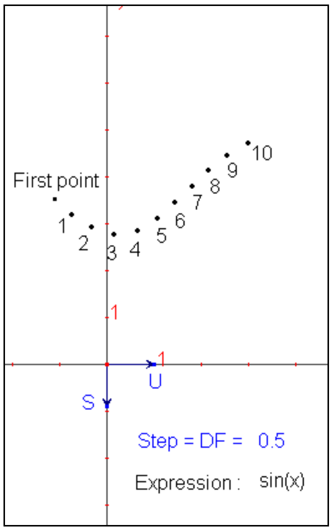
On the first screen below, we have applied this macro to a first point to get the point 1 and to the point 1 to get the point 2 and we have continued until the point 10.
So we get a chain of points approaching the curve of an antiderivative function of the y = 2x function (the accuracy of this construction is better and better when the step approaches 0)
After that, we have changed number d in the next screen below.
At last we have modified the expression into sin(x) in the final screen.



Anti2.fig



Anti3.fig

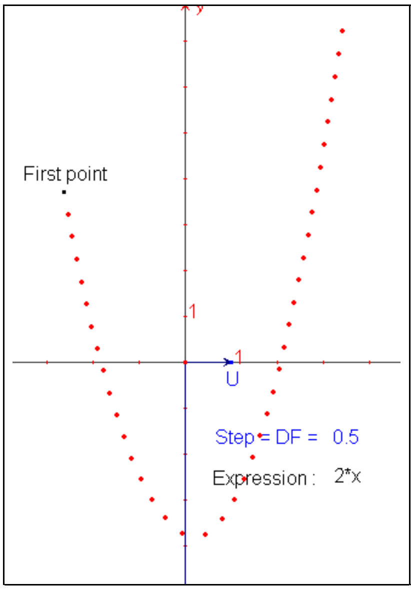


Anti4.fig

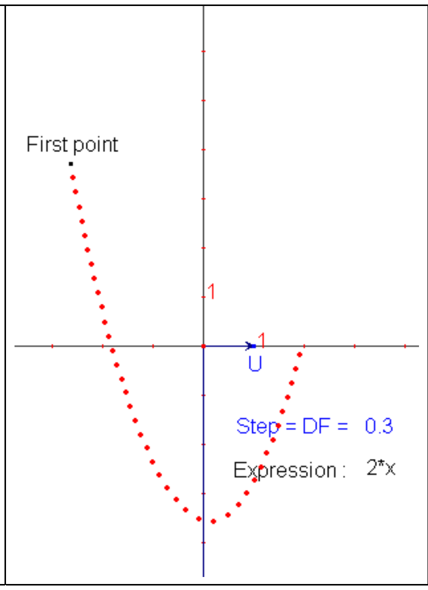
The second recorded macro "Following points 10.mac" (Euler's method)
Initial objects : point D, a system of axis, an expression and a number d.
Final objects : points 1,2, 3, 4, 5, 6, 7, 8, 9 and 10

1.3. How to draw a chain modelising the curve of the antiderivative function of any function

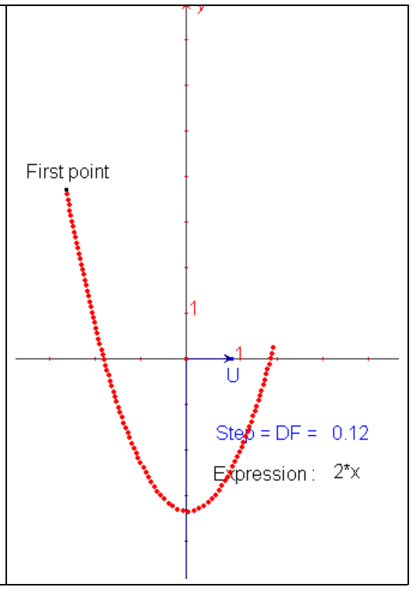
In the next screens we have applied directly the second macro to a first point to get instantaneously the 10 following points and after, we have applied again the same macro to the tenth point just constructed and so on. At last we have obtained a long chain that represents a approximation of the curve of an antiderivative function of the given expression. The accuracy of this drawing increases when number d decreases and approaches 0 : it is the Euler's method.



Anti5.fig

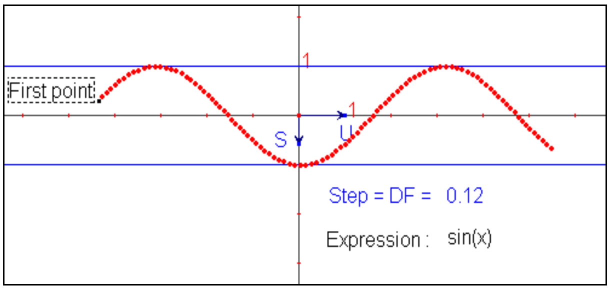


Anti6.fig

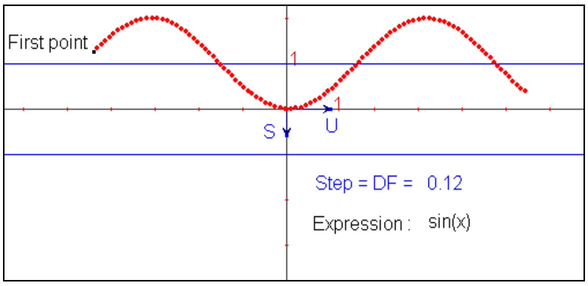


Anti7.fig

Remark : in the last screen we have applied the second macro several times more. Here are below several examples only got by changing the expression and possibly the position of the first point.



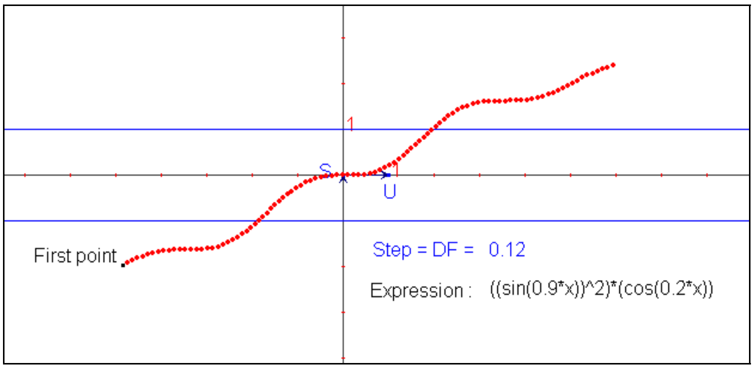
Anti8.fig



Anti8.fig

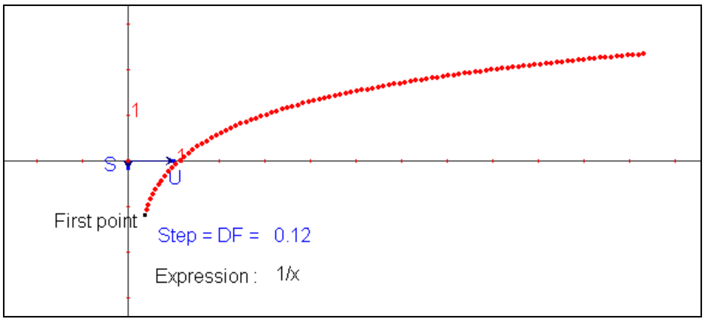
This example performed on paper would need a lot of calculation.

Anti9.fig ⟶

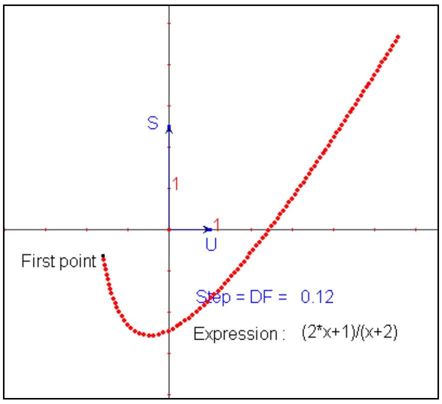


CONTS

Here is an example of an upper level:



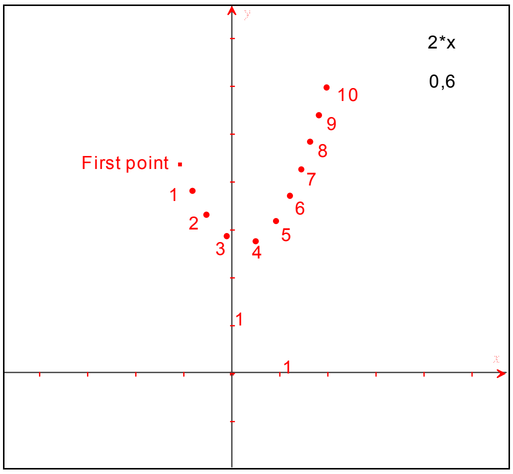
Anti10.fig



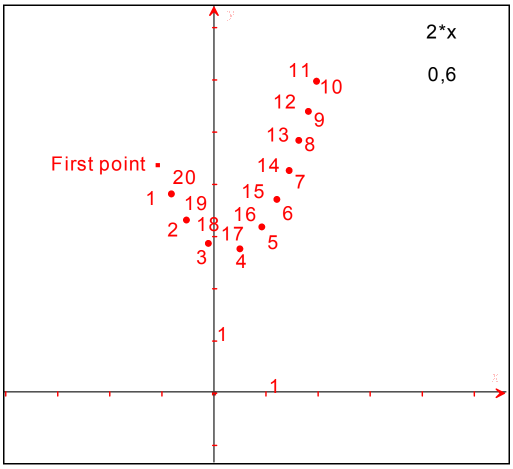
Anti11.fig

In dragging the first point we can retrieve the logarithm function and all the the antiderivative functions of 1/x

1.4. How to draw curves of antiderivative functions of any function



Anti12.fig

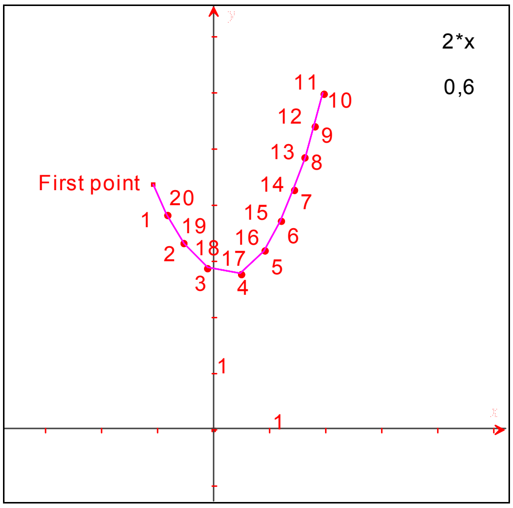


Anti13.fig

We use here the macro **"Following points 10.mac"** to get from the "First point" the chain of the ten points 1, 2, ………. 10.

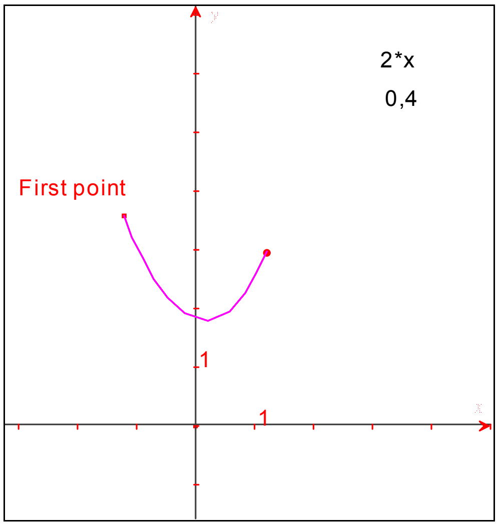
We use here again the same macro **"Following points 10.mac"** to get from the "First point" the chain of the ten points 20, 19, 18, ………11 that lie exactly on the previous points.

CONTS



Anti14.fig

We draw the polygon starting at the "First point" and linking, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20 and lastly the "First point"

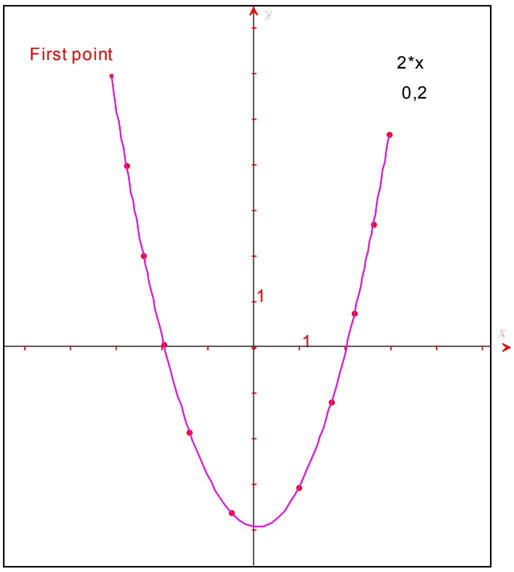


Anti15.fig

We hide all the points apart from the "First point" and point number 10 and we get the beginning of the curve of an antiderivative function of the y = 2*x function (really we would get a good approximation of this curve with a smaller value of the step)

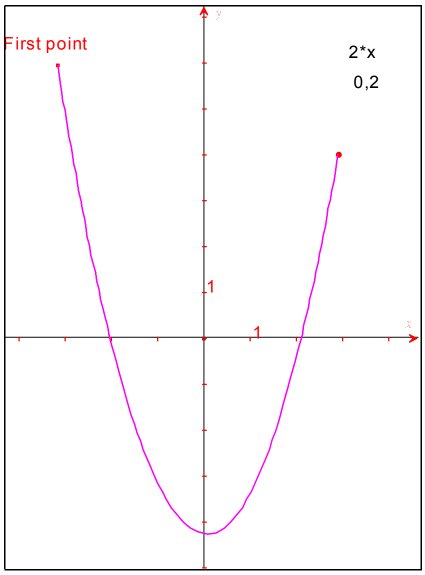
The third recorded macro "polygon10.mac" (Euler's method)

Initial objects : point D, a system of axis, an expression and a number d.
Final object: the polygon starting from "First point" and stopping at the point 10 including this last point



Anti16.fig

Here we have applied this macro ten times successively to get a longer part of the previous curve.

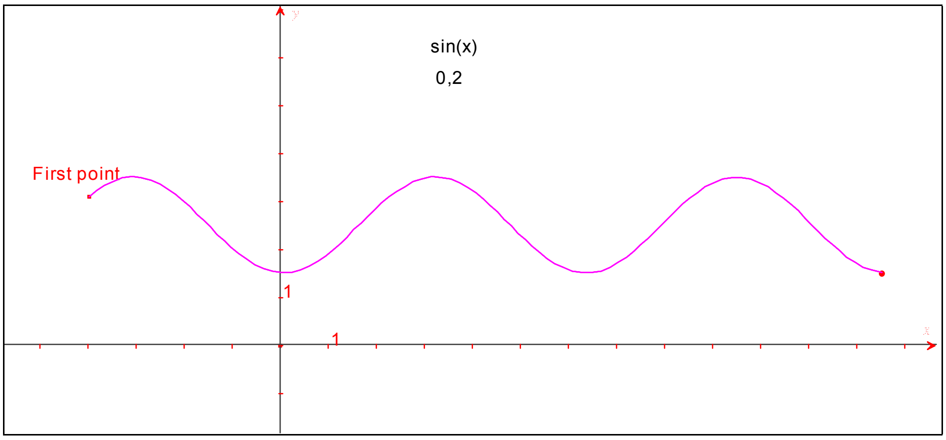


Anti17.fig

Here we have hidden all points apart from the first and the last.

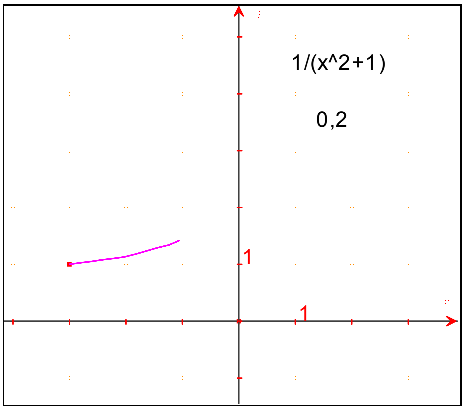
Here we change only the expression to get immediately a good approximation of the curve of the antiderivative function of the sine function.

Anti18.fig →

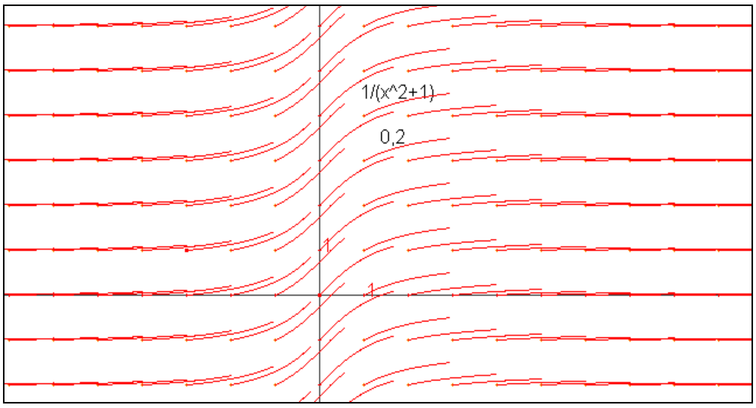


The fourth recorded macro “10polygons10.mac” (Euler’s method)
Initial objects : point D, a system of axis, an expression and a number d.
Final objects : the 10 polygons we got with “polygon10” include only two points (the first and the last points).

1.5. Field of solution curves



Anti19.fig



Anti20.fig

We apply first the macro “polygon10” to a point of the grid for the function written on the screen and we ask for the locus of this polygon when the first given point moves on the grid and we can have and idea of all the curves of the antiderivative function of the given function.

2. CURVES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS AND SYSTEMS

2.1. CURVES OF SOLUTIONS OF $\boxed{a(x).y'+b(x).y=c(x)}$

2.1.1. The 3 basic macro-constructions:

The technic we will use is the same we have already used: it will be Euler's method, using the formula ***y' = -(b.y+c)/a***. From a point of the plane we will draw an approximation of the solution passing through this point. We will create 3 macro constructions. The formulas of a, b and c are edited as "expressions". Their values are evaluated for the coordinates of point D. So vector $\overrightarrow{DS}$ is constructed such as its length is 0.25 (here, but can be modified) and its coordinates are 1 and y' calculated in applying the expression ***(c-by)/a***.

The equation we will try to solve is below

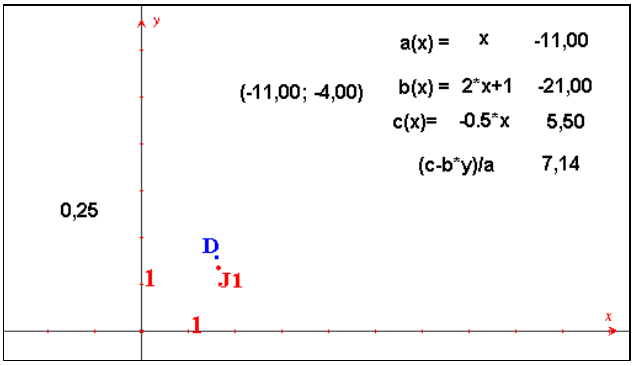
x.y'+(2x-1)y = -0.5.x

But the macro constructions we will create will be used with other coefficients.

Macro "ay'+by=c.mac"

When D is given this macro builds J1

Initial objects: expressions a, b, c and (c-b*y)/a, system of axis, accuracy (here 0.25) and staring point D
Final object: J1

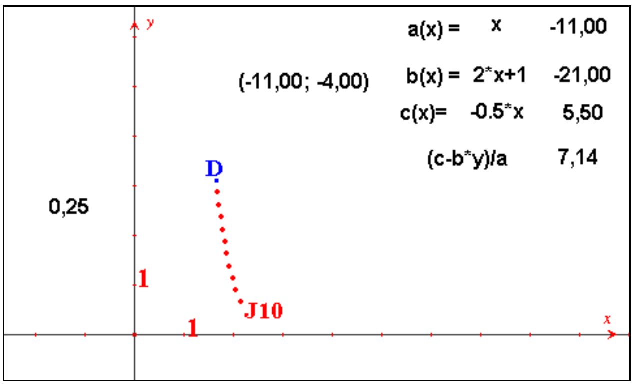


equa diff1.fig

Macro "ay'+by=c 10 points.mac"

When D is given this macro builds J1, J2, J3…,J10 (the red chain having 10 bits)

Initial objects: expressions a, b, c and (c-b*y)/a, system of axis, accuracy (here 0.25) and staring point D
Final object: J1, J2, J3…,J10



equa diff2.fig

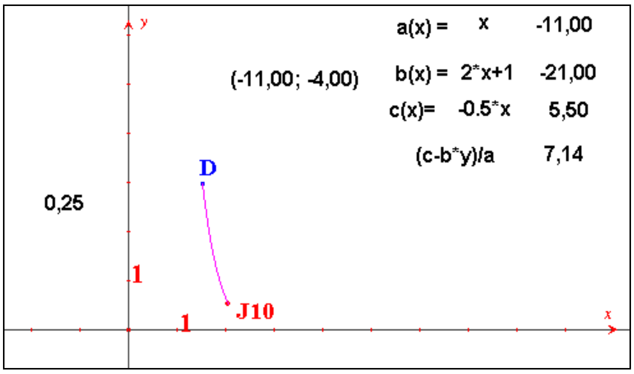
CONTS

Macro "ay'+by=c polygon 10 points.mac"

When D is given this macro builds a polygon joining the points of this chainstarting at D passing through J10 and coming back to D

Initial objects: expressions ***a, b, c*** and ***(c-b*y)/a***, system of axis, accuracy (here 0.25) and staring point D

Final object: a polygon modelising a part of the solution curve starting at D

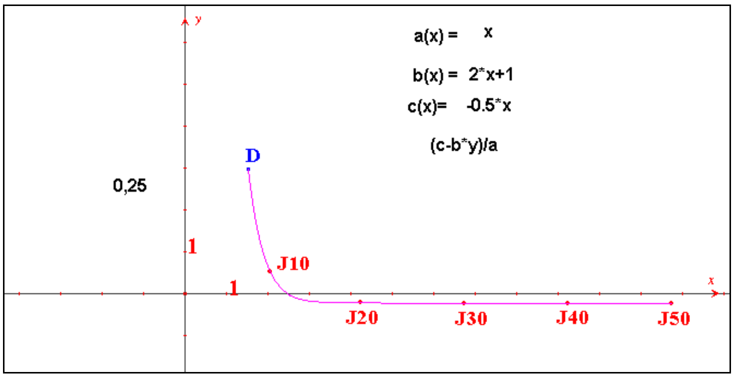


equa diff3.fig

2.1.2. Some examples of the use of the last macro

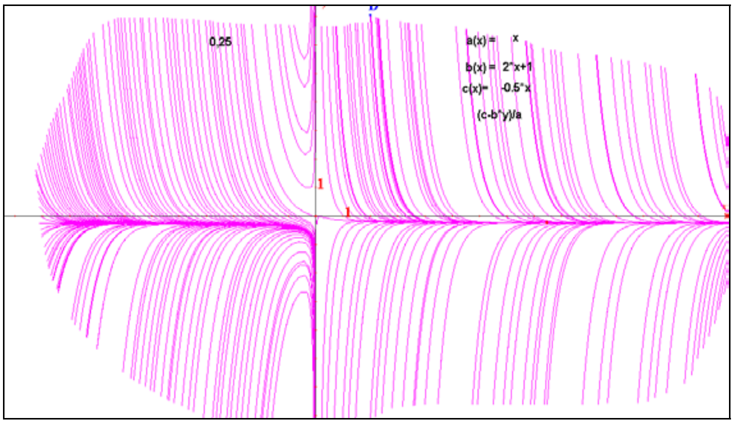
In the previous example, if we apply 4 times the last macro we get a longer curve modelising the solution passing through D.

equa diff4.fig ⟶



If we put the traces of the polygons DJ50 ON and if we drag point D everywhere in the plane, we get a visualisation of the set of solution of the given differential equation.

equa diff5.fig ⟶



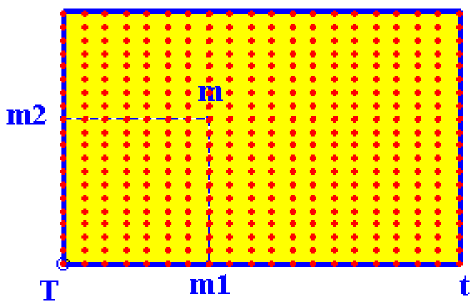
The rectangle of visualisation

We have created an expandable rectangle and inside a point m led by m1 on segment [Tt] and m2 on segment [Tt']. So we can create a grid with:

First the locus of m when m1 moves

Second, the locus of this locus when m2 moves

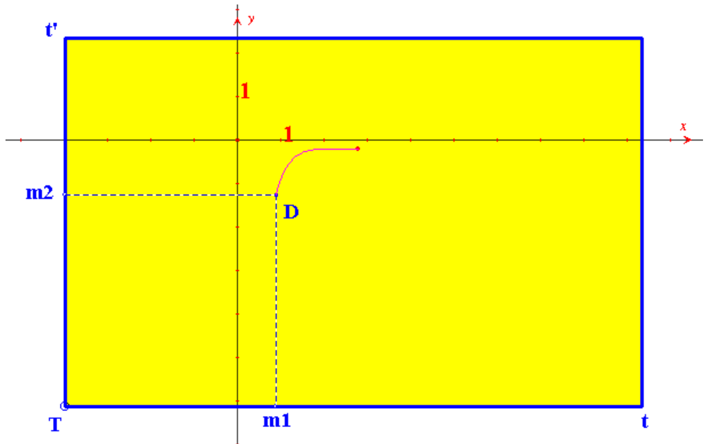
equa diff6.fig ⟶



CONTS

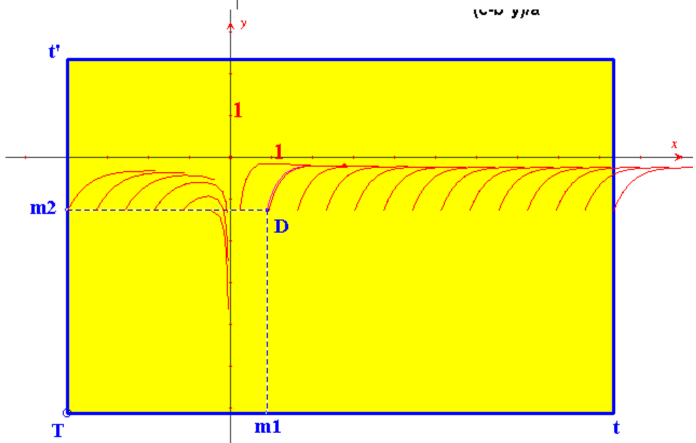
If we enlarge this rectangle, we put D on m and apply our second macro to D we get one polygon modelising a part of one solution

equa diff7.fig ⟶



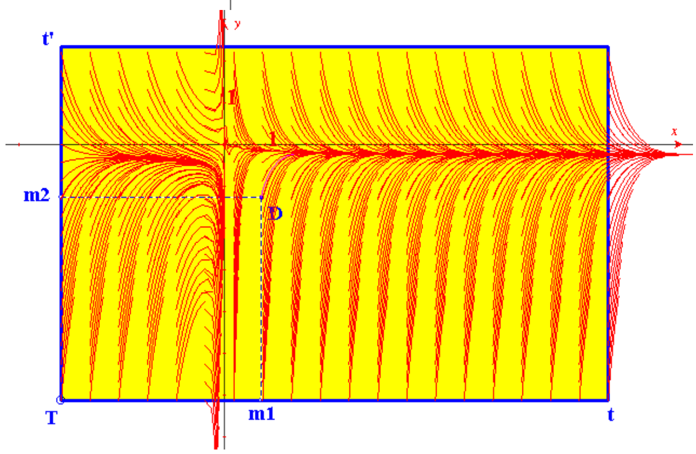
Here we have asked for the locus of the polygon when m1 moves.

equa diff8.fig ⟶



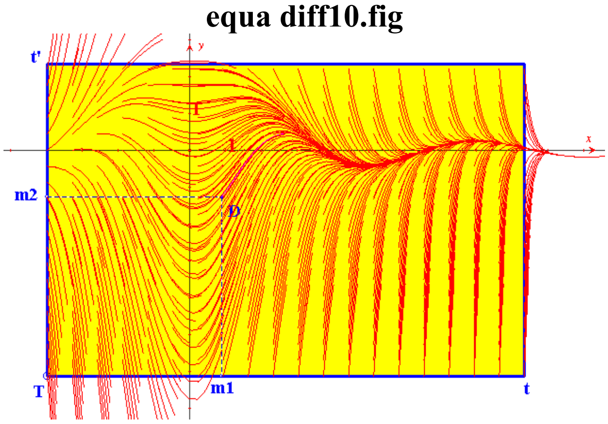
Here we have asked for the locus of the previous locus when m2 moves, to get a model of the set of solutions.

equa diff9.fig ⟶



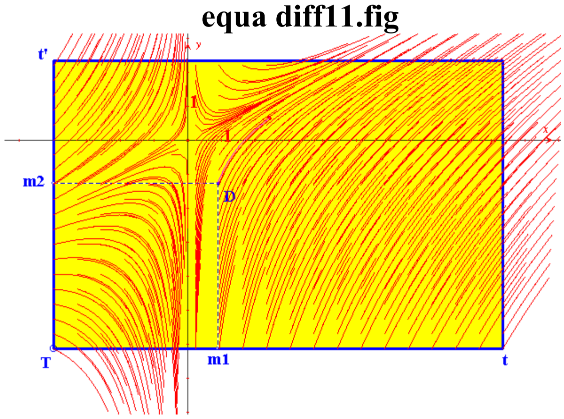
We can now change only the coefficient to get instantaneously the new set of solution (really a model)

equa diff10.fig



y'+0.5y=sin(x)

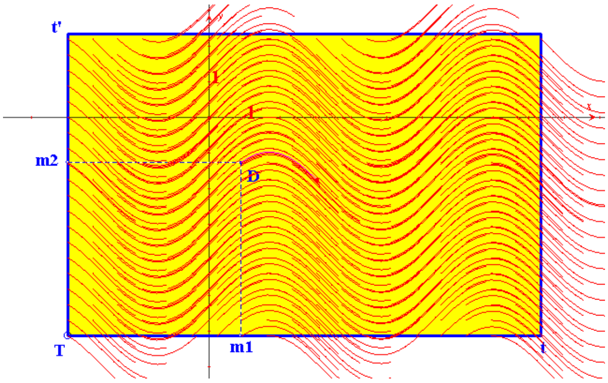
equa diff11.fig



xy'+y=x

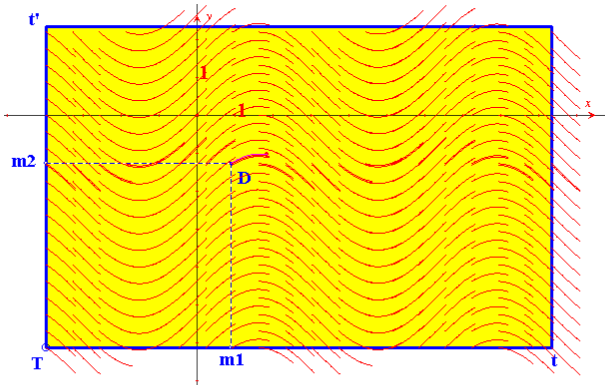
We can try to validate our model in testing it for determination of the antiderivative functions of the cosine function:

y' = cos(x) (accuracy: **0.25**)



equa diff12.fig

y' = cos(x) (accuracy: **0.10**)



equa diff13.fig

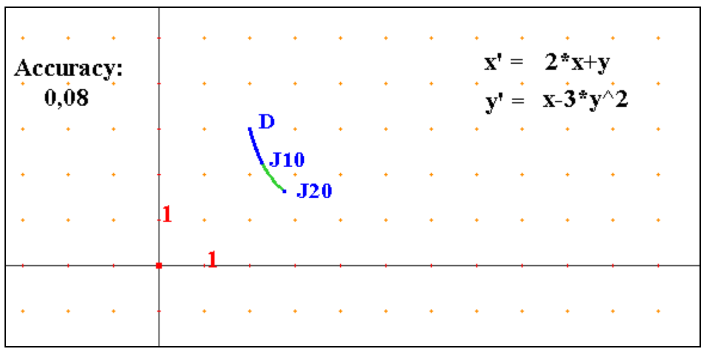
2.2. CURVES OF SOLUTIONS OF $\boxed{x' = f(x\,;y) \text{ and } y' = g(x\,;y)}$

To solve such a system the technic will still be the same with only one difference: the coordinates of vector $\overline{DS}$ will be now x' and y' evaluated with the expressions ***f(x;y)*** and g(x;y). We will also use 3 macro constructions to use them in the same way.
« **x'=f(x,y) y'=g(x,y).mac** », « **x'=f(x,y) y'=g(x,y) 10points.mac** »,
« **x'=f(x,y) y'=g(x,y) ligne10 points.mac** » and another one to build a longer solution curve:
« **x'=f(x,y) y'=g(x,y) ligne10 fois10 points.mac** »

2.2.1. A set of solution with the grid of the system of axis

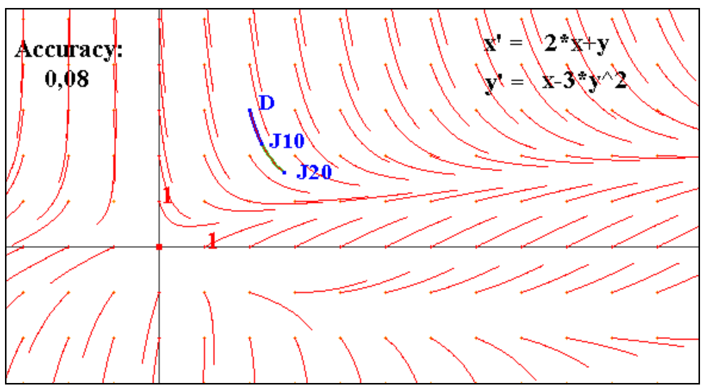
The system is edited on the Cabri page with the two expressions: ***2*x+y*** and ***x-3*y^2***.
Starting from D (chosen on the grid) we have used the third macro and from J10 we have use it a second time to get a model of a solution curve obtained with two polygons

systdiff1.fig ⟶



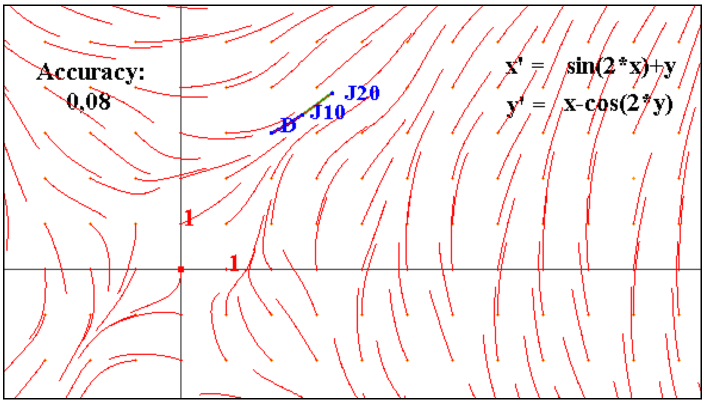
This set of solution is got by asking the loci of the two polygons when D moves on the grid.

systdiff2.fig ⟶



Here is another example where we have only changed the expressions defining the system

systdiff3.fig ⟶



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T3 congress New Orleans 2004 "How to construct curves of functions of one variable, of parametric, polar and random functions. Surfaces in several perspectives." **(CDROM)**

ATCM 2002 Melacca "How to teach Mathematics in showing all the hidden stages of a true research. Examples with Cabri."

ATCM 2003 Taïwan "Visualising functions of two variables with Cabri II Plus"

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3.4. Websites

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