Exploring Alternate Addition and Multiplication with the TI-89/92

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In our talk we will use a TI-89/92 to explore the algebraic and analytic properties of a pair of operations, an alternate addition we call circle plus ($\oplus$) and an alternate multiplication we call circle times ($\odot$). By removing the students from the environment of the addition and multiplication they are familiar with, these operations force students to pay attention to the algebraic and analytic definitions and structures since they cannot rely on their intuition about what is familiar.

The symbolic capabilities of the TI-89/92 allow students to explore these operations without getting bogged down in the algebraic manipulations. In addition, the scripts we use may provide insights into different ways the calculators can be used.

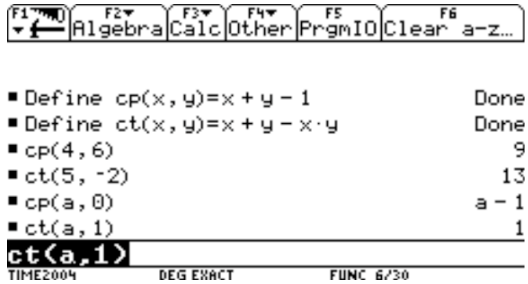
1 Defining the Operations

The circle plus and circle times operations are defined on the real numbers, although the definitions will work for any ring with unit.

$$x \oplus y = x + y - 1$$

$$x \odot y = x + y - xy.$$

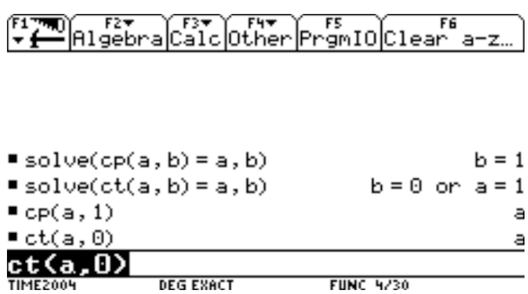
They can be defined on the TI-89/92 as a functions with two variables:



By defining the functions in this way, it is clear that they are binary operations. This can lead to a nice exploration of commutativity, associativity and the distributive properties. The last two require students to think carefully about how the order of operations work. We did not feel, however, that there was enough time in our talk to cover these ideas. For now, we will just assert without proof that the both operations are commutative and associative, and that circle times distributes over circle plus.

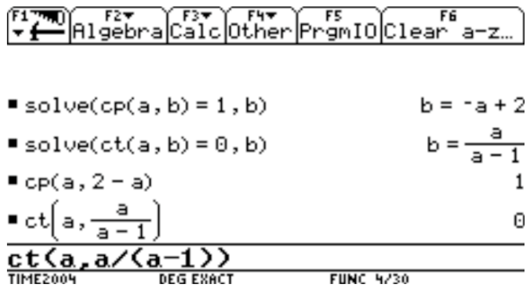
2 Identities, Inverses, and Inverse Operations

We are interested in considering the identities for these two operations, as well as whether or not elements have inverses with respect to the two operations. The last two lines in the figure above indicate that we cannot just use the identities from the more familiar operations, since $a \oplus 0 \neq a$ and $a \odot 1 \neq a$. Instead, we will rely on the calculator's solve command to find the inverses.

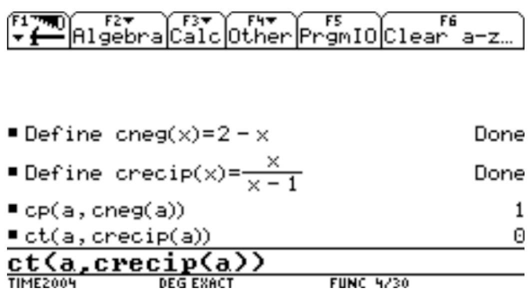


Thus we are in the odd-seeming situation where 1 is the additive identity, while 0 is the multiplicative identity. This also explains why, in the first figure, we find that $a \odot 1 = 1$. This is a variation of what we know as the zero-property of multiplication: when you multiply by the additive identity, the product is always the additive identity. Only, in this case, the additive identity is 1.

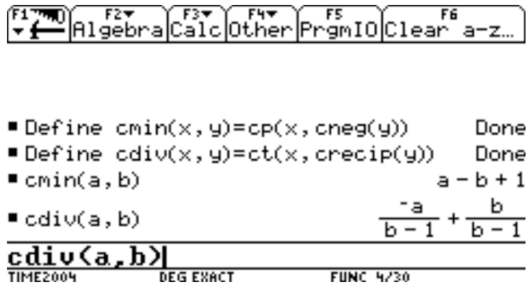
What about inverses? We can also use the calculator's solve command to determine what the additive and multiplicative inverses of numbers are, with the answers given in terms of our regular operations. We see that the additive inverse of a is $2 - a$, which is defined for all numbers. The multiplicative inverse of a is $\frac{a}{a-1}$, which is defined for all numbers except 1. This makes sense, as 1 is our additive identity.



From this, we can define two unary functions, the circle negative and the circle reciprocal.



Once we have negatives and reciprocals, we can also define the operations of subtraction and division, or rather circle subtraction ($\ominus$) and circle division ($\oslash$). We will define a circle minus b as a circle plus the circle negative of b and a circle divided by b to be a circle times the circle reciprocal of b.



Thus we could also define these operations directly from our standard operations by

$$a \ominus b = a - b + 1$$

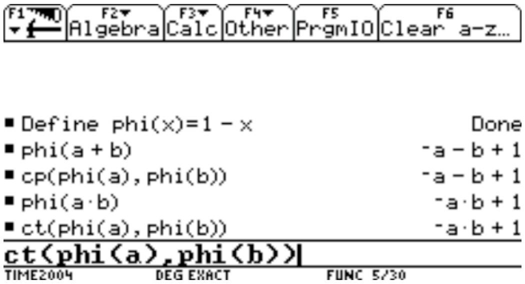
and

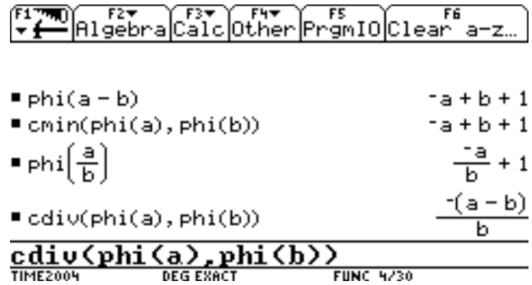
$$a \oslash b = \frac{b-a}{b-1}.$$

3 Isomorphism

It turns out that under these new operations of circle plus and circle times, the real numbers form a field. Likewise, so do the rational numbers and the integers. In fact, as rings they are isomorphic to the rings under the standard operations. The isomorphism is given by the map ϕ from $\langle \mathbb{R}, +, \times \rangle$ to $\langle \mathbb{R}, \oplus, \odot \rangle$ defined by $\phi(x) = 1 - x$.

The symbolic abilities of the TI-89/92 provide a nice way to check that this function really does preserve the respective addition and multiplication operations. We can even extend it to look at subtraction and division. Since these operations are defined in terms of addition and multiplication, it makes sense that they are also respected by the isomorphism.



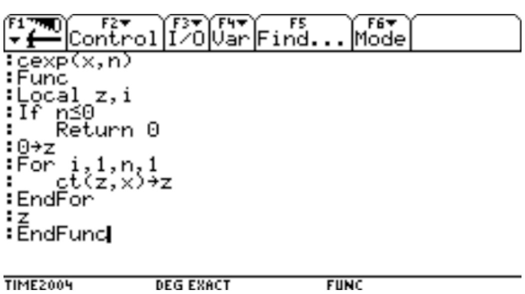


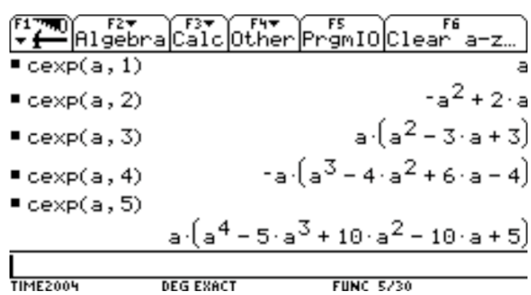
4 Exponents and Polynomials

To discuss polynomials, we must first discuss exponents and for this we need to look at what we can call circle exponentiation.

In our familiar operations, x^2 denotes $x \cdot x$. For our circle operations, x^2 would therefore mean $x \odot x = 2x - x^2$. Using the same notation for both expressions could cause some difficulty, so we will define circle exponentiation ($x^{\circ n}$), at least for whole number exponents. A recursive definition works well here: First, $x^{\circ 0} = 0$ since 0 is the identity for circle times. Then for $n \geq 0$ we define $x^{\circ n+1} = x^{\circ n} \odot x$.

We can write a function for the TI-89/92 that performs this exponentiation. The function is really a binary operation that takes the base and the exponent as its arguments.





In the second figure above, we see the first several powers of a under the circle exponent. It is possible to see a pattern that is closely related to Pascal's Triangle. While this is an interesting idea to explore, our goal is to explore the derivatives of these powers.

5 Difference Quotients and Derivatives

There is a formal defintion of the derivative as a map of the polynomial ring into itself. If $f(X) = \sum_{i=0}^{n} a_i X^i$ with $a_i \in A$ where A is a commutative ring, then the **derivative** is defined as

$$Df(X) = f'(X) = \sum_{k=1}^{n} k a_k X^{k-1}.$$

In this case, if $f(x) = x^{\circ 2}$ then $f'(x) = \circ 2x^{\circ 1}$ where $\circ 2x^{\circ 1} = x \oplus x$. While this notation is somewhat arkward, we need to clearly indicate that the derivative is not $2x$ under normal multiplication or $2 \odot x$. In fact, $\circ 2x^{\circ 1}$ denotes 2 copies of x being added under the addition of the ring. Following the definition of the derivative, for the monomials, we have

$f(x)$	$f'(x)$
$x^{\circ 2}$	$\circ 2x^{\circ 1}$
$x^{\circ 3}$	$\circ 3x^{\circ 2}$
$x^{\circ 4}$	$\circ 4x^{\circ 3}$
$x^{\circ 5}$	$\circ 5x^{\circ 4}$

Now let's consider the derivative from an analysis approach. In calculus we define the derivative as a limit, when the limit exists. The question is how to apply

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

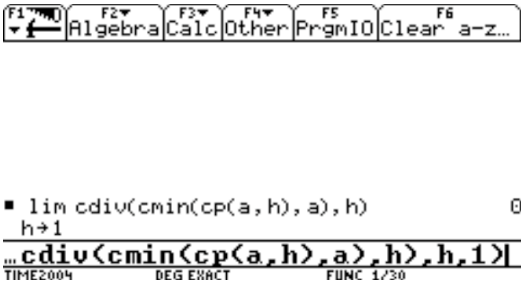
in our setting using $\oplus$ and $\odot$. At first glance, we might apply the following definition

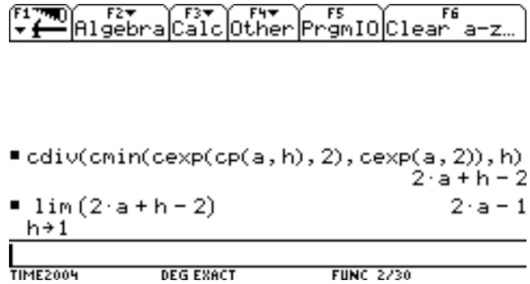
$$f'(x) = \lim_{h \to 0} (f(x \oplus h) \ominus f(x)) \oslash h.$$

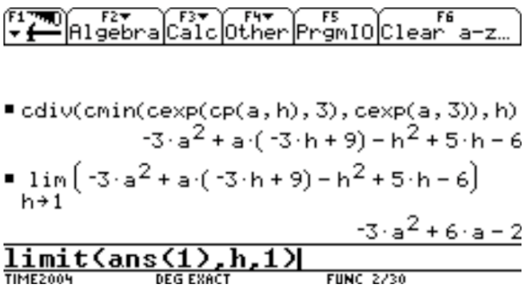
But recall that 0 is not the additive identity. Therefore, we should consider

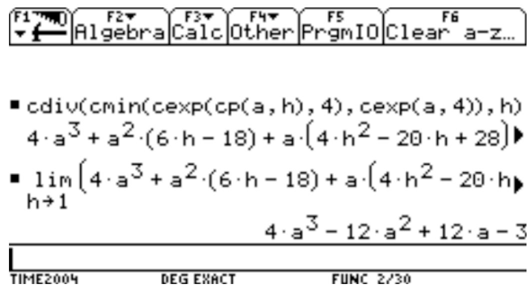
$$f'(x) = \lim_{h \to 1} (f(x \oplus h) \ominus f(x)) \oslash h.$$

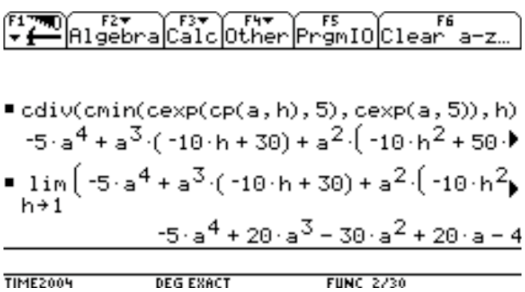
Using the TI-89/92 we can explore our conjecture.

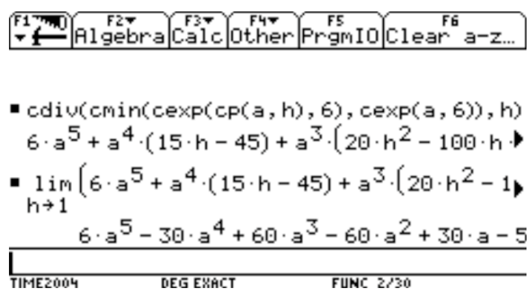












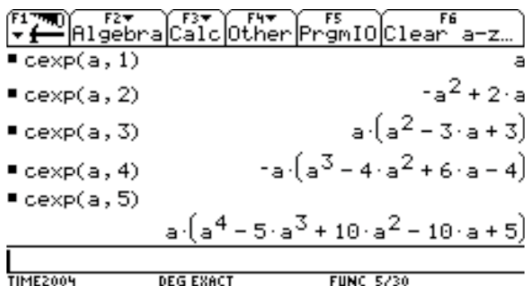
Based on the results from the TI-89/92 we have

$f(a)$	$f'(a)$
$a^{\circ 2}$	$2a-1$
$a^{\circ 3}$	$-3a^2+6a-2$
$a^{\circ 4}$	$4a^3-12a^2+12a-3$
$a^{\circ 5}$	$-5a^4+20a^3-30a^2+20a-4$

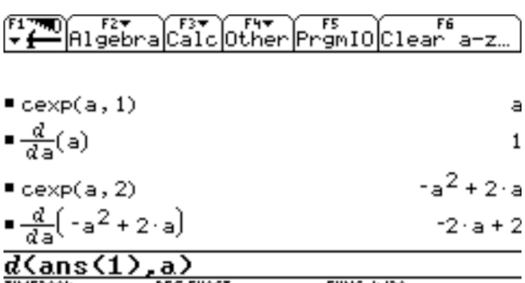
Notice that $f'(a)$ is given in its form under normal addition and multiplication. These results match our formal derivative as, by the definition of $a^{\circ n}$,

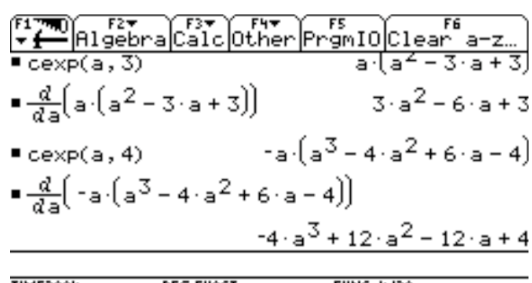
$$\begin{aligned}
\circ 2x^{\circ 1} &= 2a-1 \\
\circ 3x^{\circ 2} &= -3a^2+6a-2 \\
\circ 4x^{\circ 3} &= 4a^3-12a^2+12a-3 \\
\circ 5x^{\circ 4} &= -5a^4+20a^3-30a^2+20a-4
\end{aligned}$$

We can compare these results with the results of using the normal derivative definition under normal addition and multiplication. From before, the powers of a under $\oplus$ and $\odot$ are



Differentiating each of these powers of a we get





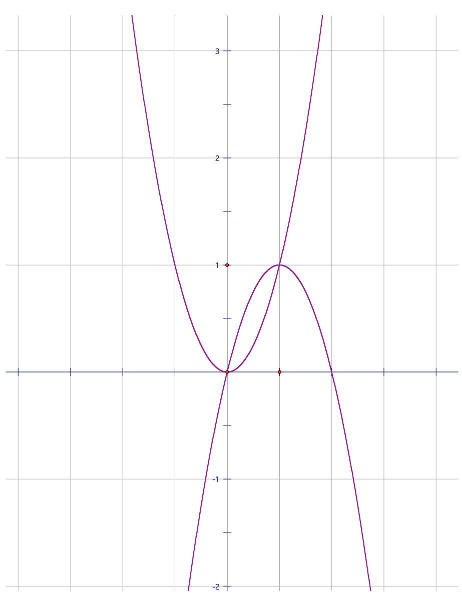
We can compare the derivatives in $\langle \mathbb{R}, +, \times \rangle$ verses the derivatives in $\langle \mathbb{R}, \oplus, \odot \rangle$

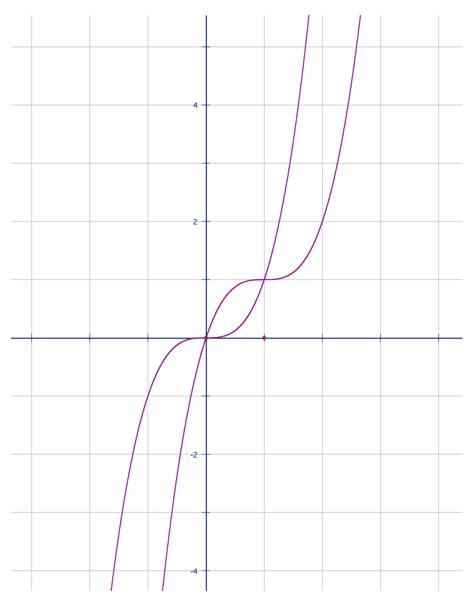
$\langle \mathbb{R}, +, \times \rangle$	$\langle \mathbb{R}, \oplus, \odot \rangle$
$-2a + 2$	$2a - 1$
$3a^2 - 6a + 3$	$-3a^2 + 6a - 2$
$-4a^3 + 12a^2 - 12a + 4$	$4a^3 - 12a^2 + 12a - 3$
$5a^4 - 20a^3 + 30a^2 - 20a + 5$	$-5a^4 + 20a^3 - 30a^2 + 20a - 4$

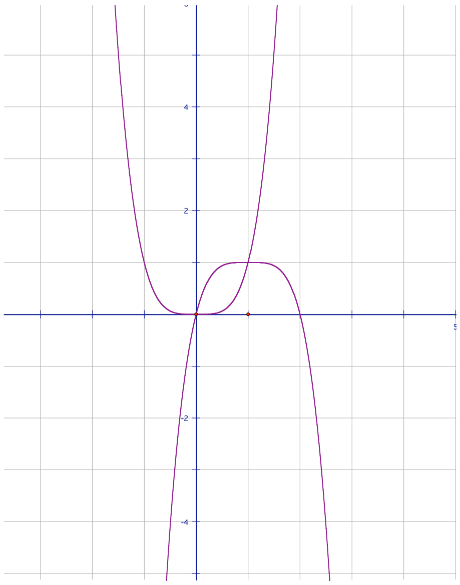
We see that the derivatives are equal under the isomorphism $\phi(x) = 1 - x$.

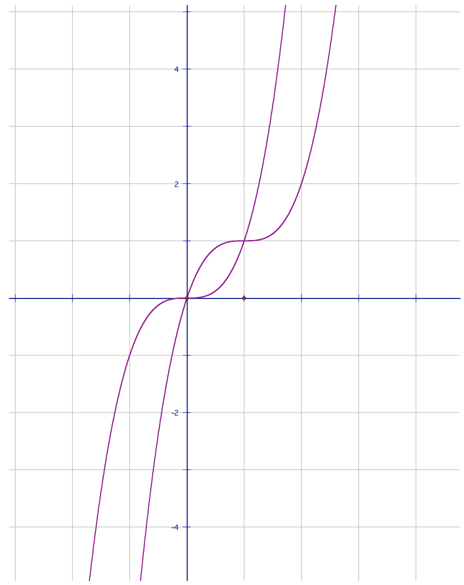
6 Transforming the Plane

A natural question to ask is whether there is any geometric meaning for the derivative in $\langle \mathbb{R}, \oplus, \odot \rangle$. In fact, we can consider the transformation of the plane given by sending (a, b) to $(\phi(a), \phi(b))$. Noting that $\phi(\frac{1}{2}) = \frac{1}{2}$, we see that $(\frac{1}{2}, \frac{1}{2})$ is a fixed point of the transformation. The transformation produces the following graphs for $x^2, x^3, x^4, x^3 - x$, along with $x^{\circ 2}$, $x^{\circ 3}$, $x^{\circ 4}$ and $x^{\circ 3} \ominus x$.









Notice that the transformation is actually a rotation of 180° about $(\frac{1}{2}, \frac{1}{2})$ and it can be seen that the slope is preserved under the transformation. By this, we mean that, for example, the slope of the graph of $y = x^2$ at a point (a, a^2) is the same as the slope the graph of $y = x^{\circ 2}$ at the point $(\phi(a), \phi(a)^{\circ 2})$. We have not yet explored the question of whether or not this preservation of slopes holds in general.

7 Conclusion

In Mathematical terms, the ideas brought forth by $\oplus$ and $\odot$ are not terribly deep. After all, they just provide an alternate set of ring operations that is isomorphic to the standard operations. Further, since the isomorphism ϕ is also an isometry, it isn't surprising that the resulting derivatives are closely aligned with the regular derivatives.

However, from a pedagogical standpoint, we have found these explorations very interesting. They allow a situation where students must rely on definitions rather than intuition, and so they provide a way for students to explore their mathematical understanding.