

Weighted Mean Approximation for Integration

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Abstract

The fundamental theorem of Calculus states that the definite integral over an interval is the difference of the anti-derivative at the two endpoints of the integration interval. However, we need to know the anti-derivative, but in too many interesting cases (for example in computing the length of curves) the anti-derivative does not exist, or is difficult to find.

In this paper we present a didactical sequence, which introduces a geometric approach for integration using the notion of Weighted Mean Approximation for computing the definite integral of any function (this approximation is exact for polynomials up to the third degree). This WMA depends only on three values of the integrand (no anti-derivative is needed). Refining the approximation opens the road to Simpson's rule for computing numerically definite integrals.

Introduction

The motivation for the didactic sequence presented in this paper originated from comments of curious high-school students while learning to compute definite integrals: "It is interesting that the definite integral for a given function depends only on the values of the primitive function at the endpoints of the integration interval; the area under the graph is not affected by the behavior of the primitive function and its derivative inside the interval of integration." This intimate connection between the definite integral and the given function is expressed in the fundamental theorem of Calculus: For $F'(x) = f(x)$:

$$\int_a^b f(x) = F(b) - F(a).$$

The following is a typical problem given to students: Find the area between the graph of the function $f(x) = 0.5x^3 - 2x^2 + 4$ and the x – axis, above it (see Figure 1). We demonstrate another "moment of curiosity" in the solution and exploration of this problem using *Derive*:

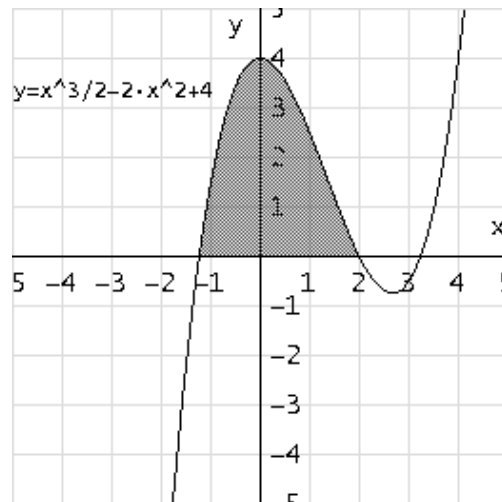


Figure 1.

#1: $f(x) := 0.5 \cdot x^3 - 2 \cdot x^2 + 4$

#2: $\text{SOLVE}(0.5 \cdot x^3 - 2 \cdot x^2 + 4 = 0, x)$

#3: $x = 1 - \sqrt{5} \vee x = \sqrt{5} + 1 \vee x = 2$

#4: $\int_{1 - \sqrt{5}}^2 (0.5 \cdot x^3 - 2 \cdot x^2 + 4) dx = \frac{5 \cdot \sqrt{5}}{3} + \frac{13}{3}$

#5: $\text{APPROX}\left(\int_{1 - \sqrt{5}}^2 (0.5 \cdot x^3 - 2 \cdot x^2 + 4) dx = \frac{5 \cdot \sqrt{5}}{3} + \frac{13}{3}\right) = \text{false}$

The students wanted to find a decimal answer for the integral, by approximating both sides. "Why is *Derive's* reaction to the Approx command -*false* !?", they wondered.

Let us approximate the left side of #4 first and then the right side:

#6: $8.060113295 = \frac{5 \cdot \sqrt{5}}{3} + \frac{13}{3}$

#7: $8.060113295 = 8.060113295$

If so, why "false"?

The conflict is resolved when we change the number of digits:

#8: NotationDigits := 12

$$\#9: \quad \text{APPROX} \left(\int_{1-\sqrt{5}}^2 (0.5 \cdot x^3 - 2 \cdot x^2 + 4) \, dx \right) = 8.06011329556$$

$$\#10: \quad \text{APPROX} \left(\frac{5 \cdot \sqrt{5}}{3} + \frac{13}{3} \right) = 8.06011329583$$

We realize that the software implements different algorithms for approximating integrals and for approximating real numbers. It is worthwhile to learn more about the algorithm for approximating integrals. Aiming to guide students in building the knowledge, and having CAS in our disposal, we designed a didactic sequence, which presents a geometrical approach for integration.

Geometric integration

Task I: The area under a linear function

(a) Given a linear function $f(x)$, show that the area under the function in the interval $[a, b]$ is the product of the length of the interval and the weighted mean of the values of the function at the two endpoints (weight = 1) and the value of the function at the midpoint (weight = 2).

(Hint: You may want to compute the area under the linear function - the trapezoid - as the sum of two trapezoids.)

(b) Now we modify the function to be a piece-wise linear function (see the following figure):

$$g(x) := \text{IF}(x \leq (a+b)/2, \text{Lin1}(x), \text{Lin2}(x))$$

Find the area under the modified function. Does the change of the function affect the solution?

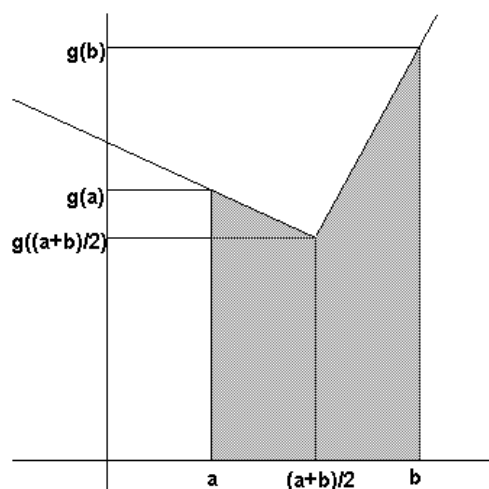


Figure 2.

Answer:

We sum the area of the two trapezoids and get:

$$\left(\frac{a+b}{2} - a\right) \cdot \frac{g(a) + g\left(\frac{a+b}{2}\right)}{2} + \left(b - \frac{a+b}{2}\right) \cdot \frac{g\left(\frac{a+b}{2}\right) + g(b)}{2} =$$

$$\frac{(b-a) \cdot \left(2 \cdot g\left(\frac{a+b}{2}\right) + g(a) + g(b)\right)}{4}$$

We see again that the area under $g(x)$ is the product of the length of the integration interval and the weighted mean of the values of the function at its endpoints and its midpoint, where the weight of the value at the midpoint is 2.

Task II: The area under a quadratic function

Given a quadratic function $qua(x)$ (see figure 3). Could you represent the area under the function as a weighted mean of the values at the endpoints and the midpoint?

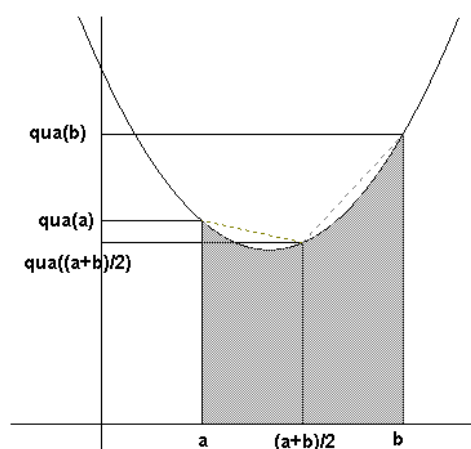


Figure 3.

Answer:

It is easy to see that the 1:2:1 weighted mean will not work here – we need to increase the weight of $qua((a+b)/2)$ if we want to have a 1:t:1 weighted mean. Students are encouraged to explore the situation using geometry or algebraic techniques. They can guess and check, or use their mathematical assistant in the following way:

$$\text{qua}(x) := A + B \cdot x + C \cdot x^2$$

$$\text{SOLVE} \left(\int_a^b \text{qua}(x) \, dx = \frac{b-a}{2+t} \cdot \left(\text{qua}(a) + t \cdot \text{qua}\left(\frac{a+b}{2}\right) + \text{qua}(b) \right), t \right)$$

$$t = 4 \vee C = 0 \vee a^2 - 2 \cdot a \cdot b + b^2 = 0$$

Hence, the area under a quadratic function is the product of the length of the integration interval and the weighted mean of the values of the function at its endpoints and its midpoint, the weights being 1:4:1, where 4 is the weight of the midpoint (for $C=0$ the function is linear, where t can be any number different from -2 . The second condition means $a=b$, and again t may be any number).

Task III: The area under a cubic function

Will the formula you got for the area under a quadratic function change in the case of a cubic function, $\text{cub}(x) := A + Bx + Cx^2 + Dx^3$?

Answer:

Surprisingly (or not) it does not change:

$$\int_a^b \text{cub}(x) \, dx = \frac{b-a}{6} \cdot \left(\text{cub}(a) + 4 \cdot \text{cub}\left(\frac{a+b}{2}\right) + \text{cub}(b) \right)$$

However, the weighted mean formula does not work for polynomials of higher degree, but it works quite well as an approximation. Before we move on to issues of numerical integration we emphasize, in the next task, the geometric perspective of the weighted mean in integration.

Task IV: Geometric integration using weighted means

Design a geometric construction to illustrate the weighted mean formula for a cubic function.

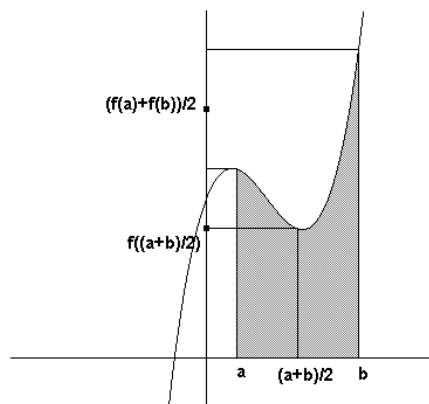


Figure 4.

Answer:

Figure 5 shows the geometric construction associated with the weighted mean formula for the area under the function $f(x) := x^3 - 3x^2 + 2x + 2$ in the interval $[0.5, 2.5]$. This task calls for incorporating geometrical knowledge and vector concepts, thus combining various mathematical topics.

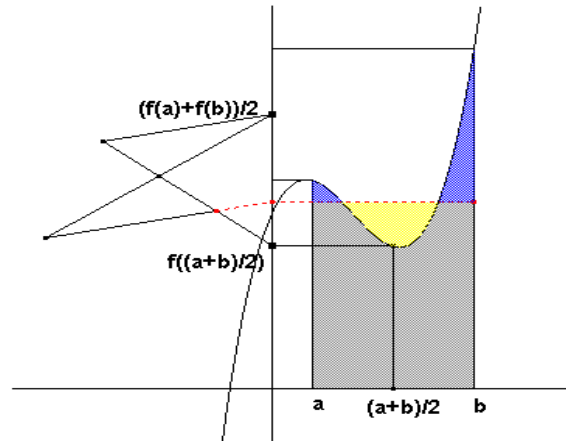


Figure 5.

In conclusion we say that the integral of any polynomial up to the third degree is a product of the length of the interval and the weighted mean of its values in the endpoints and the midpoint. In other words the **mean value** of such a polynomial over an interval equals the **weighted mean** of three values.

Weighted Mean Approximation (WMA) for a definite integral

Task V: *The difference between the definite integral of polynomials and the area obtained by the WM formula*

Does the weighted mean method work for polynomials of higher degree? Define:

$$\text{WM} := \frac{1}{6} \cdot \left(f(a) + 4 \cdot f\left(\frac{a+b}{2}\right) + f(b) \right)$$

Check the cases of $f(x) := x^4$ and $f(x) := x^5$ and compare them.

Answer:

$$\int_a^b x^4 dx - (b-a) \cdot \text{WM} = \frac{(a-b)^5}{120}$$

$$\int_a^b x^5 dx - (b-a) \cdot \text{WM} = \frac{(a+b) \cdot (a-b)^5}{48}$$

Note that the expressions for the differences were obtained by factoring the differences between the definite integral and the WMA, also note that $(a - b)^5$ is negative. The insight that we gain for $f(x) := x^4$ is that the difference depends only on the width of the interval, and it is not affected by its location. For polynomials of higher degree than 4 the difference depends on both the width and location of the interval. For other elementary functions (e.g. $\sin x$, x^x) we can use the WMA, keeping in mind that the error involved in using the WMA is proportional to the fifth power of the length of the interval. Therefore, we could improve the approximation by dividing the interval into subintervals. If we divide the interval into two equal sub intervals the error is a sum of two errors where each one them is decreased by a factor of 2^5 . Thus we get a new error, which is $1/16$ of the previous error. Doubling the number of subintervals will again decrease the error by a factor of 16 and in general, if we divide the integration interval into n subintervals the accuracy is increased by a factor of n^4 .

Integrating a function whose anti-derivative is unknown

The fundamental theorem of Calculus requires the values of the primitive function at the endpoints of the integration interval. Clearly, if we are unable to find a primitive function, we need to get a reasonable approximation of the integral using only the values of the integrand at the endpoints and the midpoint of the interval.

We shall use the WMA to approximate the length of the graph of $\sin(x)$ between $\frac{\pi}{4}$ and $5 \cdot \frac{\pi}{4}$, expressed

by the definite integral $\int_{\pi/4}^{5\pi/4} \sqrt{1 + \cos^2(x)} dx$, which is hard to compute without a CAS.

For $h(x) := \sqrt{1 + \cos^2(x)}$, we approximate $\int_{\pi/4}^{5\pi/4} h(x) dx$ by WMA(1):

$$\text{APPROX} \left(\frac{1}{6} \cdot \left(\frac{5 \cdot \pi}{4} - \frac{\pi}{4} \right) \cdot \left(h \left(\frac{\pi}{4} \right) + 4 \cdot h \left(\frac{3 \cdot \pi}{4} \right) + h \left(\frac{5 \cdot \pi}{4} \right) \right) \right) = 3.84764949045$$

$$\text{APPROX} \left(\frac{1}{6} \cdot \left(h \left(\frac{\pi}{4} \right) + 4 \cdot h \left(\frac{3 \cdot \pi}{4} \right) + h \left(\frac{5 \cdot \pi}{4} \right) \right) \right) = 1.22474487139$$

Figure 6 illustrates the results visually.

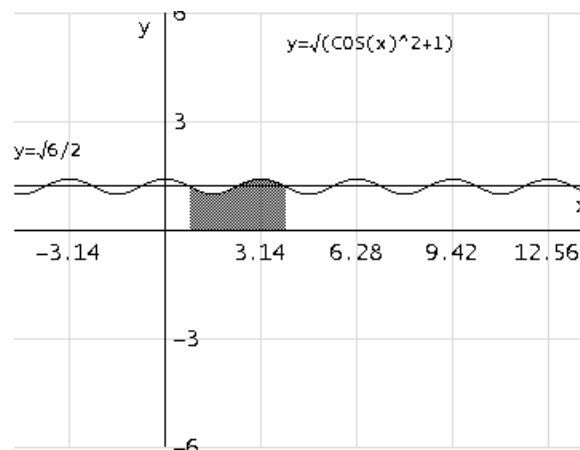


Figure 6.

Using *Derive*'s numerical integration, we get some control of our result:

$$\text{APPROX} \left(\int_{\pi/4}^{5 \cdot \pi/4} \sqrt{(1 + \cos(x)^2)} dx \right) = 3.82019778897$$

Obviously if we want a better approximation [than WMA\(1\)](#), we should divide the integration interval into sub-intervals and approximate the curve by WMA (or by quadratic functions).

We proceed by dividing the interval into two sub-intervals:

$$\int_{\pi/4}^{5 \cdot \pi/4} h(x) dx = \int_{\pi/4}^{3 \cdot \pi/4} h(x) dx + \int_{3 \cdot \pi/4}^{5 \cdot \pi/4} h(x) dx$$

Applying the method WMA(2), we get for each sub-interval:

$$\int_{\pi/4}^{3 \cdot \pi/4} h(x) dx = \int_{\pi/4}^{3 \cdot \pi/4} h(x) dx + \int_{3 \cdot \pi/4}^{5 \cdot \pi/4} h(x) dx$$

$$\int_{\pi/4}^{3 \cdot \pi/4} h(x) dx = \frac{1}{6} \cdot \left(\frac{3 \cdot \pi}{4} - \frac{\pi}{4} \right) \cdot \left(h\left(\frac{\pi}{4}\right) + 4 \cdot h\left(\frac{\pi}{2}\right) + h\left(\frac{3 \cdot \pi}{4}\right) \right)$$

$$\int_{\pi/4}^{3 \cdot \pi/4} h(x) dx = 1.68847246628 \quad \text{Approx}$$

$$\int_{3 \cdot \pi/4}^{5 \cdot \pi/4} h(x) dx = \frac{1}{6} \cdot \left(\frac{5 \cdot \pi}{4} - \frac{3 \cdot \pi}{4} \right) \cdot \left(h\left(\frac{3 \cdot \pi}{4}\right) + 4 \cdot h(\pi) + h\left(\frac{5 \cdot \pi}{4}\right) \right)$$

$$\int_{3 \cdot \pi/4}^{5 \cdot \pi/4} h(x) dx = 2.12223589443 \quad \text{Approx}$$

Combining the above we get:

$$\int_{\pi/4}^{5 \cdot \pi/4} h(x) \, dx = 1.68847246628 + 2.12223589443$$

Approx

$$\int_{\pi/4}^{5 \cdot \pi/4} h(x) \, dx = 3.81070836071$$

Doubling again the number of intervals, WMA(3) yields:

$$\int_{\pi/4}^{5 \cdot \pi/4} h(x) \, dx = \frac{1}{24} \cdot \left(\frac{5 \cdot \pi}{4} - \frac{\pi}{4} \right) \cdot \left(h\left(\frac{\pi}{4}\right) + 4 \cdot h\left(\frac{3 \cdot \pi}{8}\right) + 2 \cdot h\left(\frac{\pi}{2}\right) + 4 \cdot h\left(\frac{5 \cdot \pi}{8}\right) + \right.$$

$$\left. 2 \cdot h\left(\frac{3 \cdot \pi}{4}\right) + 4 \cdot h\left(\frac{7 \cdot \pi}{8}\right) + 2 \cdot h(\pi) + 4 \cdot h\left(\frac{9 \cdot \pi}{8}\right) + h\left(\frac{5 \cdot \pi}{4}\right) \right)$$

Approx

$$\int_{\pi/4}^{5 \cdot \pi/4} h(x) \, dx = 3.82028240611$$

The Error

The general theoretical result for the error is: $E \leq \frac{(b-a)^5}{180 \cdot 2^4} \cdot \text{Max}_{a \leq x \leq b} |f^{(4)}(x)|$.

In the case of the 4th degree: $E \leq \frac{(b-a)^5}{180 \cdot 2^4} \cdot \text{Max}_{a \leq x \leq b} |f^{(4)}(x)|$, and therefore:

$$\frac{(b-a)^5}{180 \cdot 2^4} \cdot \text{Max}_{a \leq x \leq b} |f^{(4)}(x)| = \frac{(b-a)^5}{180 \cdot 2^4} \cdot 24 = \frac{(b-a)^5}{120}.$$

In the case of the 5th degree: $\text{Max}_{a \leq x \leq b} |f^{(4)}(x)| = 120 \cdot \text{Max}_{a \leq x \leq b} |x| > 60 \cdot (a+b)$,

$$\text{So: } \frac{(b-a)^5}{180 \cdot 2^4} \cdot \text{Max}_{a \leq x \leq b} |f^{(4)}(x)| > \frac{(b-a)^5}{180 \cdot 2^4} \cdot 60 \cdot (a+b) = \frac{(b-a)^5}{48} \cdot (a+b).$$

Comparing the three results that we got by applying the WMA method for $h(x) := \sqrt{1 + \cos^2(x)}$ with the numerical approximation by *Derive* shows that the greater the number of sub-intervals, the better the approximation that we obtain:

$$\text{APPROX}(|3.84764949045 - 3.82019778897|) = 0.027451701471694116$$

$$\text{APPROX}(|3.81070836071 - 3.82019778897|) = 0.0094894282673033272$$

$$\text{APPROX}(|3.82028240611 - 3.82019778897|) = 8.4617133288927989 \cdot 10^{-5}$$

We can obtain, by reading from the graph or computing with *Derive*, that $\text{Max}|h^{(4)}(x)|$ is 7. Therefore, the error in the first case is controlled by

$$\frac{\pi}{180 \cdot 2^4} \cdot 7 = 0.7437978449$$

which is much bigger than 0.02745.... which was computed on the assumption that *Derive* knows, somehow, to compute exactly the integral.

Simpson's rule of numerical integration

Simpson's rule of numerical integration states that if you divide the integration interval into n equal sub-intervals and apply to each one of them the WMA, and sum the n approximations you get a result which strays from the definite integral by an error E , where

$$E \leq n \cdot \frac{\left(\frac{b-a}{n}\right)^5}{180 \cdot 2^4} \cdot \text{Max}_{a \leq x \leq b} |f^{(4)}(x)| = \frac{(b-a)^5}{180 \cdot 2^4 \cdot n^4} \cdot \text{Max}_{a \leq x \leq b} |f^{(4)}(x)|$$

Derive itself is using an algorithm based on Simpson's rule in order to compute definite integrals of functions with unknown anti-derivatives (a basic component of this algorithm decides how many sub intervals are needed, based on the theoretical error mentioned above).

The didactical sequence described above utilizes CAS to make numerical integration by CAS less mysterious. The power of CAS enables the to learn that we can compute the definite integral of a polynomial up to the third degree by the **weighted mean formula**, so that we have a **geometric solution** to the **geometric problem** of finding the area under a graph. Then, we find the error involved in using the method in the case of a fourth degree polynomial, and later we learn that, *in many cases, it is easier to control the upper limit of the error, which depends on the fourth derivative, than to compute the anti-derivative*. This makes the WMA a method worth refining. The outcome of this refining procedure is Simpson's rule.