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Can a Computer Algebra System improve the Mathematical Abilities of Pupils?

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1 Research questions

The experiment reported upon here is a case study on the integration of information technology into Belgian mathematics education. In Flemish secondary schools the number of pupils is going down in the sections with eight mathematics lessons a week compared to the sections with six mathematics lessons a week. To avoid a detrimental influence on their studies in engineering, mathematics, physics and IT education, the pupils need a wider mathematical baggage. Although these pupils aren't lacking in knowledge, their understanding of mathematics, their problem solving skills and mathematical creativity need greater attention.
The integration of IT in mathematics education is one of the ways in which a solution for these problems could be sought. The use of IT is generally expected to contribute to the visualization of concepts and can free students from long calculations, thus allowing them to concentrate on the development of (new) concepts and problem solving strategies. At the same time, the integration of technology also raises questions concerning the relevance of paper-and-pen techniques. For mathematics education, the use of a computer algebra system (CAS) is very interesting as it provides a complete repertoire of algebraic and graphical procedures and operations.
This brings us to the research questions of our study:

1. Can the use of computer algebra contribute to a higher level understanding of mathematics?
2. Can the development of instrumentation schemes influence the problem solving skills and the mathematical creativity of the pupils?

Other researchers ([1], [6]) had a strong hold on this second research question. Previous research ([4], [5]) on the integration of CAS into mathematics education showed that the idea of technology carrying out the calculations, so that pupils can concentrate on the development of new concepts and on their problem solving skills, is too simplistic. Therefore, as a second focus of this study, we considered the development of instrumentation schemes (which require technical skills and conceptual understanding) and its influence on the problem solving skills and the mathematical creativity of the pupils.

2 Theoretical framework

An important theoretical element in our study is the instrumental approach of the use of technology in mathematics education. We employed this instrumental approach to the use of computer algebra as a framework for understanding and interpreting the interactions between the pupils and the computer algebra systems.

The central idea of the instrumental approach is that a tool or an artifact is not automatically a useful instrument [1]. While using such a tool, the pupil has to develop instrumentation schemes. According to this view there is a triangular relationship between, the artifact (or a part of it), the mental scheme (developped by the pupil) and the task.

The development of these schemes (called 'instrumental genesis') isn't easy for the pupils. The instrumentation schemes are often complex and contain several shorter instrumentation schemes. Moreover, the instrumentation schemes integrate technical skills and conceptual insights. Difficulties in the instrumental genesis often involve both aspects. The instrumental genesis is difficult but instructive. It forces the pupils to think about their actions and solutions and according to our conviction, the instrumental genesis strongly influences the problem solving skills and the mathematical creativity of the pupils.

3 Methodology and the first research cycle

In this study we used a design research methodology. A characteristic of design research is the importance attributed to the design of instructional activities. This is a very important part of the research methodology because it forces the researcher to reflect on his choices, hypotheses and expectations. Another important characteristic of design research is the use of feedback and previous experiences for the adaptation of the learning trajectory throughout the research. This adaptation is possible because the design research is cyclic. Our complete study will contain two cycles. Each cycle contains a preliminary phase, a teaching experiment phase and a retrospective phase. We will now briefly discus each of the phases within our first research cycle.

1. *Before the preliminary phase*, we made a few choices.
   - The pupils, who are the center of our study, are last-year (or sixth year) pupils of the general secondary education with six mathematics lessons (of 50 minutes) a week. In a lot of Belgian schools, the classes are mixed; i.e. these schools use a '6+2'-principle, which means that the pupils of the six-lesson sections and the eight-lesson sections have six mathematical lessons in common. As a result of this principle, a smaller group of pupils of the eight-lesson sections will also participate in our study.

- We decided to use the current infrastructure of each school, namely a computer algebra system on a computer. Every Belgian school has several computer classes with mathematical software (Derive, Mathcad, Maple, etc.) and/or an internet connection.
- The lessons concerning the subject 'integrals' will be intensively supported with CAS, since integrals are taught in every last year class.

For this first cycle, I searched interested teachers among my ex-colleagues and on a mathematics mailing list. In total, 16 teachers of 13 different schools decided to cooperate. These teachers taught mathematics to 18 last year classes. In these classes a streaming according to their interest had already taken place. The 293 pupils (17-to 18-year-olds) were divided in different sections (Latin and Mathematics, Greek and Mathematics, Sciences and Mathematics, Languages and Mathematics, Economics and Mathematics, Latin and Sciences, Economics and Sciences etc.). 263 pupils choose a section with six mathematics lessons a week and 30 pupils chose a section with eight mathematics lessons a week.

2. In the *preliminary phase*, we developed a hypothetical learning trajectory. This involved several steps. First, we tried to determine the starting level of the pupils. Therefore, 35 pupils from three different schools made a test and I observed a few mathematics lessons supported with CAS in two groups with little experience in the use of technology in mathematics education. Secondly, we used this information to formulate the end goal and to develop a chain of mental steps towards that goal. At the same time, the design of the instructional activities that were expected to bring about this mental development, toke place. This design resulted in a package that contained:

   - a webpage for each school
   - an introduction on the use of Derive or Mathcad (if necessary)
   - 13 worksheets, 9 series of exercises and applications for making differentiation possible and 4 worksheets which are extensions on the traditional curriculum

   The concrete content of the worksheets and the series of exercises and applications are based on the official curriculum of the general secondary education. This curriculum is overfull and very intensive for the teachers and the students. We had to take in account that the teachers wanted to avoid a waste of time and that the pupils had to reach at least the 'classical' level.

   During the design of the instructional activities, we used four important didactical principles:

   - **White box - Black box**
     In a first stage, the pupils will learn the classic paper-and-pencil techniques for the calculations of definite and indefinite integrals. Once they are familiar with these paper-and-pencil techniques, they will be allowed to use the computer algebra system for all their calculations. This principle will reduce the black

box character of the CAS. The procedures of the computer algebra system will become more transparent for the pupils, because they will be able to lean on their paper-and-pencil techniques to see through the CAS-techniques.

- **Window didactics**
  The use of technology in mathematics education allows the pupils to use at the same time an algebraic and a graphical view of the same concept or problem. They can freely switch between the different windows and use those aspects they need. For example, when they want to calculate the area bounded by the graphs of two functions, they can use the graphical window to receive an overview of the problem and to control their result. The algebraic window can be used for the calculations: finding the points of intersection, the calculations of the integrals, etc.

- **Guided self-learning**
  The teachers were advised to use the worksheets for guided self-learning. This changes the role of the teacher drastically. The teachers will become a guide and will lead their pupils through the worksheets by personal interaction and classroom discussions. On the other hand, the pupils are forced to work on their own, think about the concepts and search their own solutions, of course with some help of the teacher if necessary.

- **Differentiation**
  There is a big difference between the mathematical abilities of the pupils of the six-lesson sections. The majority of these pupils are arithmeticians; they prefer number work instead of applications and problems. To meet towards these differences we wanted to make it possible for the teachers to work in a differentiated way: the teachers (or maybe even the pupils) can choose those exercises that are suitable for the pupils. The principle of differentiation and the use of a computer algebra system are a very good combination because every pupil (or every couple of pupils) can work and develop independently and on their own rhythm.

3. The second phase of the design research cycle is the *teaching experiment phase.* During the teaching experiment, we tried to collect data that reflected the learning process and provided insight into the students' way of solving problems. For this purpose, the pupils were split in an Intervention group and a Control group. This split was made according to the choice of the accompanying teachers. If the teacher preferred the traditional way of teaching (without CAS) he/she joined the Control group. The other teachers and pupils joined the Intervention Group.

   - In September 2003 we determined the mathematical skill level of all our last-year students with respect to mathematical understanding, problem solving skills and mathematical creativity.
   - Between October 2003 and May 2004, the mathematics lessons of the Intervention group concerning the subject 'integrals' was extensively supported with CAS, while the lessons of the Control group remained traditional. Six teachers who had little experience with the use of technology in mathematics educa-

tion used the worksheets that were designed in the preliminary phase. Two experienced teachers mainly used their own material.

- At the end of May 2004 both groups were tested again.

The main sources of our data were the results of the two mathematical tests, observations of student behaviour, inspection of the written materials of some pupils of the Intervention group and interviews with the teachers.

4. The final phase of the research cycle is the *phase of retrospective analysis*. A first step in this phase was the collection and analysis of our data. The data is partial numerical and partial coded. The assessment of the mathematical skill level of our pupils wasn't easy, because some pupils weren't motivated to try to solve the problems of the tests. For this reason, we decided to focus on the problems that the pupils tried to solve. All pupils handed in a complete test, which means that the written materials contained their attempts and their final answers. Because there is a big difference between a zero for a wrong answer and a zero for a missing answer, we 'cleaned' our data and for the analysis we focussed on the tryouts. The conclusions of the data analysis were translated into our first results and feed-forward for the next research cycle.

4 Data Analysis and results

4.1 The mathematical tests

The purpose of the tests was determining the mathematical skill level of the pupils with respect to mathematical understanding, problem solving skills and mathematical creativity. We didn't want to test their mathematical knowledge. For that reason, we chose questions that provided little foreknowledge and held the tests on a moment that the foreknowledge of the pupils was more or less equal: the beginning and the end of the school year.
During the marking, we had special attention for the creativity of the pupils and for their interpreting of the data, the graphs and their own calculations; miscalculations received a small penalty and obvious incompatibilities a large penalty.

In this paragraph we will give an overview of the different questions and the results of the pupils, split according to their group (Intervention group or Control group) and their number of mathematics lessons a week.

Type 1:

Question 1.1

Find:
$2003 - 2001 + 1999 - 1997 + ... + 3 - 1$.

Question 2.3

For each integer $n > 1$ we define $n! = n.(n-1).(n-2)...3.2.1$. How many zeros are at the end of 100! ?

A patient pupil could maybe complete the whole calculation, but that wasn't the purpose of these questions. A possible solution for these problems was concentrating on a smaller but similar problem, like $19 - 17 + 15 - 13 + .. + 3 - 1$ for Q.1.1 or 10! for Q.2.3.

Group	Attempted	Mean score	Group	Attempted	Mean score
Total group	95,2 %	53,6 %	Total group	86,8 %	23,9 %
Intervention group	94,9 %	46,8 %	Intervention group	86,2 %	21,8 %
6-lesson	94,6 %	46,8 %	6-lesson	85,8 %	21,8 %
8-lesson	100,0 %	53,3 %	8-lesson	100,0 %	22,2 %
Control group	95,3 %	58,2 %	Control group	87,3 %	25,3 %
6-lesson	94,6 %	56,7 %	6-lesson	86,5 %	21,9 %
8-lesson	100,0 %	66,7 %	8-lesson	91,7 %	44,7 %

Type 2:

Question 1.2

Yesterday Frederik drove 100 km from his work to his house with an average speed of 120 km/h. Afterwards, he went to refuel his car, just around the corner and this toke 4 minutes. His wife Veerle also drove 100 km from her work to their house. The first 80 km she had an average speed of 120 km/h and the last 20 km, she had an average speed of 140 km/h. She also had to refuel her car and she needed 6 minutes.
If Frederik and Veerle left at the same time, who arrived first and what was the exact difference?

Question 2.2

The average salary of the 25 employees of the videoshop *Home Alone* is 22400 Euro a year. Next week, Jef (the oldest and best-paid employee) retires. A new employee with a salary of 18000 Euro will replace Jef. The average salary will decrease to 23900 Euro a year. What was the salary of Jef?

With these (simple) problems we wanted to test if the pupils were able to translate this type of problems to pure mathematics.

Group	Attempted	Mean score	Group	Attempted	Mean score
Total group	99,3 %	79,4 %	Total group	96,8 %	64,4 %
Intervention group	98,3 %	80,4 %	Intervention group	97,4 %	62,1 %
6-lesson	98,2 %	80,3 %	6-lesson	97,3 %	61,5 %
8-lesson	100,0 %	83,3 %	8-lesson	100,0 %	83,3 %
Control group	100,0 %	78,7 %	Control group	96,4 %	66,0 %
6-lesson	100,0 %	77,2 %	6-lesson	96,5 %	63,7 %
8-lesson	100,0 %	87,3 %	8-lesson	95,8 %	79,7 %

Type 3:

Question 1.3

A regular hexagon is inscribed in a circle with radius R. How long is a diagonal, which is **not** passing through the center?

Question 2.1

In a square, the diagonals of which have length 8, a circle is inscribed. Calculate the length of the radius of that circle.

Two geometrical problems, given without a figure. The pupils had a lot of difficulties to translate the problem into a correct figure (especially Q.1.3). The calculations were mostly based on the theorem of Pythagoras or calculations in a right-angled triangle.

Group	Attempted	Mean score	Group	Attempted	Mean score
Total group	96,9 %	46,5 %	Total group	100,0 %	85,9 %
Intervention group	97,4 %	42,0 %	Intervention group	100,0 %	85,3 %
6-lesson	97,3 %	39,9 %	6-lesson	100,0 %	85,0 %
8-lesson	100,0 %	86,7 %	8-lesson	100,0 %	100,0 %
Control group	96,5 %	49,7 %	Control group	100,0 %	86,4 %
6-lesson	95,9 %	48,0 %	6-lesson	100,0 %	84,5 %
8-lesson	100,0 %	59,3 %	8-lesson	100,0 %	97,2 %

Type 4:

Question 1.4

Is $P(y) = y^8 - 256$ divisible by $y^2 - 4$? Why (not)?

Question 2.4

$x^4 + 4x^3 + 6px^2 + 4qx + r$ is divisible by $x^3 + 3x^2 + 9x + 3$. Find $(p+q)r$.

The pupils had to find a suitable translation for 'divisible by'. The pupils were rather creative. We saw a lot of different translations and methods.

Group	Attempted	Mean score	Group	Attempted	Mean score
Total group	93,1 %	59,2 %	Total group	92,2 %	59,5 %
Intervention group	94,0 %	56,1 %	Intervention group	94,8 %	59,4 %
6-lesson	94,6 %	56,1 %	6-lesson	94,7 %	58,9 %
8-lesson	80,0 %	54,2 %	8-lesson	100,0 %	77,8 %
Control group	92,4 %	61,3 %	Control group	90,3 %	59,5 %
6-lesson	91,8 %	58,9 %	6-lesson	89,4 %	57,4 %
8-lesson	96,0 %	75,0 %	8-lesson	95,8 %	71,0 %

Type 5:

Question 1.5

How many ways are there for the distribution of 10 apples among Wim, Tom and Greet, in such way that Wim has minimal 3 apples, Tom and Greet both have minimal 2 apples and Greet has maximum 3 apples?

Question 2.5

The 'total sum' of the digits of the number 12345678 is 9 because $1+2+3+4+5+6+7+8 = 36$ and $3+6 = 9$. How many years in the 21^{th} century have a total sum of 6?

These problems are creative counting problems which can be easily solved by writing down all the possibilities.

Group	Attempted	Mean score	Group	Attempted	Mean score
Total group	99,0 %	84,1 %	Total group	94,7 %	59,1 %
Intervention group	100,0 %	86,2 %	Intervention group	91,4 %	57,9 %
6-lesson	100,0 %	86,2 %	6-lesson	91,2 %	56,8 %
8-lesson	100,0 %	86,7 %	8-lesson	100,0 %	94,4 %
Control group	98,3 %	82,7 %	Control group	97,0 %	59,3 %
6-lesson	98,6 %	83,0 %	6-lesson	97,2 %	58,0 %
8-lesson	96,0 %	81,2 %	8-lesson	95,8 %	66,7 %

Type 6:

Question 1.6

Changing the dimensions of a square with surface V develops a rectangle with surface R: two opposite sides become 25% longer and two opposite sides become 25% shorter. Find $\frac{R}{V}$.

Question 2.6

Changing the dimensions of a circular disc with surface C develops an elliptic disc with surface E: one of two perpendicular diameters becomes one third longer and the other a quarter shorter. Find $\frac{E}{C}$.

With these questions we wanted to determine if the pupils could translate this type of problems in a good figure and good calculations. Unfortunately, for many pupils the % caused extra problems.

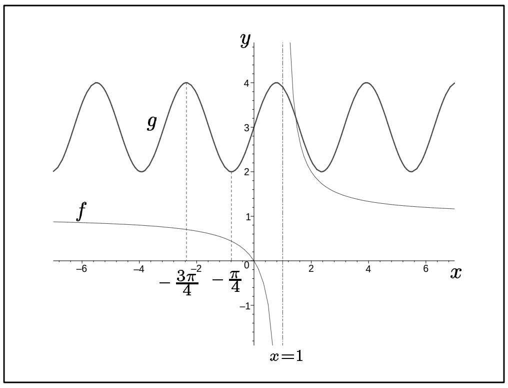
Group	Attempted	Mean score	Group	Attempted	Mean score
Total group	95,8 %	70,7 %	Total group	87,9 %	64,9 %
Intervention group	99,1 %	72,1 %	Intervention group	90,5 %	67,1 %
6-lesson	99,1 %	71,3 %	6-lesson	90,3 %	66,2 %
8-lesson	100,0 %	90,0 %	8-lesson	100,0 %	100,0 %
Control group	93,6 %	69,7 %	Control group	86,1 %	63,3 %
6-lesson	92,5 %	66,7 %	6-lesson	84,4 %	58,7 %
8-lesson	100,0 %	86,0 %	8-lesson	95,8 %	87,0 %

Type 7:

Question 1.7

Given:
$f(x) = \frac{x+A}{x+B}$ and $g(x) = \sin(Cx) + D$.

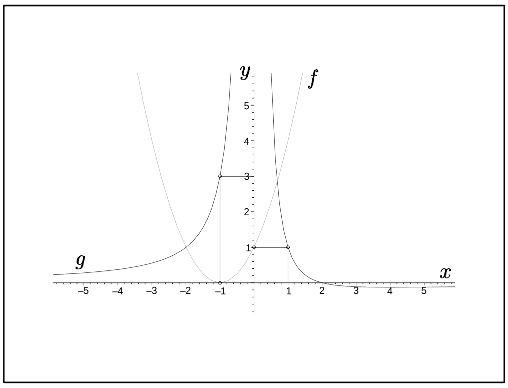
Find A, B, C and D.



Question 2.7

Given:
$f(x) = A(x+B)^2$ and $g(x) = \frac{Cx+D}{x^2}$.

Find A, B, C and D.



Q.1.7 (more than Q.2.7) turned out to be a very hard problem. A lot of pupils had difficulties reading the graphs of the functions and translating the given points into equations.

Group	Attempted	Mean score	Group	Attempted	Mean score
Total group	74,4 %	41,3 %	Total group	85,8 %	58,0 %
Intervention group	80,3 %	38,2 %	Intervention group	89,7 %	57,8 %
6-lesson	79,5 %	37,1 %	6-lesson	89,4 %	57,3 %
8-lesson	100,0 %	57,5 %	8-lesson	100,0 %	75,0 %
Control group	70,4 %	43,8 %	Control group	83,0 %	58,1 %
6-lesson	69,4 %	40,8 %	6-lesson	80,1 %	54,9 %
8-lesson	76,0 %	59,9 %	8-lesson	100,0 %	73,5 %

4.2 Numerical results

1. The pupils of the eight-lesson sections scored better for almost all the questions.

2. When we compare the pupils of IG6 (Intervention group and 6 mathematical lessons a week) with the pupils of CG6 (Control group and 6 mathematical lessons a week), we can not conclude that the pupils of IG6 are smarter than the pupils of CG6, or the other way around. The results differ from question to question.

3. The pupils of IG6 made more progress than the pupils of CG6, and this for almost all the questions:

Question type	Test 1 IG6 versus CG6	Test 2 IG6 versus CG6
1	- 10,2 %	- 0,1 %
2	+ 3,1 %	- 2,2 %
3	- 8,1 %	+ 0,5 %
4	- 2,8 %	+ 1,5 %
5	+ 3,2 %	+ 1,2 %
6	+ 4,6 %	+ 7,5 %
7	- 3,7 %	+ 2,4 %

   - The pupils of IG6 increased their lead for the questions of type 6.
   - They decreased their arrears for the questions of type 1.
   - They transformed their arrears for questions of type 3, 4 and 7 into a lead.

4. We had expected that, due to the principle of window didactics, the progress would be noticeable for the questions with a graphical aspect (type 3, 6 and 7) but the progress is more general.
   This assumption was obviously confirmed in one class group of the Intervention group. Due to accidental circumstances, the pupils had only 30 minutes (instead of the normal 50 minutes) to complete the second test. They knew that the time was to short to complete the test and for that reason they chose those questions they wanted to try. All the 18 pupils chose the questions of type 2, 3, 4, 6 and 7. Only one pupil made the question of type 5 and none made the question of type 1.

5. We tried to find a correlation between the pupils of IG6 who had the most progress. There turned out to be no such correlation: these pupils

   - have chosen different options
   - have used different computer algebra systems
   - had different teachers

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The big differences between the pupils of the six-lesson sections can probably explain the obvious progress of some pupils: some pupils are only arithmeticians, but others can reach the higher level of problem solving and mathematical creativity.
Another explanation can be found in the research of Trouche [1]. His research firstly evidenced the diversity of instrumental relationships that the pupils develop. This diversity led Trouche to introduce five extreme profiles, which he calls 'theorist', 'rationalist', 'scholastic', 'thinker' and 'experimentalist'. According to their profile characteristics, pupils develop different relationships with their graphic and symbolic calculators and use different instrumentation schemes.

4.3 The interviews

The worksheets and the hypothetical learning trajectory had a double purpose:

1. To pass on the knowledge and understanding of the concept 'definite and undefinite integral' on the pupils.
2. To confront the pupils with exercises, problems and applications, which supported this transfer of knowledge and in the meantime increase their general mathematical abilities.

We didn't provide a test to assess the knowledge and understanding of the concept 'integral' of the Intervention group but we can rely on the experiences of our teachers and the exams hold by the teachers.

- During the interviews, the teachers told that the worksheets were difficult, but not to hard for their class groups.
- The pupils understood the concept of 'definite integral', 'Riemann sums' and 'numerical integration' better than before. The teachers were very enthusiastic about these improvements.
- All the teachers agreed that using information technology is no waste of time; the pupils made more and harder exercises.
- Some teachers didn't use the principle of differentiation: they wanted to control their pupils and be sure that they reached a minimal level.
- The teachers reported very little problems:
  - Only a few pupils had problems to combine the computer algebra system with their written materials. They worked well in the computer class, but they didn't make any notes during the lesson.
  - In one class group, the teacher reported syntax problems. The pupils didn't have any experience with IT. They weren't aware of the fact that the syntax

was very strict and in the first worksheet, they had difficulties to understand that a procedure needs several arguments in a strict order.

- Some pupils were a bit lazy. They saw the teacher as a 'helpdesk' and they didn't try themselves to solve the problem but waited until the teacher helped them.
- The pupils had little problems to develop instrumentation schemes. The teachers were rather sure that this was because the pupils knew the paper-and-pencil techniques.

5 Conclusions and feed-forward

Although the results of the mathematical tests are very promising, we want to control our results with a second research cycle. An important characteristic of design research is the use of feedback and previous experiences for the adaptation of the research. We foresee the following changes:

- We are aware of the fact that the experimental phase is influenced by a lot of aspects that are hard to measure: the experiences of the teacher, the number of pupils in a class group, the infrastructure of the school, etc. To examine these aspects, we foresee more class room observations.
- Only a few teachers were willing to use the principle of differentiation. During the second cycle, we'll try to convince the other teachers of the importance of this principle. We'll strictly guide these teachers and divide the worksheets in a basic and advanced part.

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