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Magic Squares and DERIVE

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Introduction

'Mathematics is fun' is not a cry universally proclaimed by students and yet mathematical puzzles and games found in competitions and Challenges often attract the interest of students who regularly claim 'I can't do maths!'. Sadly, it would seem that perhaps the fun is disappearing from mathematics syllabuses as the pressures on students to succeed in examinations increase, and the attendance by student mathematicians at public lectures which discuss such 'recreational' mathematics is seen perhaps by teachers as of less academic value and hence of low priority.

One such recreational maths topic is that of magic squares. The subject of magic squares has a history that can be traced back almost 2500 years ago to ancient China. The number of books, journal papers and other articles written on the subject, by both established academic scholars and 'amateur' mathematicians, are far to numerous to list, but the interested reader might wish to start with the recreational works of Rouse Ball [1], Kraitchik [2], Benson & Jacoby [3], and the *Scientific American* article on magic squares and magic cubes by Martin Gardner [4]. For internet users, a useful web site to start with on magic squares and in particular some of the history of the subject, can be found at www.grogno.com/magic .

The author's own interest in magic squares was re-awakened by a challenge issued in an article in an edition of the DERIVE Newsletter in December 1998, [5], which asked 'how can one obtain the 880 different order-4 magic squares with DERIVE?' . In trying to answer this challenge, it became apparent from some investigative research on the subject that:

- magic squares as a subject is not dead. There are still many articles being published about the properties of various types of such squares and their algorithmic implementation, as the references quoted in this paper show.

- many articles are now concentrating on the algebraic properties of magic squares as illustrations of aspects of linear algebra taught primarily to undergraduate students. (see for example [6], [7], [8], [9])

- algorithms that construct various types of magic squares are well known and are implemented as built-in functions in some mathematical software. For example, MatLab has the built-in function magic(n) that returns an nxn ($n \geq 3$) matrix constructed from the integers 1 through to n^2 with equal row and column sums. The implementation of magic square algorithms in itself can be a useful exercise for students studying computer programming, whether it be in a typical programming language such as C++ or perhaps using the programming facilities of a CAS such as DERIVE 5.

Teaching and Learning Issues

In this paper, an attempt is made to put together examples of student worksheets or coursework that can be used to support and strengthen student understanding of matrix algebra and some concepts in linear algebra. A further aim, indeed perhaps a more significant aim, is that of developing students' generic mathematical skills. Skills such as the ability to conjecture hypotheses, justifying/proving such hypotheses, interpreting results, the ability to generalise, etc. are surely important in the development of potential mathematics graduates, and yet perhaps too much emphasis these days is placed on students' ability just to learn and apply routine techniques. It is the author's belief that a CAS should be an integral part of the students' and teachers' toolset in the battle to develop such skills (see [10]). Hence, the effectiveness of the proposed courseworks that follow should be judged on their potential for skills development as well as improved conceptual understanding of a particular subject domain. No novel developments of properties of magic squares are claimed in this paper, but the application of a 'fun subject' and the appropriate use of technology in the development of mathematical skills should be appealing to all teachers.

In what follows, each coursework presented is followed by a discussion of the rationale for the aspects of the coursework in terms of skills development and a brief description of more detailed sources of solution to the questions. DERIVE 5 is the computer algebra system (CAS) used in these examples, but other software platforms could be used as desired. The courseworks are such that they assume more prior knowledge of matrices and linear algebra as they go along. As the commentaries suggest, their use could be as an investigative 'lead-in' to some aspect of linear algebra using something like a constructivist approach to teaching, or as supporting exercises after some aspect of linear algebra theory has been covered in class.

Magic Squares – definitions and terminology

Of course, students will need some briefing on magic squares before undertaking any coursework. The amount of detail will depend on the teacher's requirements – part of the learning process could be to ask the students to undertake their own research into the subject prior to any coursework and perhaps to present their findings to their peers. A few definitions and terms are given here for clarity and continuity within this paper.

We define an nxn magic square to be a square array of n^2 numbers (i.e. a square matrix) whose rows, columns and two main diagonals sums are all equal and have value s (often termed the magic constant). By convention, magic squares are often thought of as having entries that are integers and maybe consecutive integers from 1 to n^2 (often termed a normal magic square). In fact, many of the theorems quoted about magic squares do not need these restrictions and in general entries may be rational, real or complex. For the purpose of this paper, integer entries will be assumed for convenience, with the special case of consecutive integers starting from 1 included as desired to emphasise some point of understanding.

Thus, for example, the following are all 3x3 magic squares

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

the first being the oldest and simplest normal magic square known, formed of consecutive integers 1..9 and magic constant value 15; the second recognised as the 3x3 identity matrix with repeated entries and with $s = 3$; the third including non-negative and repeated integer entries and with $s = 0$.

The 4x4 magic square

$$\begin{bmatrix} 15 & 10 & 3 & 6 \\ 4 & 5 & 16 & 9 \\ 14 & 11 & 2 & 7 \\ 1 & 8 & 13 & 12 \end{bmatrix}$$

is made up of the first 16 consecutive integers starting from 1, has magic constant 34 for all rows, columns and two main diagonals as required, but has the extra properties that all 'broken diagonals' (4, 10, 13, 7), (14, 5, 3, 12), (10, 16, 7, 1), (3, 9, 14, 8), (8, 2, 9, 15), (13, 7, 4, 10) similarly add up to 34. A magic square with such a property is known as pandiagonal (or diabolic, Nasik, perfect, etc.!).

Details of many other types of magic squares with extra 'magic' properties can be found in the references quoted. Magic squares obtained by appropriate reflections or rotations of a given magic square are deemed to be equivalent and are not counted in the numbers of magic squares of a given order.

The case of 3x3 magic squares is used significantly as in many examples this serves to illustrate the point. Extensions to 4x4, 5x5 and higher order magic squares are included when potential generalisations to nxn magic squares are considered. It is assumed that the CAS is used to avoid tedious matrix manipulations, even in the 3x3 case.

Coursework #1 Basic properties of magic squares

1. Consider the 3x3 normal magic square $\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$. By rotating and reflecting the elements of this matrix in an organised way, write down 7 other 3x3 magic squares that are equivalent to the given one. In each case, state clearly the nature of the rotation or reflection.

2. For the 3x3 normal magic square, the magic constant is 15. Find a formulae for the magic constant for an *nxn* normal magic square and confirm that it holds for the 4x4 case where the magic constant is 34. (Hint: the answer is most easily obtained by using the formulae for the sum of the first n^2 numbers. The SUM command in DERIVE might help here.)

3. Use trial-and-error to convince yourself that a 2x2 normal magic square involving the numbers 1,2,3,4 does not exist.

4. For the 3x3 case, write down the set of triples from the numbers 1 to 9 inclusive whose sum is 15. Use this set to explain logically why for a 3x3 normal magic square:
   i) the centre element must be a 5.
   ii) A 1 cannot appear as a corner element.

   Hence write down possible 3x3 normal magic squares. Is your list the same as your answer to 1. above?

5. Try the same process for the 4x4 normal magic square case. Is the process likely to lead to an easy identification of all the distinct 4x4 possibilities? What is the problem?

6. Writing the 3x3 normal magic square in the form $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then one can begin

   to write down equations to determine the values of the integers a, b, c, ……. such as:

   $a + b + c = 15$, $d + e + f = 15$, $g + h + i = 15$, for each of the row sums,

   and similarly for the column sums and the two diagonal sums. This gives eight equations in the nine unknowns. A ninth equation can be written knowing that the sum of the entries equals 3 times the magic constant as:

   $$a + b + c + d + e + f + g + h + i = 45$$

   Use the **Solve>System** command in DERIVE to attempt a solution to this set of nine equations in nine unknowns. Analyse and comment on your results.

7. Attempt a similar solution method as in 6. above but for a 4x4 magic square. What happens now?

Discussion of coursework #1

This is purposely aimed at students who have a little knowledge of equations, rotations, reflections and solution of systems of equations, and maybe an idea about what the term ‘matrix’ means. As such it could suit investigative work in the A-level (pre-degree) studies. Task 4 might be a challenging step into logical thinking for many students and task 5 might well be their first confrontation with a problem without a neat, closed solution. The CAS is usefully employed to provide the required sum in task 2 if the students cannot remember the formula for the sum of the first n integers starting from 1 and/or do not realise how the sum of the first n^2 numbers can be obtained.

Finally, the CAS is used to attempt a solution to a system of 9 equations in 9 unknowns as an alternative strategy to find 3x3 normal magic squares. A typical DERIVE output is shown below:

```
#1:   a :∈ Integer [1, 9]
#2:   b :∈ Integer [1, 9]
#3:   c :∈ Integer [1, 9]
#4:   d :∈ Integer [1, 9]
#5:   e :∈ Integer [1, 9]
#6:   f :∈ Integer [1, 9]
#7:   g :∈ Integer [1, 9]
#8:   h :∈ Integer [1, 9]
#9:   i :∈ Integer [1, 9]
#10:  SOLVE([a + b + c = 15, d + e + f = 15, g + h + i = 15, a + d + g = 15, b + e + h = 15, c
        + f + i = 15, a + e + i = 15, c + e + g = 15, a + b + c + d + e + f + g + h + i = 45],
        [a, b, c, d, e, f, g, h, i])

#11:  [a + i = 10 ∧ b + h = 10 ∧ c - h - i = -5 ∧ d - h - 2·i = -10 ∧ e = 5 ∧ f + h + 2·i = 20
        ∧ g + h + i = 15]
```

The variables have been declared as precisely as possible but the output solution has only confirmed that $e = 5$. Students should observe that DERIVE has only produced 7 relationships between the variables even though 9 equations were input. The brighter student should begin to suspect that ‘we didn’t really have 9 equations to start with’ and that ‘two equations could have been obtained by rearranging the other 7 somehow’. In fact, the same solution expression is obtained if the variables are each declared as any real number. Thus a bright student could deduce that the centre element of a 3x3 magic square is always equal to $s/3$.

Task 7. should soon raise the query ‘where do I get 16 (independent) equations from? The best I can do is 11. So, there are bound to be lots of answers.’

Coursework #2 Sums and products of magic squares

Confirm by inspection that a 3x3 magic square can be written in general form as:

$$\mathbf{M} = \begin{bmatrix} s/3+u & s/3-u+v & s/3-v \\ s/3-u-v & s/3 & s/3+u+v \\ s/3+v & s/3+u-v & s/3-u \end{bmatrix}$$

where u , v are arbitrary (integers) and s is the magic constant. Use this representation to determine answers to the following:

1. If **M** is a 3x3 magic square , is $\mathbf{M}^2$ also a magic square and what about other integer powers $\mathbf{M}^3$, $\mathbf{M}^4$, etc. ? Conjecture a result.

2. If any two 3x3 magic squares of the same order are multiplied together, is the result also a magic square? What if three (four, five….) magic squares are multiplied together? Conjecture a result.

3. What are the conditions on *u* and *v* above for **M** to be non-singular? Investigate a possible relationship between the singularity of **M** and whether $\mathbf{M}^{-1}$ is magic or not. Conjecture a result.

4. What are the conditions on *u* and *v* above for **M** to be normal magic square? When these values are substituted back into M, how many 3x3 normal magic squares are obtained?

5. Do your conjectures in 1-3 above generalise to higher order magic squares? Consider the set of 4x4 magic squares. Either find a counter-example that disproves a generalisation of 1-3 above or attempt a proof. It may be noted that a 4x4 magic square can be written in general form as

$$\mathrm{M} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$$

and with $d = s-(a+b+c)$; $h = s-(e+f+g)$; $\mathrm{j} = 2a+b+c+e-g+i-s$; $k = 2s-2a-b-c-e-f-i$; $l = f+g-i$; $m = s-(a+e+i)$; $n = 2s-2a-2b-c-e-f+g-i$; $p = 2a+b+e+f-g+i-s$; $q = a+b+c+e+i-s$.

The remaining exercises involve consideration of the general *nxn* magic square.

6. Show that the sum of two *nxn* magic squares is an *nxn* magic square. If the two magic squares have magic constants s_1 and s_2, then what is the magic constant of the sum?

7. Show that the scalar multiplication of an *nxn* magic square by a (real) number *k* is also an *nxn* magic square. If the original magic square has magic constant *s*, what is the magic constant of the scalar multiple?

Discussion of coursework #2

The representation given for the 3x3 magic square was quoted for example in [2]. That the middle element is *s*/3 follows can be deduced from coursework #1 above and the rest follows from the form of magic squares. DERIVE can now be used to advantage to compute powers of **M** as desired for the investigation.

Questions 1,2 and 3 have been considered for example in [11], [12], [13] and [14]. The student should be able to conjecture that if **M** is a 3x3 magic square then $\mathbf{M}^k$ is also magic with magic constant s^k for every odd positive integer k , and that if **M** is a 3x3 invertible magic square then $\mathbf{M}^{-1}$ is also magic with magic constant $1/s$. Also, the product of an odd number of magic squares is also magic. The brighter student might observe that it is worth investigating question 2 first as question 1's result is a special case of this.

Having set questions encouraging the mathematical skill of hypothesising results, the successful students should be encouraged to prove their conjectures. For example, the more able student could be expected to recognise possibilities of proof by induction or perhaps the use of the Cayley-Hamilton theorem to prove conjectures about $\mathbf{M}^k$.

For the normal magic square in question 4, s = 15 and it can be deduced that allowable values for (u, v) are (1, 3), (-1, 3), (1, -3), (-1, -3), (3, 1), (3, -1), (-3, 1), (-3, -1). These give rise to the eight 3x3 normal magic squares discussed in coursework #1 question 1.

Having considered the 3x3 case to the full, a natural question from an inquisitive student should be ‘ do my conjectures hold in general for the nxn case?’ The answer is no. For example, in [14] the case

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 2 & 2 & -1 & -1 \end{bmatrix}$$

is given. Here **M** is magic with magic constant 2, non-singular (as DERIVE will confirm quickly) but $\mathbf{M}^{-1}$ is not magic (again as DERIVE will confirm easily). More details on powers of 4x4 magic squares can be found in [13]. The general form of the 4x4 magic square is quoted in [2].

Finally, the student is encouraged to consider the general nxn case in the abstract and as a prelude to class discussion about vector spaces. These questions have been considered in [6] and [7] for example.

It should be noted that although DERIVE can readily compute $\mathbf{M}^2$, $\mathbf{M}^3$, etc. it is not easy to confirm that these are magic merely by inspection. DERIVE gives the output overleaf. A DERIVE utility program magic_check (**M**, n) can easily be constructed (especially in DERIVE 5) to sum each row, column and the two diagonals of the nxn matrix **M**, and output an appropriate message ‘magic’ or ‘not magic’. Perhaps slightly more appealing is to use the conditions for **M** to be a magic square with magic constant s quoted in [14], namely:

$$\mathbf{M}.\mathbf{X} = s\,\mathbf{X}\,, \quad \mathbf{M}^{\mathbf{T}}\,\mathbf{X} = s\,\mathbf{X}\,, \quad \text{trace}(\mathbf{M}) = s,\ \text{trace}(\mathbf{E}\,.\mathbf{M}) = s$$

where

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$\mathbf{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{E} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ for the 3x3 case.

```
input the magic square M
```

$$\#1: \quad \begin{bmatrix} \frac{s}{3} + u & \frac{s}{3} - u + v & \frac{s}{3} - v \\ \frac{s}{3} - u - v & \frac{s}{3} & \frac{s}{3} + u + v \\ \frac{s}{3} + v & \frac{s}{3} + u - v & \frac{s}{3} - u \end{bmatrix}$$

```
compute M^2
```

$$\#2: \quad \begin{bmatrix} \frac{s}{3} + u & \frac{s}{3} - u + v & \frac{s}{3} - v \\ \frac{s}{3} - u - v & \frac{s}{3} & \frac{s}{3} + u + v \\ \frac{s}{3} + v & \frac{s}{3} + u - v & \frac{s}{3} - u \end{bmatrix}^2$$

$$\#3: \quad \begin{bmatrix} \frac{s^2 + 6\cdot(u^2 - v^2)}{3} & \frac{s^2 - 3\cdot(u + v)\cdot(u - v)}{3} & \frac{s^2 - 3\cdot(u^2 - v^2)}{3} \\ \frac{s^2 - 3\cdot(u + v)\cdot(u - v)}{3} & \frac{s^2 + 6\cdot(u + v)\cdot(u - v)}{3} & \frac{s^2 - 3\cdot(u + v)\cdot(u - v)}{3} \\ \frac{s^2 - 3\cdot(u^2 - v^2)}{3} & \frac{s^2 - 3\cdot(u + v)\cdot(u - v)}{3} & \frac{s^2 + 6\cdot(u^2 - v^2)}{3} \end{bmatrix}$$

```
compute M^3
```

$$\#4: \quad \begin{bmatrix} \frac{s}{3} + u & \frac{s}{3} - u + v & \frac{s}{3} - v \\ \frac{s}{3} - u - v & \frac{s}{3} & \frac{s}{3} + u + v \\ \frac{s}{3} + v & \frac{s}{3} + u - v & \frac{s}{3} - u \end{bmatrix}^3$$

$$\#5: \quad \begin{bmatrix} \frac{s^3 + 9\cdot u\cdot(u^2 - v^2)}{3} & \frac{s^3 + 9\cdot(u^2 - v^2)\cdot(v - u)}{3} & \frac{s^3 - 9\cdot v\cdot(u^2 - v^2)}{3} \\ \frac{s^3 - 9\cdot(u - v)\cdot(u + v)^2}{3} & \frac{s^3}{3} & \frac{s^3 + 9\cdot(u - v)\cdot(u + v)^2}{3} \\ \frac{s^3 + 9\cdot v\cdot(u^2 - v^2)}{3} & \frac{s^3 + 9\cdot(u - v)\cdot(u^2 - v^2)}{3} & \frac{s^3 - 9\cdot u\cdot(u^2 - v^2)}{3} \end{bmatrix}$$

In the above, the rows and columns sum of $\mathbf{M}^2$ sum to s^2 as expected. It is the diagonal sum that fails. The brighter student will also have observed that the condition $\mathbf{M.X} = s\ \mathbf{X}$ shows immediately that a magic square has eigenvalue s. The above conditions on **M** have been used in [14] to prove the conjecture from coursework #2 question 3.

In theory, DERIVE can list all the 880 different normal magic squares of order 4 once all the appropriate values of a,b,c,d,e,f,g and i are substituted into the general form in question 5. This is essentially what has been done in [3], [15] and others.

Coursework #3 magic squares and linear algebra

In what follows, assume for convenience that the elements of the magic squares are reals although you should be aware that this is not a restrictive assumption.

1. Is the set of all nxn magic squares (with all possible values of the magic constant) a vector space?

2. Is the set of all nxn magic squares (with magic constant **not** equal to zero) a vector space?

3. Is the set of all nxn magic squares (with magic constant equal to zero) a vector space?

4. Using the representation of a 3x3 magic square given in coursework #2, write down a basis for the space in question 1. What is the dimension?

5. As in question 4 but this time for the space given in question 3.
6. Repeat questions 4 and 5 but this time using the representation of the 4x4 magic square given in coursework #2.

7. Conjecture a relation for the dimension of these spaces in question 1 and 3.

8. What relationships exist between the spaces in question 1 and in question 3 and the set of all nxn matrices?

9. For an nxn magic square with zero magic constant, consideration of the rows, columns and diagonals leads to $2n+2$ homogeneous equations in n^2 unknowns. Prove that when expressed in matrix form, the coefficient matrix has rank $2n+1$ and hence that the dimension of the space generated by these squares is $n^2 - 2n - 1$. Prove also that the dimension of the space generated by the set of nxn magic squares (with all possible values of the magic constant) is $n^2 - 2n$.

Discussion of coursework #3

The use of DERIVE for this coursework is now minimal! Nearly all the investigative work using a 'mathematical assistant' has been done in the previous two courseworks and we are now at the stage where conceptual understanding is required rather than asking a student to follow algorithmic procedures. Terms such as vector space, basis and dimension must be understood even for the 3x3 and 4x4 case. The steps in the proof required in question 9 are challenging, despite an attempt to lead the student towards a strategy by attempting the earlier questions.

The details behind questions 1,2 and 3 can be found in [6]. The answers are (yes, no, yes) respectively. The formulae for the dimensions of the spaces can be found in [7] together with the proof steps for question 9. Further magic square examples relating to aspects of linear algebra can be found in [8], and [9].
It could be argued that coursework #3 could be attempted on its own with only the general form of the 3x3 and 4x4 magic squares need to complete the exercise, and no DERIVE investigation needed at all. This may be true, especially for the student following the classical theorem – proof – corollary - application style of learning. Again, it should be emphasised that the intention at the outset is to offer the student a wider range of learning experiences and skills development and 'learning by discovery' should be a part of this. The student should be aware that consideration of simple cases should naturally lead to speculation about the general case and a desire to try and prove or disprove conjectures formally. Equally, if presented with a general result or theorem, students should be looking for special cases to consider. The CAS here is used (in the author's view) as an appropriate tool to assist in the learning development and adds to that development.

And Finally …..

The choice of magic squares as a subject domain was primarily because it is an easy problem to understand if not to solve. It is not immediately obvious whether the students' interest in magic squares will increase or decrease as the above courseworks are attempted. However, the author's interest in magic squares started with a challenge as described earlier and for those students still interested in a challenge, here is another one!

Magic squares regularly appear in mathematical puzzle books. In his book, Dudeney, [16], an author famous for such works, posed the following problem.

' can you construct a square of sixteen different numbers (positive integers) so that it shall be magic whether you turn the square upside down or not? You must not use a 3,4 or 5 as these figures will not reverse; but a 6 reverses to a nine, a 9 into a 6, a 7 into a 2 and a 2 into a 7. The 1,8 and 0 will read the same both ways. The magic constant must not be changed by reversal.'

An answer is given in the book. The challenge is ' How is it done? Is there a solution method other than trial and error? Is the solution unique? Can DERIVE or any other CAS help at all?'

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and many others……………………..!!!