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Steiner's Porism: An Activity Using the TI-92

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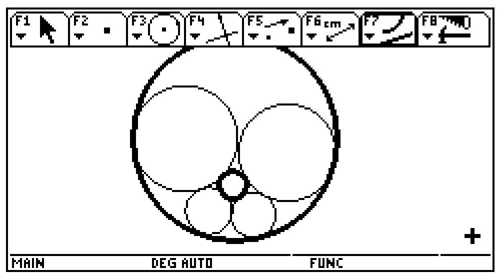
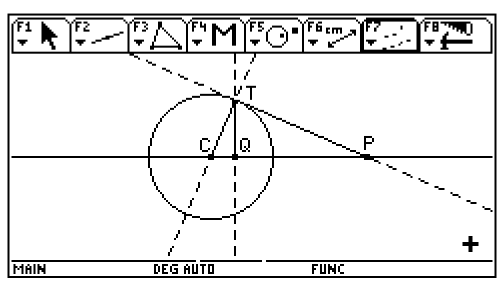
Suppose you are given two circles, one inside the other. Suppose you start drawing circles whose centers lie in the annular region between the two circles (outside the innermost circle and inside the outermost circle) which are tangent each to the next and tangent to the two original circles. Then, it may or may not happen that, when you complete the ring of circles, the last circle you draw will be tangent to the first. If it does, you will have drawn a ring of circles, all lying in the annular region between the two original circles, each tangent to two other such and all tangent to the original two circles. (See the figure, where a ring of four such circles has been drawn.)



Steiner's Porism is the statement that if you can draw such a connected ring of circles in the annular region between the two original circles, then it doesn't matter where you place the first circle. No matter where you start, the ring will be successfully completed (with the same number of annular circles.)

The usual proof of this theorem is via inversion. The figure is inverted to make the original two circles concentric, where the result is obvious. The objective of this paper is to show how the TI-92 calculator can be used to demonstrate the theorem and illustrate the main points in its proof. Of course, in a classroom, this would be done interactively and over the course of several days.

We start with a quick review of inversion in the plane. Suppose ***O*** is a circle with center C and radius r. The function which sends a point P, not equal to C, to a point Q, with the properties



1. $CP \cdot CQ = r^2$,
2. Q lies on the ray $\overrightarrow{CP}$;

is called the <u>inversion</u> in ***O***. Inversions have many properties in common with reflections (in lines) and, in fact, inversions are sometimes called reflections in circles. It is easy to construct the point Q when the circle and P are given. The above figure should point the way.

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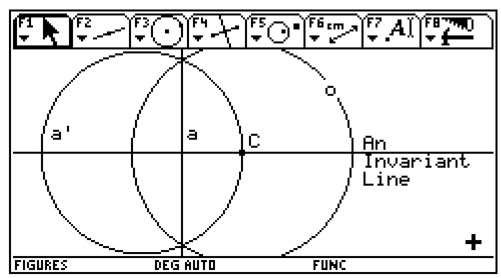
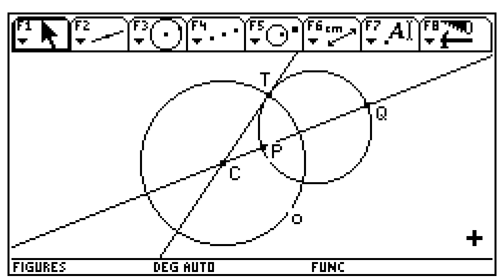
Inversion in ***O*** is an involution of the punctured plane. (The center *C* does not have an image under inversion.) It is conventional to extend the plane by adding an ideal point ("at infinity") which is then thought of as being interchanged with the center of any circle under inversion in that circle. The extended plane is called the inversive plane. Inversion in any circle then becomes an involution on the inversive plane. It is not hard to see that the fixed set of any inversion is the circle of inversion itself.

For our purposes, the most important property of inversions is that they preserve the set of circles and lines, called the set of inversive circles. That is, if ***A*** is a circle or a line, then its inversive image ***A***', in any circle, is also a circle or a line. However, whereas reflections send circles to circles and lines to lines, inversions may interchange the two types of inversive circles. Whether an inversive circle ***A*** inverts (in ***O***) to a line or a circle depends on whether *C* (the center of ***O***) lies on ***A***. If it does, then ***A*** inverts to a line; if not, it inverts to a circle. This is true whether ***A*** itself is a circle or a line.

A second principle, related to the first, is that an inversive circle ***A,*** not equal to ***O***, is invariant under an inversion in ***O*** if and only if it is orthogonal to ***O.*** Here, orthogonality of circles is measured along tangent lines. For example, two circles are orthogonal, if their tangent lines are perpendicular at their intersections. It follows from this principle that a line never inverts to another line under inversion. For if it contains the center of the inversion, it must be orthogonal to the circle of inversion and therefore be invariant under the inversion, whilst, if it does not, then it needs invert to a circle.



It also follows that a circle ***A*** is invariant under an inversion in ***O*** if and only if ***A*** contains at least one inversive pair of points *P* and *Q*. For suppose *P* and *Q* lie on ***A*** and neither equal *C*. Suppose *T* is the foot of the tangent line dropped from *C* to ***A.*** Then, by similar triangles, $CP \cdot CQ = CT^2$. (The quantity $CP \cdot CQ$ is called the power of *C* with respect to the circle *PQT*, which term being due to Steiner.) It follows that ***A*** and ***O*** are orthogonal if and only if *P* and *Q* are inversive images of each other in ***O***.



Moreover, it is known that inversion preserves the orthogonality of inversive circles. That is, if ***A*** and ***B*** are orthogonal, then so are their images ***A′*** and ***B′*** under inversion in ***O***. This is true regardless of whether ***A′*** and ***B′*** are circles or lines.

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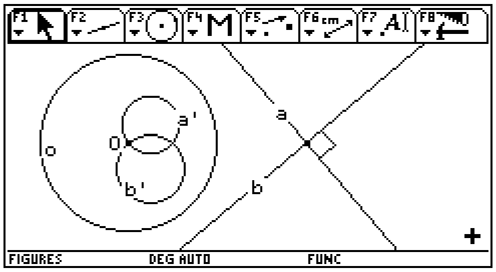
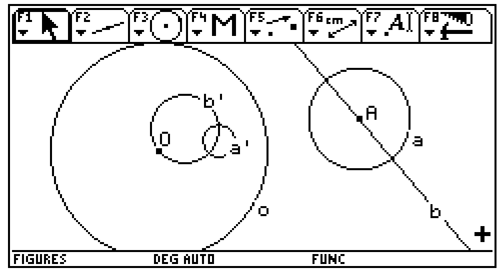
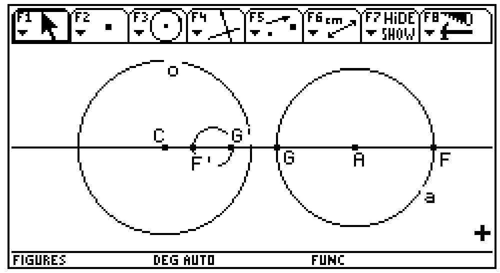
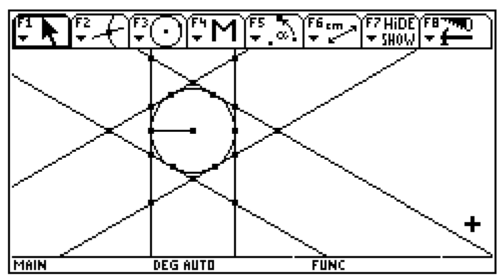




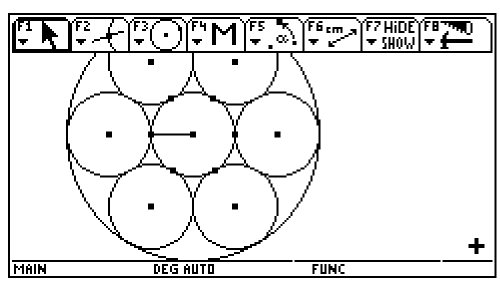
Using these ideas, it is not hard to construct to construct the inverse of circle ***A*** in ***O***, in the case that ***A*** does not contain *C*, the center of ***O***. First, assume also that *A*, the center of ***A***, is not equal to *C*. Suppose the line through these two centers meets ***A*** in the two points *F* and *G*. It follows from the above, that *F'* and *G'*, the inversive images of *F* and *G*, are the endpoints of a diameter of ***A'***. The TI-92 has a function to invert points in circles. We can use this feature and the above observations to write a macro to invert circles that are not concentric with the circle of inversion. (The concentric case is easier.)

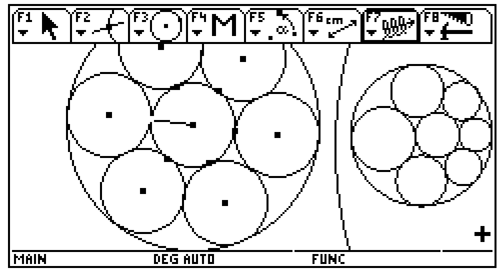


Let's use inversion and the TI-92 calculator to construct an illustration of Steiner's Porism. Start with a new geometry application. We want to start with an easy situation. So let's look at the problem of constructing six circles in a ring between two concentric circles. Start with a small circle and construct a point on it. The point will be used to animate the figure later. Next draw a segment connecting the point on the circle to its center. This segment will help you to locate the parameter point later. Now draw a tangent line to the circle at the parameter point and rotate it five times around the center of the circle, each by 60 degrees. Construct their intersection points and the points where they are tangent to the circle. Animate the figure, using the parameter point to check that the whole system will rotate around the center of the circle.



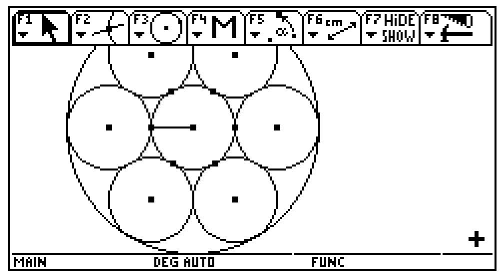
Now construct the six circles centered at the outer points of intersection and tangent to the original circle. Construct the circle concentric with the original circle and tangent to the six outer circles. Hide the construction elements and again check that the animation still works. Do not hide the marker segment.





Next, use the Show Page Tool to construct a circle centered at the extreme right edge of the page and is large enough to almost intersect the annular figure from the previous step. Invert all eight circles in this large circle and animate the resulting figure. The six inside circles should move so as to remain tangent both to the stationary inside and outside circles and to each other.

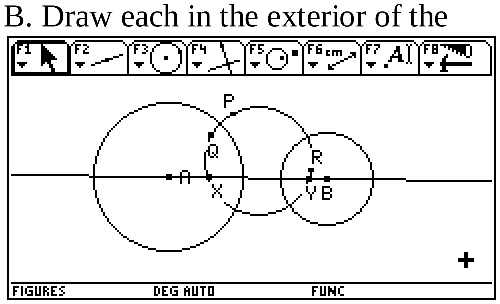
Now our problem is to do this in reverse. If two non-concentric circles, one inside the other, are given, when is it possible to find such an annular ring of circles? Clearly, the first step would be to invert the two circles into a concentric situation, if possible, where the problem is much easier and depends only on the two radii involved and not on the position of the center of the innermost circle.



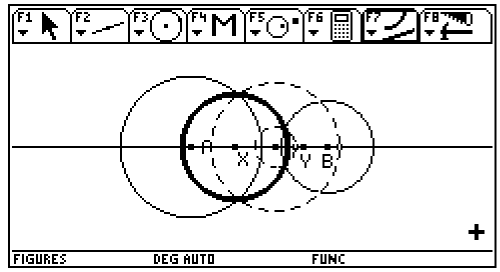
For example, in the above case, it is not hard to see that the ratio of the radii of the outermost circle to the innermost is 3:1. Whenever this is the case, it will be possible to fill the annular region between them with six circles, at least when the innermost and outermost circles are concentric. So, if we want to fill the annular region between two circles with six circles, we need to know if it always possible to invert the circles into a concentric situation. Also, can we guarantee that the ratio between the radii of the resulting pair of concentric circles is 3:1?

The answer to the first question is that given *any* pair of non-intersecting circles, it is always possible to find a circle ***O***, inversion in which results in a pair of concentric circles. The answer to the second question is more complex. We will use the TI-92 to explore these two questions.

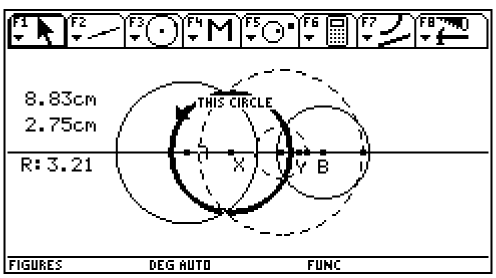
Start by opening a new geometry application and construct two non-intersecting, non-congruent circles ***A*** and ***B***, centered at A and B. Draw each in the exterior of the other. Construct the line of centers, $\overleftrightarrow{AB}$. This is an inversive circle, which is orthogonal to both ***A*** and ***B***. Hence it inverts, under any inversion, into another inversive circle orthogonal to both ***A***' and ***B***'. We can find another such by inverting a point *P* in both ***A*** and ***B***, obtaining *Q* and *R*, and constructing the circle through *P*, *Q* and *R*. Call the intersection points of these two inversive circles *X* and *Y*.

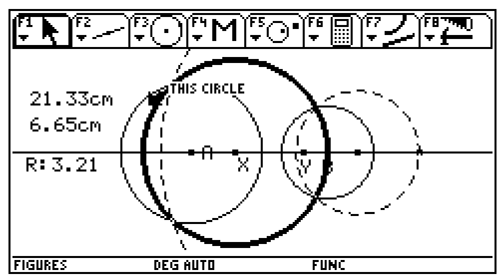


Now suppose our job accomplished. That is, suppose we have inverted ***A*** and ***B*** into concentric circles ***A***' and ***B***'. Then, since the circle *PQR* and the line $\overleftrightarrow{XY}$ must invert to a pair of inversive circles orthogonal to both ***A***' and ***B***', they actually must invert to a pair of lines, concurrent at the mutual center of ***A***' and ***B***'. This can only happen if *X* or *Y* is the center of inversion. Conversely, if *X* (respectively, *Y*) is taken to be the center of inversion, then both the circle *PQR* and the line $\overleftrightarrow{XY}$ will invert to lines meeting at the inversive image of *Y* (respectively, *X*.) Moreover, ***A***' and ***B***' will be a pair of inversive circles orthogonal to both of these lines - that is, a pair of circles concentric at the intersection of the lines. In the figure, the circle of inversion is in bold and ***A***' and ***B***' are dotted. Note that ***A***' and ***B***' are concentric at *Y'*.



Now measure the circumferences of ***A*** and ***B*** and compute their ratio, using the Calculate Tool. Do the same with ***A***' and ***B***'. Notice that they are (probably) not the same. Now change the size of the circle of inversion. You should notice that, although both ***A***' and ***B***' change sizes, the ratio of their radii does not.

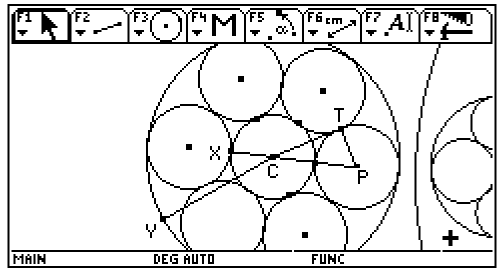




The absolute value of the natural logarithm of this ratio is called the inversive distance between ***A*** and ***B.*** (The logarithm is used to change the multiplicatively varying ratio to an additively changing coordinate-like quantity. The absolute value is used, as usual, to ensure that the quantity is positive, like a distance.) It turns out that if we were to use the point Y as the center of inversion, we would obtain the reciprocal of this ratio. So, the inversive distance is an invariant of the pair of non-intersecting circles.

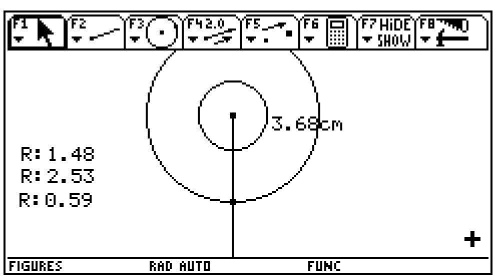
This situation is similar to that which arises when defining the slope of a line. The slope only apparently depends on the segment of the line chosen to measure it. Similarly, the inversive distance only apparently depends on the circle of inversion used to measure it. Of course, the center of the circle of inversion must be one of the two points *X* and *Y*, the only possible centers that yield concentric circles after the inversion. The points *X* and *Y* are called the limiting points of ***A*** and ***B,*** a term taken from the topic of coaxal pencils of circles.

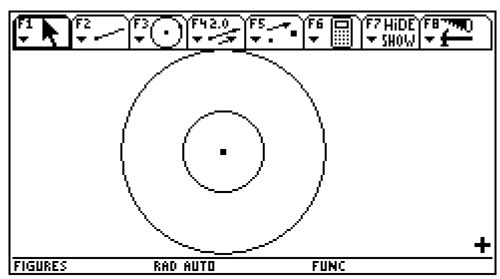
Let's use this idea to construct an annular ring of seven interior circles. First, we need to compute what the required ratio must be. Suppose n interior circles can be drawn between two concentric circles. (In the figure, n is six.) . Suppose, in the figure, $CY = s$ and $CX = r$. Then $CP = \frac{r+s}{2}$ and $PT = \frac{s-r}{2}$.



Hence, $\sin(\frac{\pi}{n}) = \frac{s-r}{s+r}$. Solving for the ratio, we get $\frac{s}{r} = \frac{1+\sin(\frac{\pi}{n})}{1-\sin(\frac{\pi}{n})}$.

In order to construct an annular ring with seven interior circles, we first need to construct a pair of concentric circles whose radii are in the appropriate ratio. Open a new geometry application and construct the circle that will serve as the innermost of your two concentric circles. Construct its radius and calculate the number $\frac{1+\sin(\frac{\pi}{7})}{1-\sin(\frac{\pi}{7})}$. (Make sure that your calculator is in radian mode first.) Multiply these numbers together and use the Measurement Transfer Tool to construct the outermost circle. Now hide everything except the two circles and their center.





Place another point on the innermost circle. This point will be used to animate the figure later. Now draw another ray from the center through the point and construct the point where the ray meets the outer circle. Use the points where the ray meets the two circles to construct the first of the interior circles. Draw a diameter for this first circle. (It will be useful to distinguish this circle from the others during the animation later.) Finally, rotate this circle around the annulus to produce the required figure.

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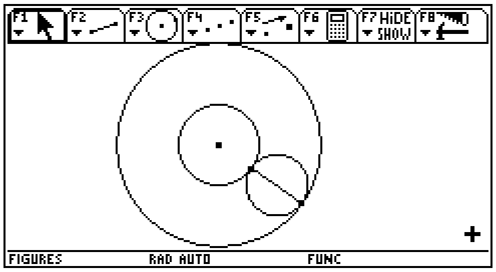

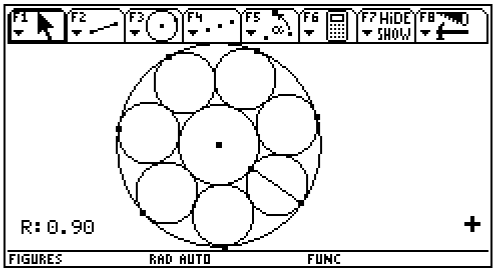

Now use the Show Page Tool to move to the far end of the page and construct a large circle that extends close to the previous drawing. Then move back and invert the entire construction, one circle at a time, in the large circle. The entire figure can then be animated, using the innermost of the two diameter points.