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Computerized Mathematics

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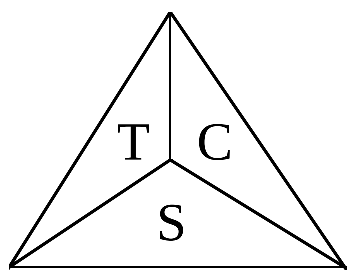
My experience of using TI-92 (as well as "DERIVE") on calculus lessons in high school convinced me that a "gradual revolution" has started in the teaching process. Its moving forces are computer technology and software development. As a result, the mathematical education will be different, it will become more human; many purely technical and boring problems will disappear. However, I do not think that we will come to teacher's and students' paradise. Some current problems will go away, but new ones will appear. It's clear that we will teach to different things and in a different way. The question is what exactly has to be changed. The answer to this question will have to be found already in the next century.

Few people share my position. The two important problems usually raised by the opponents are:

1. What is the minimum set of the ideas that would give students the modern picture of mathematics?
2. What is the minimum level of the mathematical technique that would allow students to solve simple problems independently, without TI-92?

It is clear that this list of problems is not complete. But the goal of this conference, in particular, is in formulating the questions that are being raised by the use of TI-92 in high school mathematical education.

Let me now turn to my experience. The analysis of this experience has lead me to the following picture:



This picture reflects the list of participants in the teaching process: S - student, T - teacher, C - computer (TI-92). I was able to find certain connections between the participants, and I will try to illustrate them with some examples.

1.Teacher - Computer (TI-92)

Let us ask ourselves: what is changing in the work of the teacher who uses TI-92 in the teaching process?

1.1. *The contents of the problems changes*

Indeed, can we ask students to solve an equation, take a derivative or an integral, plot a graph etc. if TI-92 can do this in a few seconds? (When I say TI-92 does something I mean of course some software which is installed inside it) We can, but not for a long time. Hence, ideally one has to pick such problems, that make students helpless without TI-92 and TI-92 "helpless" without the students. I like to talk about the combination of "human and computer intellects" - I can translate this nice phrase and give some examples.

Example 1. It is getting possible to carry out computational experiments of the following type:

a) the number 444...4888...89 (there are *n* digits 4 and *n*-1 digits 8 here) is a perfect square. First, it's being offered to check at what *n* this is true and then one has to prove it.

b) to prove the divergence of the harmonic series; one has to show first that some its partial sum can not be greater than 1000, and then prove its divergence.

Example 2. We are interested in the effect of some parameter on the shape of the curve. I offered my students to investigate the following curves: $y = x^3-ax$, $y = ax + x^{-2}$, $y = x + ax^{-2}$, $y = \cos x \cos ax$, $r = a^{\varphi}$ (in the polar coordinates, $a > 0$).

Example 3. One has to figure out how the shape of some function changes when one adds something to it.

a) Replacing the function or its variable by their absolute values. After we finished considering the plot of f(x), we study the plots of these functions: $|f(x)|$, $f(|x|)$, $|f(|x|)|$

b) Multiplication the function sin (1/x) by x^n ($n \in N$), i.e. switching to the function $y = x^n \sin(1/x)$. It seems even more interesting to multiply sin (1/x) by x^{α} ($\alpha \in R$) and try to define it at $x = 0$ in such a way that it becomes continuous and smooth.

Note: I used the following functions as additional terms: e^x, e^{-x}, sin x ...

1.2 *The accents in the teaching process change*.

What was not important before is getting important now.

Example 1. One has to solve the equation $x^2 = 10x$. Suppose the student plots the graphs of both the function on the right and the function on the left. Within the limits of the display (e.g., from -5 to 5) he will see only one crossing point, corresponding to $x = 0$. To find another crossing point, corresponding to x = 10, he has to know that it exists. Hence, it is important to know from the beginning how many solutions the given equation has and therefore, when studying the properties of the functions, a special attention has to be paid to investigation of their monotonicity as well as their behavior at the infinity.

Example 2. - very strong. It stops being necessary to solve the inequalities of the type $f(x) > 0$, because now one can plot the graph of the function f(x) on the display, find all zeros using this graph, and the rest is obvious.

1.3. *The contents of the theoretical course changes*

It would be strange if students thought about TI-92 as if it were a magician. Suppose, for instance, TI-92 gave us all the solutions of the fifth order equation. I do not know how it did that, I can only guess, but I try to explain my students how it could do that. If one develops this idea, then it is easy to imagine the appearance of the course of "computerized mathematics" that will be oriented on explaining the students how TI-92 "works". We will have to tell students about algorithms, iteration, approximation, uncertainties...

I have to point out here one very important thing. TI-92 can save a lot of time in the process of studying the current canonical course of mathematics in Russian high schools. How should one spend the remaining time? If this time is spent on mathematics, than it would make sense to study many other important things that TI-92 can not do or things that haven't found yet the appropriate place in the high school curriculum. But maybe we should reduce the students' school load and let the child spend this time the way he wants. I believe these arguments are already sufficient to justify all the huge financial expenses needed for "computerization" of the high school education.

1.4. *The teaching methods change*

New problems arise: what can we let TI-92 do, what do we have to do ourselves, when and how should we "connect" TI-92 to the student?

Example 1. It is one thing when I show the formula for solving the quadratic equation, and - it is clear - the student has to solve a certain number of these equations manually. But it is a different thing when at the end of the high school education the student can afford solving these equations by simply pressing the buttons on the keyboard of TI-92. The question is when should we turn the switch on.

I would like to note that the process of solving these small practical teaching problems goes continuously, and therefore - you think, do experiments, make mistakes and get happy when you succeed, and at the end you gain more professional experience.

Example 2. Suppose, TI-92 “gives” us a picture.

a) If this is a graph of some function, then all the extrema are clearly seen. The student can use this information (somewhere). But what if the extrema are not seen - this happens when the scale is too small. Shell one go to a bigger scale? But it may take forever. What if there are no extrema in principal ?At what point can the student say that there are no extrema?

b) Suppose, we deal with the curve represented parametrically, e.g. x = cos t, y = sin t. A closed loop appears on the display, and the student decides that it looks so similar to the circle that it is indeed a circle. Can he stop here?

Example 3. Suppose, we have to solve the equation sin 4x + cos 10x = 0. TI-92 gives us the answer only as a decimal number. Is that enough? I believe we do not need the answers that can not be represented by a number, e.g. arcsin (1/3). These expressions are important only in theoretical problems.

To analyze the changes in my own professional activity associated with TI-92 and to formulate my point of view on these changes I had to figure out what the computer is on math lessons. This is not a simple question and I do not think I have found the final answer. So far it is like this: the computer in the school mathematical education is like some instrument for a physicist. We can live without it, but solving problems will take much longer or we will not be able to solve them at all. But then, I think, one has to ask computer the right questions and critically analyze the answers. We have to teach students to do this

.

Student - Computer (TI-92)

What changes for a student who has an access to TI-92? He/she is forming a new culture - the culture of communication with the mathematical instrument, which is related to the general mathematical culture.

The growth of student’s mathematical culture depends on the level of understanding of several things: what kind of questions can we ask TI-92, what is the right way to ask questions; how should we interpret the obtained result, what kind of mathematics can explain the result given by TI-92? I saw the lack of this understanding in one very good school. Using graphical calculators the students could easily plot the graphs of the second order curves. And when I offered them to plot the graph of the equation $x^2 + y^2 = a$, they started to press buttons.

2.1. What can we ask TI-92 to do?

Unfortunately, not everything.

Example 1. It is not worth asking TI-92 to solve problems on identity transformations because it is unclear how to formalize this task - “simplify the expression”.

Example 2. We were not able to solve certain (typical for our course of mathematics) trigonometric equations, e.g.

a) $\sin^3 x + \cos^3 x = 1$.

b) $|\cos x - \sin x| + \sin x + \cos x = 0$.

c) $|\mathrm{tg}\, x + \mathrm{ctg}\, x| = 2 - \mathrm{tg}^2 x$.

d) $2\sin^2 x + \sin x \cos x - \cos^2 x = 1$.

It is especially strange that we could not solve the last equation because it is easy to reduce it to the quadratic equation.
Of course, the number of such examples can be increased.

Example 3. TI -92 hasn’t given us any result when calculating the limits of some sequences, e.g. $\lim \frac{n! + (n+1)!}{(n+2)! + (n+3)!}$. It is also took a long time for TI-92 to calculate the following limit: $\lim \frac{\ln n}{n^{26}+1}$. But if one removes 1 in the denominator, TI-92 gives the result very quickly. For students, the removal of 1 is very natural.

2.2. What is the right way to ask questions?

Example 1. Suppose, one has to solve the equation $\sin 4x + \cos 10x = 0$. Using TI-92 one can solve this equation graphically on different intervals. The choice of the interval is related to the level of understanding of this problem.

Example 2. Suppose, we are interested in the behavior of some function at the infinity. How can we get an answer to this question from TI-92 - it can deal only with finite intervals?

Example 3. We have to solve the inequality $(x-1)^3(x^2+x+1)^5 < 0$. What do we have to ask TI-92 to do?

2.3. How shall we interpret the results?

Example 1. Let us ask TI-92 to plot the graph y = 1/x -30 in the standard window (from -1 to 1). What will we see? A straight line...

Example 2. The analytical solution of the cubic equation resulted in two answers. What shall we do next?

2.4. The importance of the result foreseeing

Working with TI-92 students start to understand its features and, as time goes by, get adjusted to these features. Taking into account the features of TI-92, the students have to learn to foresee the results - as in any other situation when one has to work with some research instrument.

1) There is no graph yet on the display but the student has to “see” it already. He has to “see” it and be able to draw it approximately, without any details because otherwise one does not need TI-92. It is worth spending a lot of time on teaching students to foresee the results. One can ask students to do the following. They can be offered to draw the graphs of some functions approximately and then check the drawings using TI-92.

2) The equation has just been written down but the student has to think already about the number of possible solutions.

2.5. The importance of mathematical techniques.

Can we release students from learning mathematical techniques? In principle - no, but we can certainly reduce the quantities in which we require students in Russian schools to do this. Students have to know how to use certain mathematical methods in simplest situations or, maybe just have an idea about these methods. But the ability to easily solve transcendental equations or to plot graphs using derivatives - this is a “waist of taxpayers’ money”. And so much children’s time is lost...

2.6. Computer ideology

It is important to realize that computer software is being constantly improved. That is why it is necessary to learn the computer ideology on the whole, i.e. to understand what computer does in principle, regardless of the software being used. I will give only one example. Computer plots graphs point by point. That is why the search of the extrema of some function requires the human analysis of the picture which is being displayed.

Student - Teacher

What kind of opportunities does computer (not only TI-92) provide for a teacher in the everyday work? What problems arise in this case?

3.1 Computer allows the teacher to organize the research activities of students more efficiently. In particular, the student starts using the computer in the dialog with himself. For instance, the student can ask: "What if ..." and then check the answer using computer.

Example 1. I was able (using "DERIVE") to familiarize students with many curves of the third and the forth order. The appearance of the graph in the beginning of the research process provided the guidance for the further student's work. That is the thing. In the "normal" situation we first try to study the properties of some function using its equation and then we plot its graph. But now everything is exactly the opposite: looking at the graph student starts seeing things that he later has to prove. What happens sometimes is that using computer the student sees things that he never was able to find analytically before. Suppose, for example, one studies the properties of the Cartesian Folium described by the equation $x^3 + y^3 = 3xy$. It is very clear from the graph that: 1) in the first quarter there is a point which is separated by the largest distance from the origin; 2) the curve is symmetric relative to the line $y = x$; 3) there is an oblique asymptote.

In the process of studying the properties of various curves some students even came up with very interesting topics for mathematical investigation or programming.

It is clear that TI-92 provides similar possibilities.

Example 2. Suppose we have n linear functions f_1, f_2, ..., f_n. Let us now consider the function $|f_1| +|f_2|+...|f_n|$. One has to find the dependence of the number of the break points of this function on n.

3.2. This is another important question - the use of TI-92 on tests and exams. Shall we tell students not to use it? But why did we teach them to do this then? Shell we allow students to use TI-92? But what will the boss say? The realization is not simple either. Once I carried out the following experiment. I "gave" TI-92 a 4-hour exam for the 9th grade of the mathematical school. It solved 5 problems out of 6 in 10 minutes. The sixth problem was a text problem, and it was simply impossible to offer it. It is clear that if we allow students to use TI-92 on exams we have to change the content of the exams.

I gave the first exam of this sort (student+computer) to the students in the 10th grade. The students were allowed to use the computer and its answer right after receiving the problem sheets. After that they had to obtain this result on the paper. Another version - the student can solve the problem by himself and then check the result using computer. It is very surprising that some students did not use the computer at all. Some of them were afraid of loosing time, some were not sure that the computer could give the right answer.

Next time, when I gave an exam to the students in the 9th grade, I tried to pick such problems that would make the use of the computer natural or even necessary. The exam was done in 4 hours. I also asked students to point out when exactly they used the computer (I can show this to anyone who is interested).

3.3. TI-92 changes the dialog with the student. The student does not want to know anymore if he solved an equation correctly; his questions become more interesting and more mathematical.

Summary

In summary, I would like to emphasize that the problem of creating the course of "computerized mathematics" - the course in which students study mathematics using computers (TI-92) - becomes very important. I believe that the first attempts of creating this course are being made already. But the amount of work that has to be done is enormous and will probably involve people all over the world.