Proceedings of the Third International DERIVE/TI-92 Conference

Learning Mathematics through Investigations with the TI-92

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Abstract

Learning mathematics is an active process and good teachers should find ways of involving their students in the teaching and learning process. Investigations and problem solving in mathematics are two ways of creating an active learning environment.

This workshop will provide TI-92 activities for teachers to use in their classrooms to introduce new topics in mathematics to their students. Participants in the workshop will be expected to work through the activities during the session and discuss their feelings about using such tasks with their students.

Outline: Discussion of good ingredients of mathematics teaching and learning.
Investigational activities for delegates to work through.
Discussion of appropriateness of using such tasks.

What are the Good Ingredients of Teaching and Learning Mathematics?

There is much discussion in many countries about how teachers should teach mathematics and how students should learn mathematics. There is a movement in the UK away from students working in groups towards whole class teaching. It is generally accepted by many educators that good learning of mathematics should contain the following six ingredients:

A: exposition by the teacher
B: discussion between ‘student and teacher’ and between ‘student and student’
C: practical work
D: consolidation and practice of basic skills and routines
E: investigations
F: problem solving

Perhaps I remember lots of A and D from my schooldays, probably a little of F but not much of B, C and E! In many of our schools in the UK ‘whole class teaching’ in which a teacher is involved in exposition and students consolidate their skills with ‘hundreds of example’ was replaced by students working from a text book in small groups supposedly at their own ‘level’. Now our teachers are being encouraged to return to the former by government edict.

In this workshop I want to propose combining the skills of investigations together with hand-held technology to introduce new mathematics to students. There have been many studies on the use of graphic calculators and computer algebra systems in mathematics teaching and learning (see for Mayes, 1994, *The International DERIVE Journal*, **1 No 2**, pp21-38 and Jaworski (ed.), *Technology in Mathematics Teaching*, published by Chartwell-Bratt). These studies (and most others) use worksheets or laboratory based experiments as additional tasks to traditional teaching. There are many laboratory based workshop resources to be used as an add-on to a mathematics course. But how do we fully *integrate* technology into the learning process? I would argue that a structured investigational approach to introducing new topics is one way.

The tasks in this workshop are taken from the textbook: *Learning Mathematics through the TI-92* by Berry, Graham and Watkins and published by Chartwell-Bratt.
The following case studies give a flavour to the workshop activities.

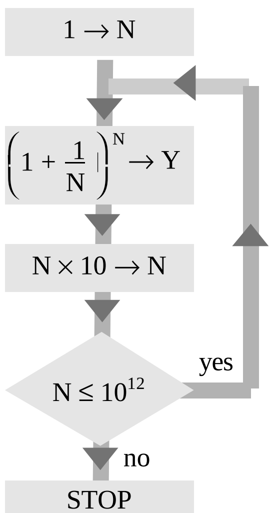
INTRODUCING EXPONENTIALS

This case study shows the use of an investigation and problem solving task to introduce the exponential function and its application. Encouraging students to explore mathematics in this way throughout the course will then lend itself to a discovery or investigational method of introducing more advanced topics such as calculus.

We begin with a calculator activity to explore the function $\mathrm{f}(n) = (1+\frac{1}{n})^n$.

Use this flowchart to create a program that allows you to complete the table.

Describe what happens to the sequence of values in the right hand column as n gets larger.



N	$\left(1+\frac{1}{N}\right)^N$
1	2
10	2.593 742 46
100	

From this activity students discover the important result, $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n$ = 2.7182818284455... which we then denote by e and is related to the e^x function key on the calculator. This calculator task is followed by an exploration on interest rates and annual percentage rates (APR's).

Imagine you have £100 to invest for one year at a nominal annual rate of interest of 8%.

How much would your investment be worth after one year if interest is compounded

(a) annually, (b) quarterly, (c) monthly, (d) weekly, (e) daily?

What are the corresponding annual percentage rates (APR's)?

The results of this exploration lead to the formula

$$A = P(1+\frac{r}{n})^n$$

where A is the value of the investment after 1 year, 100r is the percentage interest rate and n is the number of compoundings per year. A natural extension is to ask what happens to the formula as n→∞.

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A second calculator activity investigates this.

Amend the flowchart to investigate the limit of the sequence $\left(1+\dfrac{x}{n}\right)^n$ as $n\to\infty$ for various values of x.

The TI-92 will evaluate the limit for us. Figure 1 shows a screen dump of the activity providing the link between the limit and e^x.

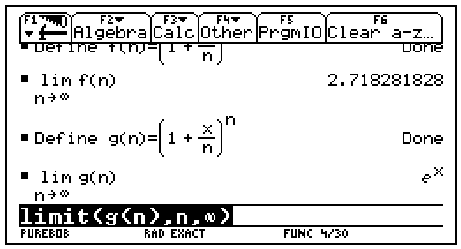

Figure 1 The limit definition of the exponential function.

We can now introduce the exponential function e^x and explore some of its properties.

INTRODUCING DIFFERENTIATION

It is interesting to look at many text book introductions to differential calculus. The approach is usually through a description that the derivative is the slope of the tangent to a graph. Then the algorithmic rules for finding derivatives are often given and the student is encouraged to learn these rules by doing many examples. In some textbooks students are encouraged to draw a few graphs of quadratic and cubic polynomials, draw tangents at various points and find the slope of the tangent to ‘prove the rule’. Students have little experience of exploring the algebraic (i.e. limit) definition of the derivative perhaps because their algebraic skills are too weak. For example, give some 17 year old students the following to evaluate by hand and watch the errors!

$$\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}.$$

Now technology can be used as a powerful tool to do all these things. Since using DERIVE in this way my first year calculus students can at last go forward with both a geometric view of differentiation as the slope of a tangent and an algebraic view as a rate of change. After all it is the latter concept that is more often used than the former.

The screen dumps in figure 2 show a simple program for drawing the graph of a function, the tangent at various points and the calculation of the slope of each tangent.

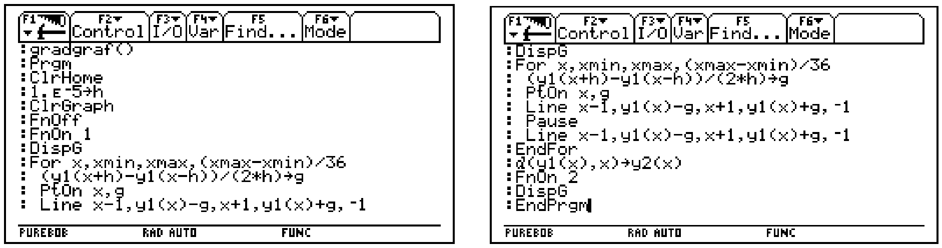

Figure 2 Introducing differentiation

For each point the slope of the tangent is plotted as a point in the *x*-*y* plane as (*x*-value, value of slope). The final step is to draw a line or curve through these points. This is the graph of the derived function. From this graph students can investigate the rules for standard functions. Figure 3 shows the complete activity for the functions x^2 and sin(*x*).

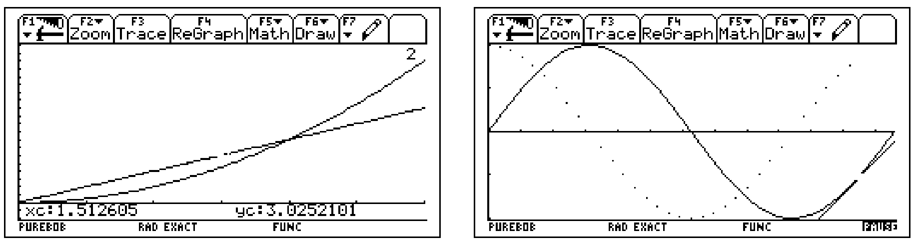

Figure 3 Exploring the derivative of x^2 and sin(x) graphically

From the graphs students deduce that

$$\frac{d}{dx}x^2 = 2x \text{ and } \frac{d}{dx}\sin(x) = \cos(x)$$

An activity of this type introduces or reinforces the graphical concept of a derivative. With the TI-92 we can go one step further and introduce or reinforce the algorithms for differentiating the basic functions using the limit definition. Figure 4 shows screen dumps for the evaluation of

$$\lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$$

for f(*x*) = x^3 and f(*x*) = sin(*x*).

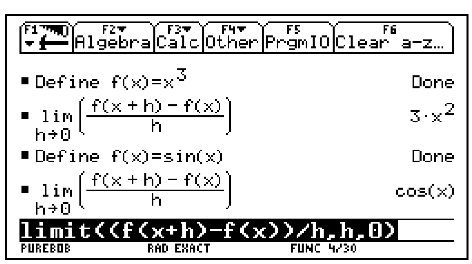

Figure 4 Exploring the derivative algebraically

With technology we can explore the derivative of many more functions using the limit definition. Students could also investigate the rules of differentiation in this way.

INTRODUCING INTEGRATION

So often integration is introduced only as the reverse process of a differentiation. This of course leads to the algorithmic rules for integration but misses the important concept of integration as a limit of a summation. It is easy to understand why this approach as occurred. Without technology as an aid the evaluation of an infinite summation is not easy. But now we can approach the topic of integration from a summation and show that the rules are in fact the reverse process.

One of the more useful applications of integration is the limit of a sum and this approach to the concept often starts with finding the area under a graph using thin rectangular strips. Figure 5 shows a typical diagram of this activity found in textbooks.

A familiar line of argument leads to the expression

$$\text{AREA} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f\,(i * h) * h\right) \text{ where } h = \frac{x}{n}$$

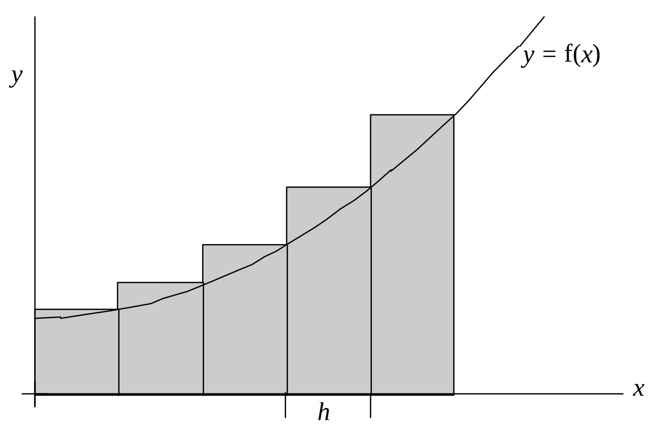

Figure 5 Approximating the area under a graph

Without technology the link between this algebraic formula and the algorithmic rules between this algebraic formula and the algorithmic rules that 'integration is the reverse process of differentiation' is difficult to demonstrate. Now, with the TI-92 we can provide the link by investigating this summation for different functions $f(x)$. The screen dumps in Figure 6 show the results of two explorations

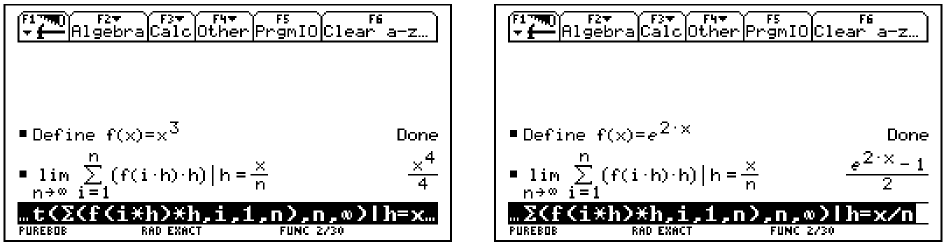

Figure 6 Investigating integration algebraically

Could this lead to a resequencing of calculus with integration coming before differentiation? One advantage would be to unlink differentiation and integration in the student's mind so that they see them as distinct, separate concepts.

2.5 PROBLEM SOLVING

An important part of the school mathematics curriculum for all children is to develop the ability to apply and use mathematics. This is often called mathematical modelling.

Experience suggests that developing and assessing these skills, especially formulating a model, is not easy. It is important for students to see modelling at work and then formulate models for themselves. It is with technology that we can begin to concentrate on the formulation and validation parts of the problem solving process. But there is added value in using technology to explore algebraic models. Consider the following problem on traffic flow.

Traffic queues often form on motorways when three lanes of traffic are forced to reduce to two lanes or one lane because of road repairs. This is a major frustration to motorists and often extends journey times by hours.
Consider the problem of reducing two lanes of traffic to one. What speed of traffic achieves a maximum flow through the motorway section?

A full solution will not be given but the steps in the solution are outlined below:

The objective is to maximise the number of cars per hour passing through the motorway section; this is the flow rate which we designate as *F*. Suppose that vehicles enter the section *t* seconds apart. Then the number of cars entering per second is $1/t$ and the flow rate is $F = \frac{3600}{t}$.

Suppose that the distances between the fronts of two vehicles is *d* metres. At a speed of *v* ms^{-1} the quantities *t*, *v* and *d* are connected by the equation $d = vt$.
Hence the expression for the flow rate becomes

$$F = \frac{3600v}{d}$$

The distance *d* is made up of two parts, the length *L* of the vehicle and the separation between the vehicles *s*, so that $d = s + L$ and the expression for the flow rate becomes

$$F = \frac{3600v}{s+L}.$$

Modelling the separation distance

The British Highway Code recommends the following distances between vehicles for different speeds.

Shortest stopping distances - in metres and feet								
Speed (mile/hr)	Thinking distance (m)	Thinking distance (ft)	Braking distance (m)	Braking distance (ft)	Overall stopping distance (m)	Overall stopping distance (ft)	On a dry road, a good car with good brakes and tyres and an alert driver will stop in the distances shown.
20	6	20	6	20	12	40	Remember these are shortest stopping distances. Stopping distances increase greatly with wet and slippery roads, poor brakes and tyres, and tired drivers.
30	9	30	14	45	23	75	
40	12	40	24	80	36	120	
50	15	50	38	125	53	175	
60	18	60	55	180	73	240	
70	21	70	75	245	96	315	

First consider a 'numerical approach' to solving this problem. This is unfortunately the modern way of our upper secondary school mathematics curriculum - in problem solving introduce numbers at an early stage instead of developing an algebraic approach.

Using the data from the British Highway Code, it is clear from the table that the thinking distance in feet is the same as the speed in miles per hour (u). With the aid of algebra and/or graphs it can be seen that the braking distance is $u^2/20$. Hence, $s = u + \frac{u^2}{20}$ and if we assume that all the vehicles are average sized family cars then L = 13 ft. The mathematical problem is to find a maximum of the function

$$F = \frac{3600v}{13 + v + \frac{v^2}{20}}$$

Figure 7 shows a TI-92 screen for this activity.

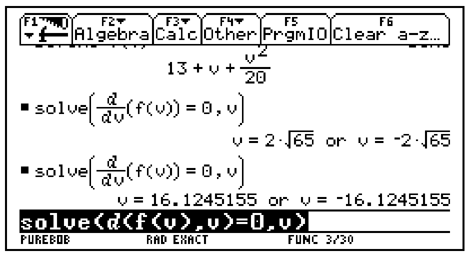

Figure 7 Finding a maximum of $F(v)$

The conclusion is that the vehicles should travel at about 17 mph (or 27 km/hr on European roads!).

The next step in the modelling process is to criticize and improve the model. It is unlikely that vehicles will travel at the ‘recommended safe stopping distances’. So it is the formula for stopping distances that will need to be revised.

It is common for students to argue that the ‘thinking distance’ can be reduced or even ignored. However as we shall see this will lose some essential information.
Consider, instead, an algebraic model in which we take the separation distance as the expression

$$s = av + bv^2$$

Now the mathematical problem is to find a maximum of the function

$$F = \frac{3600v}{L + av + bv^2}$$

Figure 8 shows a TI-92 screen for this activity.

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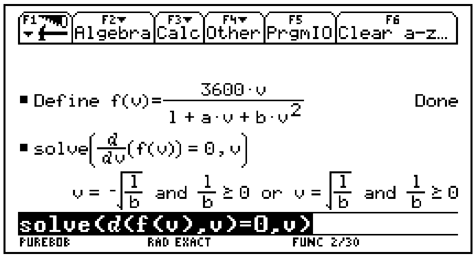

Figure 8 Finding a maximum of $F(v)$

The algebraic approach leads to the solution $v = \sqrt{\dfrac{L}{b}}$ (surprising for many students that the value of v for maximum flow rate is independent of the thinking distance. This example demonstrates the important skill for the mathematician of being able to investigate algebraically situations involving parameters. If one of the roles of mathematics is to explore problems, then, like statisticians who use technology to explore realistic data sets, mathematicians must start to realise that technology should be used at all levels of learning to explore more realistic problems.