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FUNCTIONS OF MATRICES WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract

The goals of this workshop are to illustrate how Derive may be used to compute functions of matrices and to illustrate some of the properties and applications of matrix functions. One particular interesting application of matrix functions is the computation of the solution of a system of differential equations by finding the matrix exponential function.

Functions of Matrices

A simple way to envision a matrix function is by first considering a polynomial with coefficients taken from the real numbers:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

If we replace the scalar, x, by the square matrix, $\mathbf{A}$, of order n, we obtain the polynomial matrix function

$$p(x) = a_n \mathbf{A}^n + a_{n-1} \mathbf{A}^{n-1} + \cdots + a_1 \mathbf{A} + a_0 \mathbf{I}$$

where the identity matrix **I** is required in the last term in order to preserve closure of addition since each of the other terms of the sum is a square matrix of order n.

1. *The polynomial function $g(x) = 2x^3 + 3x^2 - 4x - 3$ has a corresponding matrix function. Our approach in finding G(**M**) for the matrix **M** in statement #3 below is simply to substitute **M** into the polynomial expression in statement #2 using DERIVE.*

    a. Use the **T**ransfer-**L**oad-**D**erive command sequence to enter the file *exl.mth* from the **a:** drive.

    $$\#2:\quad G(x) := 2 \cdot x^3 + 3 \cdot x^2 - 4 \cdot x - 3 \cdot x^0$$

    $$\#3:\quad m := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

    b. **M**anage-**S**ubstitute the matrix in statement #3 into G(x) for x and **S**implify the result.

We need not restrict matrix functions to polynomials. In fact, we can find the matrix representation of any function with a corresponding analytic scalar form when A is a matrix of order *n* with eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m | m \le n\}$, such that $\max_{i=1,2,\cdots,m}(|\lambda_i|) < R$ by replacing

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad |x| < R \qquad \text{with} \qquad f(\mathbf{A}) = \sum_{k=0}^{\infty} a_k \mathbf{A}^k$$

An Algorithm for Computing Functions of Matrices

Our goal is to determine an algorithm which will allow us to express any matrix function $f(\mathbf{A})$ corresponding to an analytic scalar function $f(x)$ as a finite sum which is polynomial in **A**. If we begin by assuming $f(\mathbf{A})$ has the power series expansion above and the characteristic polynomial of **A** is $p(\lambda) = \lambda^n - p_{n-1}\lambda^{n-1} - \cdots - p_1\lambda - p_0$, then by the Cayley-Hamilton Theorem, **A** is a zero of $p(\lambda)$; hence, $\mathbf{A}^n = p_{n-1}\mathbf{A}^{n-1} + p_{n-2}\mathbf{A}^{n-2} + \cdots + p_1\mathbf{A} + p_0\mathbf{I}$, and we can write all powers of **A** as a linear combination of power of **A** which are less than *n*. (For example, $A^{n+1} = AA^n$, but A^n can be written in terms of powers less than *n*, and after **A** is distributed through that sum the highest power will be A^n, which can be written in terms of powers less than *n*. After similar terms are combined, the highest power will be less than *n*.) Thus, for any matrix **A**, we can write

$$f(\mathbf{A}) = r_{n-1}\mathbf{A}^{n-1} + r_{n-2}\mathbf{A}^{n-2} + \cdots + r_0\mathbf{I}$$

where the coefficients $r_0, r_1, \ldots, r_{n-1}$ are to be determined. If λ is any eigenvalue of **A** and **x** is its corresponding eigenvector, then $f(\lambda)$ is an eigenvalue of $f(\mathbf{A})$; thus,

$$f(\mathbf{A})\mathbf{x} = (r_{n-1}\mathbf{A}^{n-1} + r_{n-2}\mathbf{A}^{n-2} + \cdots + r_0\mathbf{I})\mathbf{x} = (r_{n-1}\lambda^{n-1} + r_{n-2}\lambda^{n-2} + \cdots + r_0)\mathbf{x} = f(\lambda)\mathbf{x}$$

hence,

$$f(\lambda) = r_{n-1}\lambda^{n-1} + r_{n-2}\lambda^{n-2} + \cdots + r_0$$

But λ is an arbitrary eigenvalue; therefore, we can substitute in turn the *n* eigenvalues of **A** and obtain *n* equations for the unknowns $r_0, r_1, \ldots, r_{n-1}$, which in turn determine $f(\lambda)$.

If **A** has less than *n* distinct eigenvalues, the above approach will not yield *n* independent equations. But given an analytic function $f(x)$ and a polynomial $p(x)$ of degree *n*, there exists an analytic function $q(x)$ and a polynomial $r(x)$ of degree $n-1$ or less such that $f(x) = p(x)q(x) + r(x)$. Differentiating this expression we note for the multiple root α_m that both $p(\alpha_m)=0$ and $p'(\alpha_m)=0$; thus, $f'(\alpha_m) = r'(\alpha_m)$. This procedure can be duplicated until we exhaust the multiplicity of the root α_m. Once we have obtained $r(x)$ we can write $f(\mathbf{A}) = p(\mathbf{A})q(\mathbf{A}) + r(\mathbf{A})$, and by the Cayley-Harmlton Theorem, $p(\mathbf{A}) = 0$; therefore, $f(\mathbf{A}) = r(\mathbf{A})$.

Algorithm: Given the analytic function, *f*, and the square matrix **A** of order *n* with eigenvectors, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ we compute $f(\mathbf{A})$ as follows:

1. If the n eigenvalues of **A** are all distinct, then
   a. solve the system of equations: $r_{n-1}\lambda_i^{n-1} + r_{n-2}\lambda_i^{n-1} + \cdots + r_1\lambda_i + r_0 = f(\lambda_i)$ $i=1,2,\ldots,$n for $r_0, r_1, \ldots, r_{n-1}$
   b. and set $f(\mathbf{A}) = r_{n-1}\mathbf{A}^{n-1} + r_{n-2}\mathbf{A}^{n-2} + \cdots + r_1\mathbf{A} + r_0\mathbf{I}$

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2. Otherwise, (at least one eigenvalue is repeated).
   a. for each eigenvalue   with multiplicity $m > 1$, differentiate $r_{n-1}x^{n-1} + r_{n-2}x^{n-2} + \cdots + r_1 x + r_0 = f(x)$, $m - 1$ times to obtain m equations for $f(\lambda_j)$, $f'(\lambda_j)$, $f''(\lambda_j)$, . . . , $f^{(m-1)}(\lambda_j)$ in r_0, r_1, . . . , r_{m-1} ,
   b. supplement these equations with $r_{n-1}\lambda_i^{n-1} + r_{n-2}\lambda_i^{n-1} + \cdots + r_1\lambda_i + r_0 = f(\lambda_i)$ for each eigenvalue of multiplicity 1 and solve the resulting system,
   c. and set $f(\mathbf{A}) = r_{n-1}\mathbf{A}^{n-1} + r_{n-2}\mathbf{A}^{n-2} + \cdots + r_1\mathbf{A} + r_0\mathbf{I}$

2. *The algorithm can be used to calculate the matrix equivalent of the scalar function* $f(x) = \sqrt{x}$ *. (Another interesting example is:* $f(x) = \dfrac{1}{x}$*.)*
   a. Use the **T**ransfer-**L**oad-**D**erive command sequence to enter the file *ex2.mth* from the **a:**drive

$$\#1:\quad M := \begin{bmatrix} 6 & 4 & 4 \\ 10 & 19 & 10 \\ -12 & -19 & -10 \end{bmatrix}$$

$$\#2:\quad \begin{bmatrix} p^2 & p & 1 & \sqrt{p} \\ q^2 & q & 1 & \sqrt{q} \\ r^2 & r & 1 & \sqrt{r} \end{bmatrix}$$

$$\#3:\quad a \cdot x^2 + b \cdot x + c \cdot x^0$$

   b. Use the EIGENVALUES command to find the three eigenvalues of **M**: $w = 2$, 4, and 9.

   c. Since the eigenvalues are distinct, we will follow the outline of the first option of the algorithm: (1) Use the **M**anage-**S**ubstitute command sequence to replace *p*, *q*, and *r* in statement #2 with the three eigenvalues respectively. (2) Solve the system of equations resulting from this substitution using the ROW_REDUCE commands. (3) Use the **M**anage-**S**ubstitute command sequence to replace *x* in statement #3 with M, and *a*, *b*, and *c* with the three values in the last column of the solution of the above system of equations. (4) Our algorithm can be checked by squaring this result and noting that the answer is equal to **M**.

3. *Given the matrix* ***M*** *in statement #1 below, we can find expressions for sin(****M****) and cos(****M****) and show that these functions behave in a similar fashion to their corresponding scalar functions for example,* $\sin^2 \mathbf{M} + \cos^2 \mathbf{M} = \mathbf{I}$*. Our approach will be to find the matrix representation of the function* $e^{i\alpha} = \cos\alpha + i\sin\alpha$*, and compute cos(****M****) and sin(****M****) by finding the real and imaginary parts of* $e^{i\mathbf{M}}$

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a) Use the **T**ransfer-**L**oad-**D**erive command sequence to enter the file *ex3.mth* from the **a:** drive.

$$\#1:\quad M := \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 3 \\ -5 & -6 & -4 \end{bmatrix}$$

$$\#2:\quad \begin{bmatrix} p^2 & p & 1 & \hat{e}^{\hat{\imath}\cdot p} \\ q^2 & q & 1 & \hat{e}^{\hat{\imath}\cdot q} \\ r^2 & r & 1 & \hat{e}^{\hat{\imath}\cdot r} \end{bmatrix}$$

$$\#3:\quad a\cdot x^2 + b\cdot x + c\cdot x^0$$

b) Use the EIGENVALUES command to find the three eigenvalues of the matrix **M** in statement #1: $w = 1, -1,$ and 2.

c) Since the three eigenvalues are unique, we use the first option of the algorithm: Set up a system of three equations based on $ax^2 + bx + c = e^{ix}$ by **M**anage-**S**ubstituting the three eigenvalues into the augmented matrix in statement #2 (replacing p with 1, q with –1, and r with 2), and **S**implify the result.

d) Solve the system obtained in step c) using the ROW_REDUCE command, and **S**implify the result.

e) According to the algorithm, $e^{i\mathbf{M}} = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{M}^0$; therefore, **M**anage-**S**ubstitute **M** for x in statement #3 and enter the values in the last column of the result of step d) for a, b, and c, respectively. When this result is **S**implified, we have an expression for $e^{i\mathbf{M}}$ **.**

f) To find expressions for cos(**M**) and sin(**M**), compute the real and imaginary parts of $e^{i\mathbf{M}}$, respectively.

g) We can confirm the fact that these matrix functions behave in the same manner as their corresponding scalar functions by validating identities. For example,

i) $\sin^2 \mathbf{M} + \cos^2 \mathbf{M} = \mathbf{I}$

ii) $\sin 2\mathbf{M} = 2 \sin\mathbf{M} \cos\mathbf{M}$

iii) $\cos 2\mathbf{M} = 2\cos^2 \mathbf{M} - \mathbf{I} = \mathbf{I} - 2\sin^2 \mathbf{M}$

4. *Given the coefficient matrix* **M** *in statement #1 below we can write a linear system of` differential equations in the form* $\dot{\mathbf{x}} = \mathbf{M}\mathbf{x}$, *where* **x** *is a vector with components dependent on a variable t and* $\dot{\mathbf{x}}$ *is the derivative of* **x** *(taken element by element). Since the derivative of* $\mathrm{e}^{i\mathbf{M}}$ *is* $\mathbf{M}e^{i\mathbf{M}}$, *it follows that* $e^{i\mathbf{M}}$ *is a solution to the matrix equation* $\dot{\mathbf{X}} = \mathbf{M}\mathbf{X}$; *hence, each column of* **X** *is a solution to* $\dot{\mathbf{x}} = \mathbf{M}\mathbf{x}$. *Also, the dimension of the solution space of* $\dot{\mathbf{x}} = \mathbf{M}\mathbf{x}$ *where* **M** *is an n×n matrix, is n and the columns of* **X** *are linearly independent; therefore, the general solution of*

$\dot{\mathbf{x}} = \mathbf{M}\mathbf{x}$ *is* $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$ *Where* $e^{i\mathbf{M}} = \mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n]$. *Our approach will be to use the algorithm to find a representation for* $e^{i\mathbf{M}}$*, and then write the general solution to our system of differential equations as a linear combination of the columns of the resulting matrix.*

a) Use the Transfer-Load-Derive command sequence to enter the file *exl.mth* from the **a:** drive.

```
            ┌ -1    1   0 ┐
#1:  M :=   │ -12   5   8 │
            └  9   -3  -7 ┘

         ┌  2           p ┐
         │ p    p  1  ê   │
         │                │
         │              p │
#2:      │ 2·p  1  0  ê   │
         │                │
         │              p │
         └  2   0  0  ê   ┘

          2
#3:  a·x   + b·x + c
```

b) Use the EIGENVALUES command to find the single eigenvalue of **M**: $w = -1$. It follows that $(-1)t$ is the single eigenvalue of $t\mathbf{M}$.

c) Since $t\mathbf{M}$ only has one eigenvalue, the second option of the algorithm applies: Set up a system of three equations based on $ax^2 + bx + c = e^x$ and its first two derivatives by **M**anage-**S**ubstituting $-t$ (the single eigenvalue of $t\mathbf{M}$) for p in statement #2, and **S**implify the result.

d) Solve the resulting system of equations obtained in step c) using the ROW_REDUCE command, and **S**implify the result.

e) According to the algorithm, $e^{t\mathbf{M}} = a(t\mathbf{M})^2 + b(t\mathbf{M}) + c(t\mathbf{M})^0$; therefore, **M**anage-**S**ubstitute $t\mathbf{M}$ for x in statement #3 and enter the values in the last column of the result obtained in step d) for a, b, and c, respectively. When this result is **S**implified, we have an expression for $e^{t\mathbf{M}}$.

f) If $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are the three columns of $e^{i\mathbf{M}}$, then the general solution of the system of differential equations $\dot{\mathbf{x}} = \mathbf{M}\mathbf{x}$ can be written as $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, which can be formed by finding $e^{t\mathbf{M}}\mathbf{c}$ where $\mathbf{c}$ is an arbitrary constant vector.

5. *Solutions for the nonhomogeneous system of differential equations* $\dot{\mathbf{x}} = \mathbf{M}\mathbf{x} + \mathbf{G}(t)$ *can be given which satisfy the coefficient matrix* **M** *in statement #3, the vector* $\mathbf{G}(t)$ *in statement #4, and the initial condition vector* **n** *(which we want our solution to satisfy when* $t = 0$*) in statement #5 below. Our approach will be to first find the general solution to the system, which can be written in the form,* $\mathbf{p} = e^{t\mathbf{M}}\left(\int e^{-t\mathbf{M}}\mathbf{G}(t)dt + \mathbf{c}\right)$*, where* **c** *is an ordinary constant vector. Our algorithm will be used to find an expression for* $e^{t\mathbf{M}}$*, and in turn for* $e^{-t\mathbf{M}}$*, as an intermediate step in computing this expression. The particular solution satisfying the initial condition can then be found by setting* $\mathbf{p}(0) = \mathbf{n}$ *and solving for* **c**.

a. Use the **T**ransfer-**L**oad-**D**erive commands to enter the file *ex5.mth* from the **a:** drive.

$$\#3:\quad m := \begin{bmatrix} 36 & 41 \\ -34 & -38 \end{bmatrix}$$

$$\#4:\quad G(t) := \begin{bmatrix} \mathrm{SIN}(t) \\ \mathrm{COS}(t) \end{bmatrix}$$

$$\#5:\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b. Use the algorithm to find a representation for $e^{t\mathbf{M}}$ by (1) finding the eigenvalues of **M** ($-1\pm 5i$); (2) solving the system of equations in statement #6 after substituting t times the first eigenvalue for p and t times the second eigenvalue for q; and (3) substituting into statement #7 values for x ($t\mathbf{M}$) and a and b (the two values in the rightmost column of the result from solving the system of equations in statement #6).

$$\#6:\quad \begin{bmatrix} p & 1 & \hat{e}^{p} \\ q & 1 & \hat{e}^{q} \end{bmatrix}$$

$$\#7:\quad a\cdot x + b\cdot x^{0}$$

c. Determine the general solution for this nonhomogeneous system of differential equations by evaluatng the integral, $\mathbf{p} = e^{t\mathbf{M}}\left(\int e^{-t\mathbf{M}}\mathbf{G}(t)dt + \mathbf{c}\right)$, where **c** is the arbitrary vector in statement #8.

$$\#8:\quad \begin{bmatrix} c1 \\ c2 \end{bmatrix}$$

d. Find the particular solution satisfying our initial conditions by solving $\mathbf{p} = \mathbf{n}$ with $t = 0$) for $c1$ and $c2$, and substitute these values into the general solution p. Finally, plot [t, X1(t)] and [t, X2(t)] for $t > 0$.

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6. *The system of differential equations*

$$\ddot{x} = -2\dot{x} - 5y + \sin(3t)$$
$$\dot{y} = \dot{x} + 2y$$

can be made into a first order system which can be solved by the techniques of Example 5 by making the substitutions $u_1 = x$, $u_2 = \dot{x}$, *and* $u_3 = y$:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(3t) \\ 0 \end{bmatrix}$$

a. Use the **T**ransfer-**L**oad-**D**erive commands to enter the file *ex6.mth* from the **a:** dnve.

$$\#1:\quad M := \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\#2:\quad \begin{bmatrix} p^2 & p & 1 & ê^p \\ q^2 & q & 1 & ê^q \\ r^2 & r & 1 & ê^r \end{bmatrix}$$

$$\#3:\quad a\cdot x^2 + b\cdot x + c\cdot x^0$$

$$\#4:\quad G(t) := \begin{bmatrix} 0 \\ \mathrm{SIN}(3\cdot t) \\ 0 \end{bmatrix}$$

$$\#5:\quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\#9:\quad \begin{bmatrix} c1 \\ c2 \\ c3 \end{bmatrix}$$

b. Use the algorithm to find an expression for $e^{t\mathbf{M}}$: (1) Find the eigenvalues of **M** (0, *i*, and –*i*). (2) Solve the system of equations in statement #2 after substituting *t* times the three eigenvalues for *p*, *q*, and *r* respectively. (3) Substitute into statement #3 values for *x* (*t***M**) and *a*, *b*, and *c* (the three respective values in the rightmost column of the result from solving the system of equations in statement #2 and **S**implify.

c. Find $e^{-t\mathbf{M}}$ by replacing *t* with –*t* in $e^{t\mathbf{M}}$, and then evaluate the general solution $\mathbf{p} = e^{t\mathbf{M}}\left(\int e^{-t\mathbf{M}}\mathbf{G}(t)dt + \mathbf{c}\right)$, where **G**(*t*) is the vector in statement #4 and **c** is the arbitrary vector given in statement #9.

d. Set **p** = **n** (where **n** is the vector in statement #5 representing the initial conditions for the system) and *t* = 0. Solve this system for *c*l, *c*2, and *c*3, and then substitute these values back into the expression for **p**.

e. Finally, plot [*t*, X1(*t*)], [*t*, X2(*t*)], and [*t*, X3(*t*)] for *t* > 0, where X1(*t*)= *x*, X2(*t*)= $\dot{x}$, and X3(*t*) = *y*, are the three components of the particular solution, respectively.

7. *The system of differential equations*

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$$\dot{x} = y$$
$$\dot{y} = -101x - 2y$$

can be made into a first order system which can be solved by the techniques qf Example 3 by first writing the system as a matrix equation.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -101 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

a. Use the **T**ransfer-**L**oad-**D**erive commands to enter the file *ex7.mth* from the **a:** drive.

$$\#1: \quad m := \begin{bmatrix} 0 & 1 \\ -101 & -2 \end{bmatrix}$$

$$\#2: \quad \begin{bmatrix} p & 1 & \hat{e}^{p} \\ q & 1 & \hat{e}^{q} \end{bmatrix}$$

$$\#3: \quad a \cdot x + b \cdot x^{0}$$

$$\#6: \quad \begin{bmatrix} c1 \\ c2 \end{bmatrix}$$

$$\#7: \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

b. Use the algorithm to find an expression for $e^{t\mathbf{M}}$. (1) Find the eigenvalues of **M** ($-1+10i$ and $-1-10i$). (2) Solve the system of equations in statement #2 after substituting t times the two eigenvalues for p and q respectively. (3) Substitute into statement #3 values for x (tM) and a and b (the two values in the rightmost column of the result of solving the system of equations in statement #2) and **S**implify.

c. The general solution of our system of differential equations can now be written as the linear combinations of the columns of $e^{t\mathbf{M}}$ found in b) above. To find the particular solution corresponding to the vector in statement #7 when t = 0, multiply $e^{t\mathbf{M}}$ by the arbitrary vector in statement #6, and set the result equal to the vector in statement #7. After setting $t = 0$, solve for cl and c2 and substitute these values into the product of $e^{t\mathbf{M}}$ with the vector in statement #6.

d. Finally, plot [t, Xl(t)] and [t, Yl(t)] for $t > 0$, where Xl(t) and Yl(t) are the two components of the particular solution.

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Reference: Robert J. Hill and Thomas A. Keagy, Elementary Linear Algebra with *Derive*, Chartwell-Bratt, 1995, pp 227-301 and 339-349.