Systems of Orthogonal Circles and Poincarè Geometry, on the TI-92

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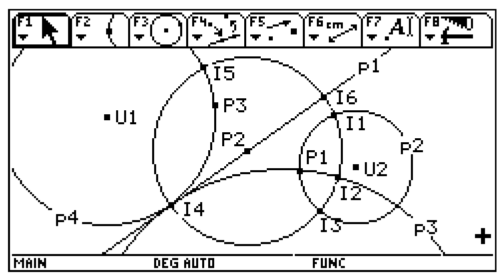
When we encounter hyperbolic geometry for the first time, we are struck by the strange relations that can be found among straight lines. Some of these are: parallel lines converge, two lines may possess at most one common perpendicular, equidistant curves are not lines at all and there are many (not just one) lines through a point which do not intersect a given line. In fact, these relationships seem so bizarre and unfamiliar, that we might believe that such a geometry can't really exist. For this reason models of hyperbolic geometry are constructed using Euclidean geometry, so that any contradictions deriving from the hyperbolic axioms would already be inherent in Euclidean geometry.

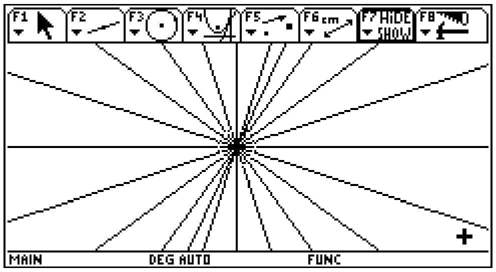
Among the many models of hyperbolic geometry is the disk model attributed to Henri Poincarè. The model is based on a fixed circle called the base circle. The points of the geometry are the points strictly interior to the base circle and are called Poincarè-points - or P-points. The set of all P-points constitutes the Poincarè plane. The points on the base circle are called Ideal-points (or I-points) and the points exterior to the base circle are called Ultra-ideal (or U-points.) The Poincarè-lines (or P-lines) in the geometry are those circles which are orthogonal to the base circle along with those lines which contain the center of the base circle (and therefore also orthogonal to it.) The purpose of this talk is to illustrate many of the relations among Poincarè-lines in this geometry, by exploiting the analogous relations, which hold among circles in the Euclidean plane. We will use the TI-92 calculator throughout.

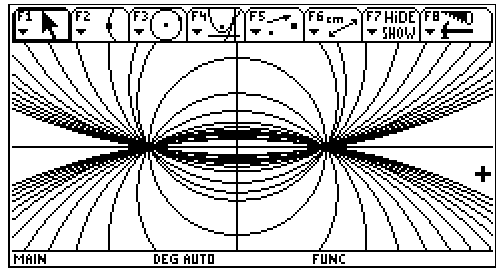


Our point of view in this paper is that many of the properties involving points, lines and circles in the hyperbolic plane (P-points, P-lines and P-circles in the Poincarè model) can be seen clearly by studying their counterparts in the inversive version of the Euclidean plane. The inversive plane is the standard Euclidean plane to which an ideal point, called the "point at infinity", has been affixed. The inversive plane is topologically a sphere and the usual stereographic projection is a specific mapping realizing the homeomorphism.

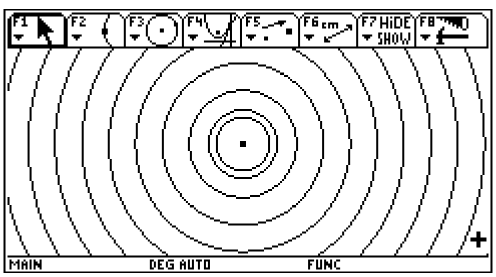
The advantage to using the inversive version of the Euclidean plane is that we can fit circles and lines under one rubric - that of inversive circles. An inversive circle is an ordinary circle if and only if it does not contain the ideal point. Otherwise, it is an ordinary Euclidean line. Inversive circles are preserved by inversions in circles. That is, if a circle or a line is inverted in a circle (called the circle of inversion), then the result is either a circle or a line. We will consider pencils of inversive circles in the inversive plane There are three cases of interest.

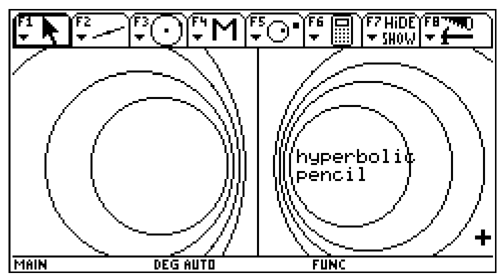
The first case is that of a maximal collection of inversive circles that meet in two inversive points. This is called an intersecting (or elliptic) pencil of inversive circles. In a special version of this case, one of the inversive points is the ideal point and the pencil is a concurrent pencil of straight lines. The general case can be obtained either directly or by inversion of the special case.



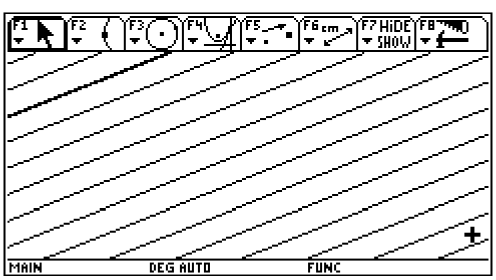


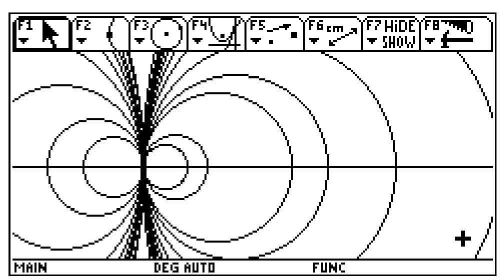
The next case consists of a maximal collection of circles, no two of which intersect. This is called a non-intersecting (or hyperbolic) pencil of inversive circles. The special version of this case is a pencil of concentric circles. The general case can most easily be obtained as an inversion of the special case.



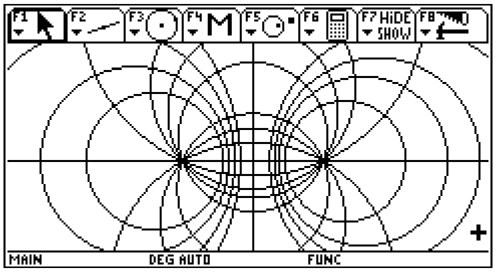


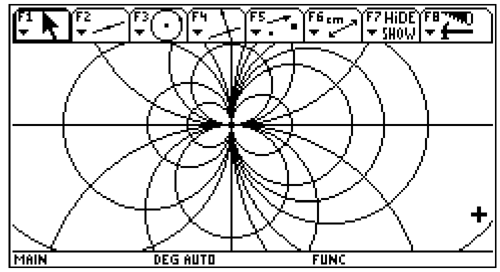
Finally, there are the tangent (or parabolic) pencils of inversive circles. A tangent pencil consists of a maximal collection of inversive circles, all of which have exactly one point in common. It is easy to construct such a family directly, but it can also be constructed as an inversion of its special case, that of a pencil of parallel lines.





For any pencil of inversive circles, there is a complementary pencil consisting of all those inversive circles that are orthogonal to all the members of the original pencil. The complement of any hyperbolic pencil is elliptic and vice-versa. The complement of a tangent pencil is another tangent pencil. For example, the complement of a pencil of concentric circles is a concurrent pencil of lines. The complement of a parallel pencil of lines is another parallel pencil, and so forth.

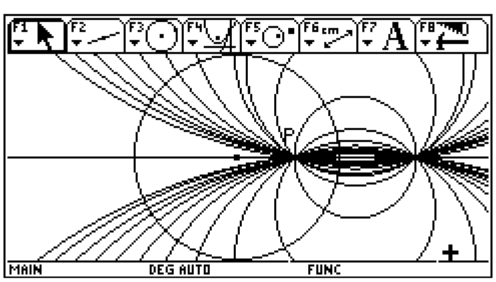


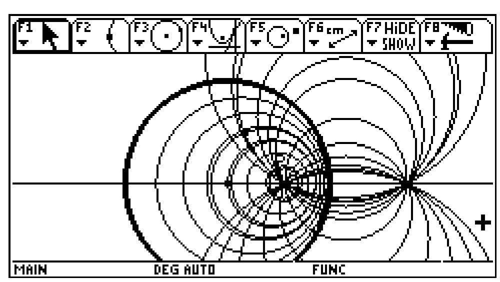


It is easy to see that any pencil of inversive circles, which is not one of the special versions, contains exactly one straight line, which is called the radical axis of the pencil. Moreover, the line of centers of the circles in the pencil is orthogonal to every inversive circle in the original pencil and therefore is a member of its complement. In fact, it is not hard to see that the line of centers of one pencil is the radical axis of its complement. For this reason, such pencils of inversive circles are sometimes called coaxal pencils.

It turns out that any two distinct inversive circles belong to a unique pencil of one of the three above types, which type depending, of course, on whether the inversive circles meet in one, two or no points. It follows that that there is always an infinity of inversive circles orthogonal to any two inversive circles.

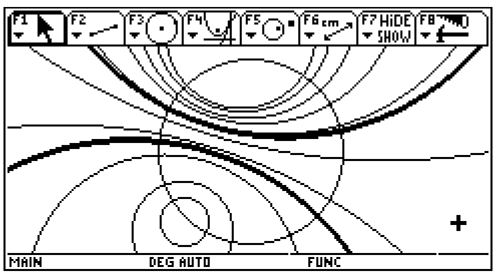
We can exploit these ideas when studying the Poincarè plane. For example, suppose we start with a pair of intersecting P-lines. That is, suppose we are given a pair of inversive circles (ordinary circles or lines) which are both orthogonal to the base circle and which meet at a P-point. It is easy to see that such a pair of inversive circles must meet again outside the base circle. So a pair of intersecting P-lines are members of an intersecting pencil of inversive circles. Moreover the base circle being orthogonal to both, must be in the complementary non-intersecting pencil.

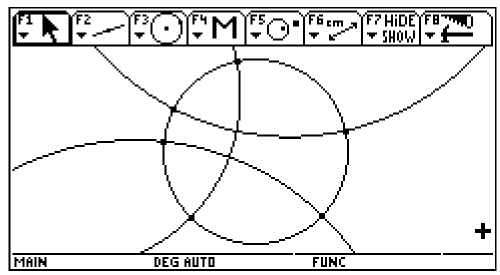




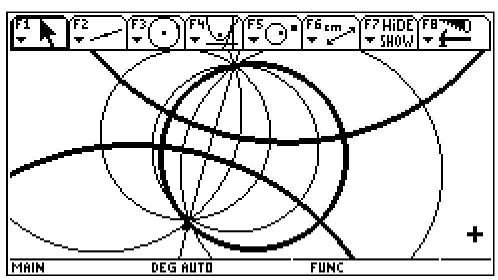
All other members of this complementary pencil either lie entirely inside or entirely outside the base circle. Hence there is can be no P-line perpendicular to both of the original P-lines. This is similar to the Euclidean case, where no triangle can contain two right angles. Moreover, since every inversive circle in the pencil of the original two inversive circles is orthogonal to the base circle, the entire pencil consistd of P-lines (called a concurrent pencil of P-lines.) Those inversive circles in the complementary pencil that lie entirely inside the base circle are called <u>P-circles</u> and the pencil of such is called a concentric pencil of P-circles.

Next, consider the case of a pair of non-intersecting P-lines. That is, suppose we are given a pair of inversive circles, both orthogonal to the base circle and which don't meet. They must belong to a unique non-intersecting pencil of inversive circles and, again, the base circle must belong to the complementary intersecting pencil. Then there must be a unique member of that complementary pencil that is also orthogonal to the base circle. (This is true for any member of any intersecting pencil, since it is obvious for the special case of a concurrent pencil of lines and inversion preserves orthogonality.)

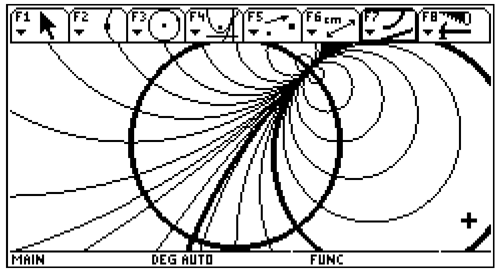




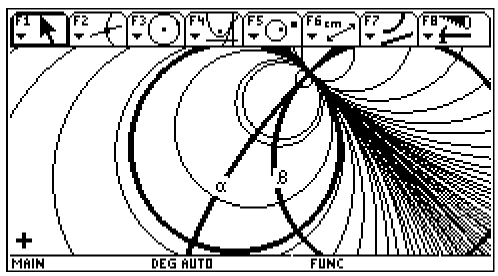
Hence, that unique inversive circle is also a P-line and is the only P-line perpendicular to both of the original non-intersecting P-lines. This is different than the Euclidean situation, where there would be infinitely many such common perpendiculars. Note that again, the non-intersecting pencil of inversive circles consists entirely of P-lines (called a non-intersecting pencil of P-lines.) Those members of the complementary intersecting pencil other than its one P-line and the base circle itself are called <u>equidistant curves</u>.



We have considered the cases where two P-lines meet in two points (only one being a P-point) and in no points. Suppose they meet at one point. It is clear that such a point would needs be on the base circle. These two P-lines will belong to a tangent pencil of inversive circles and, as usual, the base circle will belong to the complementary tangent pencil. Since each inversive circle in this complementary pencil meets the base circle only once, none can be a P-line. Hence, there are no P-lines perpendicular to both of the original P-lines.



Again, this is different from the Euclidean case. For, since these P-lines do not meet at a P-point (they meet at an I-point instead) we would expect an infinite number of common perpendiculars, but we find none at all. Again the tangent pencil of inversive circles containing the original two P-lines consists entirely of P-lines. This pencil is called a parallel pencil of P-lines. Those members of the complementary pencil which, except for their one point in common with the base circle, lie entirely inside the base circle are called limiting curves.



Limiting curves get their names from being the limits of P-circles, constrained to contain a given point whilst their radii tend to infinity. In the Euclidean case, such a curve would be a line. They can also be considered to be limits of equidistant curves.