Modeling With Discrete Dynamical Systems Using Derive

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Abstract

In this workshop we introduce modeling with discrete dynamical systems (difference equations). We will focus on the use of Derive for Windows as a very simple and yet powerful tool to explore and investigate discrete dynamical systems with very interesting and diverse applications in the life and social sciences, and economics. The participants will work on carefully designed activities to explore the qualitative behavior of solutions of the following systems: homogeneous and nonhomogeneous linear dynamical systems, and nonlinear dynamical systems which include logistic models. Dynamical systems of several equations will be investigated for realistic applications.

The study of discrete dynamical systems does not require a sophisticated mathematics background. The materials for this workshop can be introduced to undergraduates in some courses such as finite mathematics, discrete mathematics, liberal arts mathematics for non-mathematics majors, and in courses such as linear algebra, mathematical modeling, and discrete dynamical systems for mathematics, science, and engineering majors. This material was successfully introduced to high school students at the Summer Young Scholars Program "Explorations in Mathematics and Biology" at College Misericordia.

Nonhomogeneous Linear Difference Equations

Suppose that the kidneys remove 20% of a drug in the blood every four hours. Assume that a patient takes an initial dose of drug followed by a dose of 20 mg of the same drug every four hours.
Let A_n = the amount of drug in the blood at the end of n 4-hour intervals. We have

$$A_{n+1} = A_n - 0.2\,A_n + 20$$
$$A_{n+1} = 0.8\,A_n + 20 \tag{1}$$

Equation (1) is a linear non-homogeneous difference equation (discrete dynamical system).
An analytical solution of the difference equation (1) can be easily derived as follows:

$$A_1 = 0.8\,A_0 + 20$$
$$A_2 = 0.8\,A_1 = 0.8\,(0.8\,A_0 + 20) + 20 = (0.8)^2 A_0 + 20(0.8) + 20$$
$$\ldots\ldots$$
$$A_n = (0.8)^n A_0 + [20(0.8)^{n-1} + 20(0.8)^{n-2} + \ldots + 20]$$
$$A_n = (0.8)^n A_0 + 20(1 - (0.8)^n)/(1 - 0.8) \tag{2}$$

Activity 1. Find a numerical solution (n, A_n), n = 0, 1, 2, . . ., 25 and graph it for each of the

following initial dose of the drug (the three graphs in a single coordinate system):
i) 140 mg   ii) 70 mg   iii) 100 mg

The analytical solution (2) of the difference equation (1) can be written as a function, A(n):

Author A(n) := 140(0.8^n) + 20(1 – 0.8^n)/(1 – 0.8)

To find the numerical solution,

Author VECTOR ([n, A(n)], n, 0, 25)
Approximate
Plot

Adjust the range of the axes. The students will discover that A(n) = 100 for any value of n when A(0) = 100. The constant value 100 is called an equilibrium value (solution) or fixed value. To determine the equilibrium solution E analytically, put $A_{n+1} = A_n = E$ in the difference equation (1).
The students repeat this activity for different values of A_0. They will conclude that for any $A_0 > 100$ or $A_0 < 100$, A_n approaches 100. In this case the equilibrium solution is called stable or an attractor.

Iteration in DERIVE

The numerical solution of a difference equation can be generated by iterating the given difference equation. This is very helpful especially if an analytical solution can not be found as the case of nonlinear difference equations. Equation (1) can be written in the form

$$A \leftarrow 0.8A + 20 \tag{3}$$

Which means that A is replaced by 0.8A + 20.

The utility file **RECUREQN.MTH** contains some pre-defined routines to calculate the needed iterations for a recurrence equation. To load this file, click on

File   Load   Utility   RECUREQN.MTH

Because A was defined as a function it can not be used in the iteration, so either free A or use another symbol such as x. For example to generate $[A_0, A_1, \ldots, A_{25}]$ for the recurrence equation (3) with $A_0 = 140$ do the following:

Author ITERATES (0.8A + 20 , A , 140, 25)
Approximate

Since we are interested in the ordered pairs (n, A_n) for n = 0, 1, 2, . . . , 25, we will do the following:

Author ITERATES ([n + 1, 0.8x + 20], [n, x], [0, 140], 25)

Approximate
Plot

Activity 2. Assume that the rate of growth of deer in a region is 4% per year and the state restricts hunting to 8,000 deer every year. Suppose that P_n is the deer population after n years.

a) Write a difference equation that represents the relationship between P_{n+1} and P_n.

b) Find an analytical solution of the formed difference equation in (a).

c) Find the equilibrium solution E analytically.

d) Find the numerical solution (n, P_n) , n = 0, 1, 2, . . . , 20, and graph it for each of the following initial deer population:

i) 250,000 ii) 150,000 iii) 200,000

Describe the above graphs.

The students repeat this activity for different values of the initial population P_0. They will realize that for any P_0 greater than or less than the equilibrium solution E, P_n diverge from E. In this case the equilibrium solution is called unstable or a repeller.

First-order Nonlinear Difference Equations

The exponential growth model is represented by the difference equation

$$P_{n+1} = P_n + aP_n = (1 + a)P_n = RP_n \qquad (4)$$

where P_n is the population after n periods, a is the growth rate, and R is the growth factor.

The logistic growth model is represented by

$$P_{n+1} = P_n + aP_n - bP_n^{\,2} = (1 + a)P_n - bP_n^{\,2} = RP_n - bP_n^2 \qquad (5)$$

Where b is a very small positive number compared to a.
Equation (5) is a nonlinear difference equation. Equation (5) can be transformed into the equation

$$x_{n+1} = Rx_n (1 - x_n) \qquad (6)$$

which is called a quadratic map. In general there is no analytical solution for equation (5) or (6), however we will use iteration and cobweb method to generate and investigate solutions.

Activity 3. Consider the following logistic growth model of bacteria

$$P_{n+1} = 1.669351P_n - 0.00105292\, P_n^{\,2} \qquad (7)$$

where n is measured in hours and the initial population is 9.6. Find the numerical solution of the difference equation (6) for n = 0, 1, 2, . . . , 20, and graph it.

Author ITERATES ([n + 1, 1.669351P – 0.00105292P^2], [n, P], [0, 9.6], 20)
Approximate
Plot

To realize that the constant b must be very small with respect to the constant a in equation (5), and to explore the effect of b on the solution, the students find the solutions of (5) for some selected different values of a and b.

The following activity is designed to explore and visualize concepts such as monotonic, oscillatory, and periodic solutions and stable/unstable equilibrium values of dynamical systems.

Activity 4. Find the numerical solution of equation (6) for n = 0, 1, 2, . . ., 40 and graph it (connect the points on the graph) and describe the graph for each of the following cases:

I) R = 2.9 , and $x_0 = 0.2$
ii) R = 3.4, and $x_0 = 0.2$
iii) R = 3.55, and $x_0 = 0.2$
iv) R = 3.7, and $x_0 = 0.2$
v) R = 3.7, and $x_0 = 0.2001$

The students will observe that for any R in the interval $0 < R \le 1$ and any initial condition $0 \le x_0 \le 1$ the solution of (6) approaches 0. They will conclude that for any $1 < R \le 3$ and any $0 < x_0 < 1$, the solution of (6) approaches the equilibrium value (R - 1)/R. For any $3 < R \le 3.4495$ the attracting fixed point bifurcates into two points (attracting two-cycle). Similarly the students explore the attracting four-cycle for $3.4495 < R < 3.56$. They will discover that the dynamical system behaves in an interesting way, when $R > 3.56994$, say R = 3.7. The solution of (6) does not seem to have any pattern. It does not approach an attracting fixed point or a periodic cycle. The students will discover that the solutions of a chaotic dynamical system depend sensitively on the initial conditions in addition to R. This fact can be observed easily by comparing the graphs of solutions in iv) and v) which show that a very small change in the initial value x_0 causes a very different type of behavior.

Cobweb method is a graphical technique that help students explore the qualitative behavior of discrete dynamical systems. Students use a written DERIVE code to plot the cobweb diagrams of difference equations. For example, they produce cobweb diagrams of the difference equations in Activity 4 and compare the results from the graphs of the numerical solutions and the cobweb diagrams.

Systems of Difference Equations

Many situations involve more than one dependent quantity and therefore such situation can be modeled by a system of difference equations. We will investigate systems of two or more difference equations with interesting applications such as predator-prey model and competing species. A system of difference equations, such as a system of linear difference equations, might be translated into a matrix equation and therefore matrix algebra can be used to determine the analytical solution. DERIVE iteration routines provide very elegant method to find a numerical solution to a general system of difference equations. Phase plane portrait and time series analysis for these systems will be nicely created.

Activity 5. (Predator-Prey Species)
The following nonlinear system models the populations x_n of the prey and y_n of the predator after n periods:

$$x_{n+1} = c_1 x_n - c_2 x_n y_n \qquad (8)$$
$$y_{n+1} = - c_3 y_n + c_4 x_n y_n \qquad (9)$$

where c_1, c_2, c_3, and c_4 are positive constants. Consider this system for different values of the constants that depend on the growth rates of the prey and predator, the intrinsic rate of increase of the prey in the absence of the predator, and the rate of decline of the predator in the absence of the prey. The students use DERIVE to iterate for large value of n to conduct the time series analysis, graph it and graph the phase plane. They repeat this activity for different initial values of x_0 and y_0 to predict the long term behavior.