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Chaos investigation with DERIVE

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Summary

The aim of the workshop is to illustrate some of the latest achievements in the nonlinear dynamics at the popular level in DERIVE. The new physical result for nonlinear, nonstationary systems is presented - the structure of attraction basins in period doubling bifurcation systems with varying control parameter.

1. Main Notions of Nonlinear Dynamics

The logistic map defines the simplest nonlinear difference equation

$$x_{n+1} = r\, x_n\, (1-x_n),\ n=1,\dots,N,$$

where x_n is the value under consideration, so-called iterative value. This equation is nonlinear and its solution has a very interesting behavior for some values of the control parameter r. We will see that if r is equal to 3.56 the equation results in chaotic solution. Theoretical foundations of the problems are analyzed in detail in papers [1-5], which contain a voluminous bibliography.

If the solution of the logistic map does not change with iterations, then we say that the system is in *a stationary state*. Fixed point is the value of x such as $x_n= x_{n+1}= x^*$. In this case the limit of x is called *an attractor* . The periodical solution with period 2 (2 consequently changing constant values x_n and x_{n-1} is called an *attractor of period two* for the logistic map. This name *attractor* is obvious, because any sequence started from [0,1] will reach these two values by definition.

It may be demonstrated using this program:

```
" Main Notions of Nonlinear Dynamics"
"Shortcut"
D(v):=DIMENSION(v)
"Logistic equation iterations"
LOGIST(l,x0,n):=ITERATES(l*x*(1-x),x,x0,n)
"One line"
ATR_AUX(x1,x2):=[[x1,x2],[x2,x2]]
"Append lines"
ATTRACTOR_AUX(m):=APPEND(VECTOR(ATR_AUX(element(m,i),element(m,i+1)),i,
D(m)-1))
"Attractor Path matrix"
ATTRACTOR(r,x0,n):=[x,r*x*(1-x),ATTRACTOR_AUX(LOGIST(r,x0,n))]
"Only in Approximate mode"
Precision:=Approximate
```

Exercise 1.

- Plot logistic map for stationary (r=2.5) case.
- Approximate function ATTRACTOR(r,x_0,n) for r=2.5, x_0=0.04, n=10.
- Plot the result in Beside 2D window. Centered at x=0.5, y=0.5 using Scale x=0.25 y=0.25.
- Increase number of iterations and plot the result until no plot change.
- Try other start points.

Result: All iterations attract to 1 point $x^*=x_n=x_{n+1}=1-1/r$. (Fig.1, a)

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Exercise 2.

- Plot logistic map for orbit of period 2 (r=3.2).
- Approximate function ATTRACTOR(r,x_0,n) for r=3.2, x_0=0.04, n=50.
- Switch to 2D window, Delete All previous plots & Plot the highlighted one.
- Increase number of iterations and plot the result until no plot change.
- Try other start points.

Result: Double period Bifurcation. 2 stable states instead of one. (Fig.1, b)

Exercise 3.

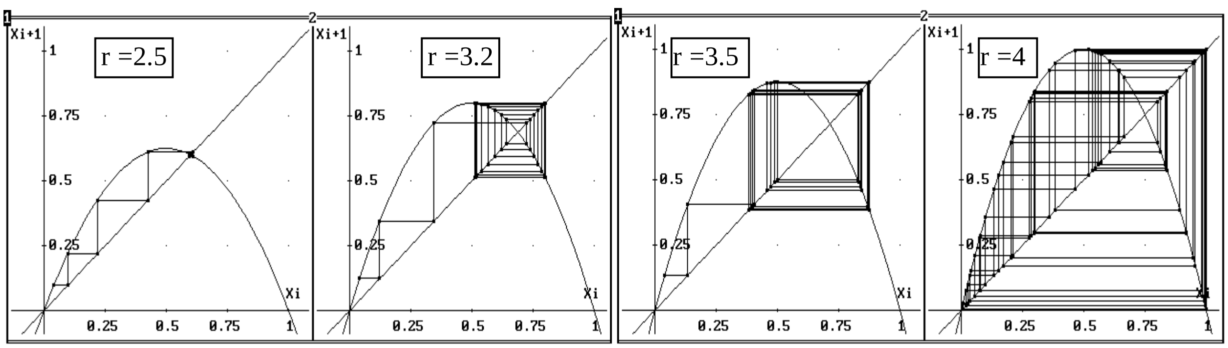
- Plot logistic map for orbit of period 4 (r=3.5).
- Approximate function ATTRACTOR(r,x_0,n) for r=3.5, x_0=0.04, n=50.
- Switch to 2D window, Delete All previous plots & Plot the highlighted one
- Increase number of iterations and plot the result until no plot change.
- Try other start points.

Result: Orbit of period 4. 4 stable states instead of 2. (Fig.1, c)

Exercise 4.

- Plot logistic map for orbit of period 4 (r=4). (Fig.1, d)
- Approximate function ATTRACTOR(r,x_0,n) for r=4, ,x_0=0.04, n=50.
- Switch to 2D window, Delete All previous plots & Plot the highlighted one
- Increase number of iterations and plot the result until no plot change.
- Try other start points.

Result: Chaos. (Fig.1, d)



a) Stable point: $x_n=x_{n+1}$ b) Orbit of period 2 $x_n=x_{n+2}$ c) Orbit of period 4 $x_n=x_{n+4}$ d) Chaos

Fig. 1. Attractors in the logistic map.

2. The model of a dynamic system with variable control parameter

The nonlinear system acquires new stable equilibrium states in bifurcation points. We consider the situation, in which two equivalent final states arise in the system, that is, states which have identical energies, but are different in some non-energetic aspect. It can be phase (period-doubling bifurcations in logistic map) or polarization (polarization states in nonlinear optics). If we plot all limiting values of x as a function of the control parameter r, we will get the bifurcation diagram.

In real system the control parameter r is not constant, r is changed due to some reason. Hence it is important to model such system: a system with parameter r which is changed at slow rate. Let us use the same logistic map but with minor modification:

$x_{n+1} = r_n\, x_n\, (1- x_n),$

$r_{n+1}= r_{n+}\, s,$

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r_n will be changed for each iteration. The rate of growth or decay of the control parameter r is value "*s*". We will consider only such systems in which rate *s* is much smaller then control parameter *r*: $s<<r$.

Let us plot the bifurcation diagram in case *r* is changed. There is no need to calculate 100 iterations at each value of parameter *r*. We can simply plot the current value *x* as a function of variable control parameter *r*, because *r* is changed at each step. If control parameter is changed, there is a new phenomenon: some delay in bifurcation. Control parameter goes through bifurcation point, but solution continues to be in an unstable state. Only after some time (or after a few iterations) the system arrives at a new stable state. The similar phenomenon is observed during the reverse process when the control parameter is decreased. If we compare results obtained for the increasing and decreasing of control parameter through bifurcation point, we will observe a *hysteresys* in the behavior of the system under consideration.

The calculations we have made were performed with precision up to 6 digits. Our software DERIVE permits to do calculation with much better precision, for example 40 digits. In our study the numerical errors are of a paramount importance. In some paper the similar calculations are made with accuracy up to 80 digits. Let us do a few test calculations, which will demonstrate the importance of a different precision.

It may be demonstrated using this program:

```
"Dynamic Bifurcations""Shortcut"D(v):=DIMENSION(v)
"Logistic Map. r -control parameter, x0 -initial value, n iterations"
LOGIST(r,x0,n):=ITERATES(r*x*(1-x),x,x0,n)
"Logistic Map with Sweep Parameter. dr - parameter rate"
LOGIST_D(r,x0,n,dr):=ITERATES([u SUB 1+dr,(u SUB 1+dr)*u SUB 2*(1-u SUB 2)],u,[r,x0],n)
"Logisic Map with Sweep Parameter & Initial value = Stationary one"
L_D(r,n,dr):=LOGIST_D(r,1-1/r,n,dr)
"Matrix: 1st column - vector of scalar r, 2nd column - vector v."
H2(v,r):=VECTOR([r,v SUB i],i,DIMENSION(v))
"m last elements of vector v"
H(v,m):=VECTOR(v SUB (D(v)-i_),i_,0,m-1)
"Makes n iteration for x0 & r, returns m last values."
H1(r,x0,n,m):=H2(H(LOGIST(r,x0,n),m),r)
"Bifurcation Diagram. Parameter range [r1,r2] step dr"
"x0- Initial Point. n - number of iterations. m - number of last values."
W(r1,r2,dr,x0,n,m):=APPEND(VECTOR(H1(r,x0,n,m),r,r1,r2,dr))
"Use only Approximate Mode"
Precision:=Approximate
```

After loading your program you have to specify some settings:

Exercise 5. Bifurcation Diagram.

- Approximate function W(r_1,r_2,dr,x_0,n,m) for r_1=2.8, r_2=3.8, dr=0.01, x_0=0.1, n=100, m=4.
- Plot the result in Beside 2D window Centered at x=3.5, y=0.5.
- Increase the number of iterations to 500. Approximate and Plot once more. (Fig. 2, 1)
- Switch to Algebra window. Remove the result to free the memory.

Exercise 6. Hysteresis and Dynamic Bifurcation Diagram

- Approximate function L_D(r,n,dr) for r=2.8, n=300, dr=0.004. Plot the result.

- Reverse parameter change direction. Approximate function L_D(*r*,*n*,*dr*) for *r*=3.8, *n*=300, *dr*=-0.004. Plot the result. (Fig. 2, 2)
- Switch to Algebra window. Remove the result to free memory.

Exercise 7. Rate Dependence

- Change rate *dr* and number of iterations n to 0.002 & 250 respectively. Approximate L_D() and Plot the result.
- Try other *dr* & *n*. Compare Hysteresis loops size.
  Exercise 8. Noise Effect
- Repeat the above exercise using increased precision (20 digits). Note the curve shift.
- *Result:* Increased precision results in less round off errors and qualitatively changes the result: the curve stops shifting with rate decrease.

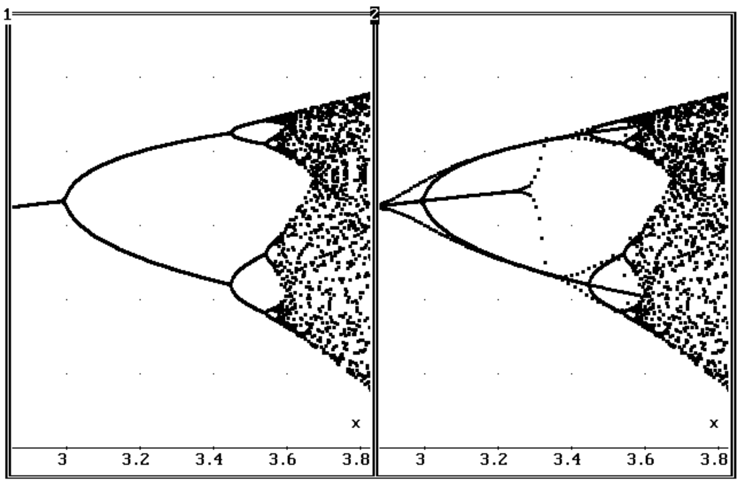

Fig. 2. Bifurcation diagram of the logistic map: 1 -. bifurcation diagram for $x_{n+1} = r\, x_n\, (1-x_n)$, 2 - hysteresis in dynamic bifurcations for $x_{n+1}=r_n x_n(1-x_n)$.

3. The Basins of Attraction for Final Steady States

The bifurcation diagram of the logistic mapping for a variation of *r* within the interval from r_1=2.8 to r_2=3.8 shown in Fig. 2 (1). The fist period-doubling bifurcation occurs at the critical value $r=r_{c1}$=3.0. For $r>r_{c1}$ the initial branch $x^*(r)$ becomes unstable, and the system enters one of the two possible stationary states $\bar{x}$ or $\underline{x}$.The quantity x^* serves as the unstable point of this mapping. For $x> x^*$, the system arrives at the state $\bar{x}$ and for $x>x^*$ and for $x< x^*$ at the state $\underline{x}$. A determination of the attraction zones of the states $\bar{x}$ and $\underline{x}$ constitutes the subject of our investigation in this paper.

Let us consider what final steady states $\bar{x}$ or $\underline{x}$ arise in the system after bifurcation if the control parameter is changed at rate $S=dr/dn$ (*n* is the iteration number) and initial value is equal to x_0. We plot $\bar{x}$ in black and $\underline{x}$ in white in axes x_0 - *S*. The consideration is made for different noise level in the system. Numerical modeling reveals several quite unexpected phenomena.

First of all, every final steady states $\bar{x}$ or $\underline{x}$ has its own basin of attraction on the axes of initial value x_0. These basins have a complicated structure.

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The second result is that in the presence of noise the boundaries of the attraction basins become fuzzy, so that strong noise equalizes the probabilities of arriving at the final steady states.It may be demonstrated using this program:

```
Precision:=Approximate
N(s):=FLOOR((3.2-2.8)/s)+1
FINAL(x0,s):=ELEMENT(ITERATE([r*x*(1-x),r+s],[x,r],[x0,2.8],N(s)),1)
REC(x0,s):=IF(FINAL(x0,s)>0.6875,[x0,s])
step_x0:=0.025
LINE(s):=VECTOR(REC(x0,s),x0,step_x0,1-step_x0,step_x0)
 STRIP(beg_s,fin_s,step_s):=APPEND(VECTOR(LINE(s),s,beg_s,fin_s,step_s))
```

After loading your program you have to specify some settings:
Exercise 9. Plot the Basins of Attraction of Final Steady States

- Approximate function STRIP(beg_s,fin_s,step_s) for beg_s=0.01,fin_s=1,step_s=0.03.
- Plot the result in Beside 2D window ,Scale: 0.5, 1; Axes: 7, 7; Center : x=1, y=1
- Delete All previous plot. Switch to Algebra window. Remove the result to free the memory.
- Approximate function STRIPE(beg_s,fin_s,step_s) for small rates s: beg_s=0.016, fin_s=0.029, step_s=0.0005.
- Plot the result in Beside 2D window , Scale: 0.5, 0.015; Axes: 5, 7; Center : x=1, y=0.015

Result: Attraction basins of the first, $\bar{x}$ and of the second, $\underline{x}$ steady states in period doubling bifurcation systems are plotted.

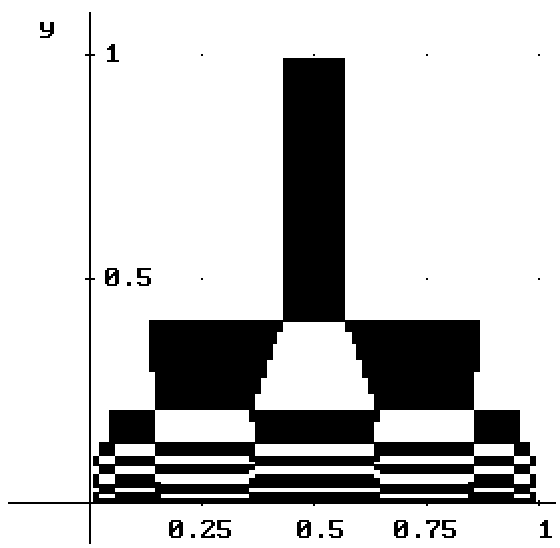

Fig. 5. Division of the "initial coordinate – rate" plane into attraction regions of the states $\underline{x}$ (black cells) and $\bar{x}$ (white cells).

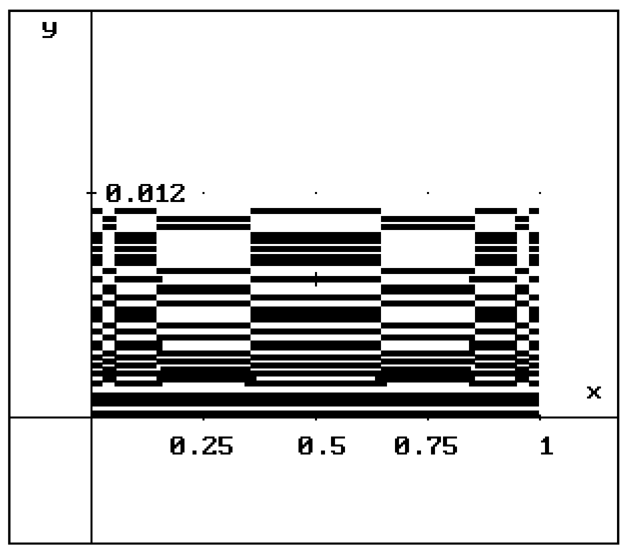

Fig. 6 Fine structure of attraction regions of the states $\underline{x}$ (black cells) and $\bar{x}$ (white cells for small rates s.

4. Conclusion

We have shown, that DERIVE can be successively applied to the numerical modeling of complicated nonlinear systems. DERIVE is a very prospective tool for student scientific research. High precision of calculations supported by DERIVE provide unique opportunity for the investigation of chaos in nonlinear systems.

Contents

5. References

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