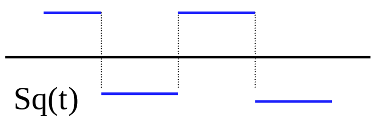
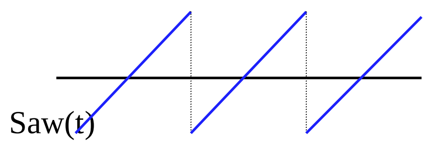
INTRODUCING FOURIER SERIES WITH DERIVE.

Alex J. Lobregt
Hogeschool van Utrecht
Department of Electrical Engineering
alobregt@et.fnt.hvu.nl
Utrecht

In Electrical Engineering courses functions such as the square wave Sq(t) and the sawtooth Saw(t), shown below, are frequently used.





These functions may well be approximated by a so-called Fourier Series.

$$F_N(t)=a_0+a_1\cos(\omega t)+b_1\sin(\omega t)+\ldots\ldots\ldots\ldots\ldots\ldots+a_N\cos(n\omega t)+b_N\sin(n\omega t)=$$

$$=\tfrac{1}{2}a_0+\sum_{n=1}^{N}(a_n\cos(n\omega t)+b_n\sin(n\omega t))$$

Where the Fourier-coefficients a_n and b_n are :

$$a_n=\frac{2}{T}.\int_T f(t).\cos(n\omega t)dt \qquad ; \ b_n=\frac{2}{T}.\int_T f(t).\sin(n\omega t)dt$$

The introduction of this Fourier series involves a great deal of computing effort, which taxes the calculating skills of many a student to the limit.
After all, to solve Fourier Series problems the student must

- be familiar with techniques of integration and be able to handle the typical features of convergence,
- have sound understanding of complex numbers,
- know the basics of trigonometry as this is a prerequisite for handling sinusoidal functions.

In view of the above, it will come as no surprise that teachers should like to keep tedious, time-consuming calculation work to a minimum, if only to avoid computing errors from slowing down the learning process and wasting valuable time which could otherwise be spent on applying what has been learned.

It has been some years now that the course dealing with Fourier Series was extended to include a Derive laboratory. The Derive laboratory has managed to enhance the students' enthusiasm for the subject considerably.
This is in no small measure attributable to Derive's graphical features; one picture is worth a thousand words.

When first introduced to Fourier series, students learn to apply the following built-in function :

$F_N(t)$= Fourier(f(t),t,a,b,N)

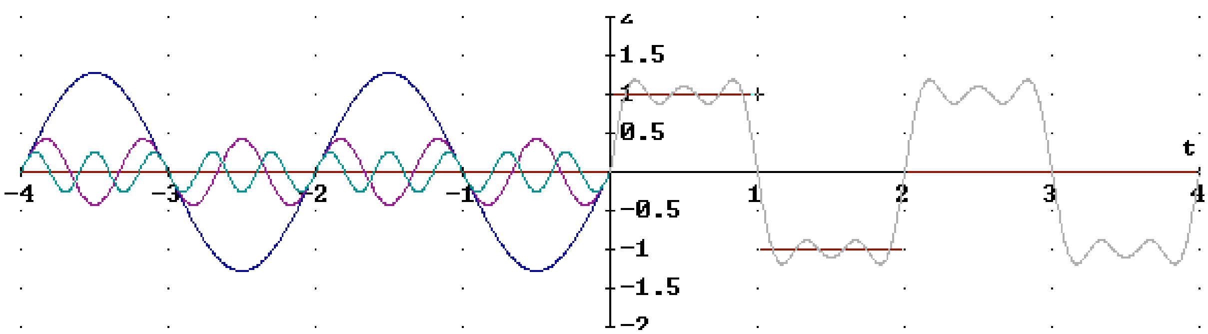
for example $\mathrm{Sq}(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \end{cases}$ period 2

than a=0 , b=2

#1 chi(0,t,1)-chi(1,t,2)
#2 Fourier(#1,t,0,2,5) simplify

$$\frac{4\sin(5\pi t}{5\pi} + \frac{4\sin(3\pi t)}{3\pi} + \frac{4\sin(\pi t)}{\pi} \qquad (1)$$

Derive allows them to draw the individual harmonic functions and their summation side by side, which gives an excellent insight into the structure of the Fourier-Series.



After this introduction students are keen to make a thorough study of the underlying theory and eventually are able to calculate the following Fourier-coefficients using Derive:

$n \in \mathtt{integer}(0, \rightarrow)$

$$\int_0^2 (\mathrm{chi}(0,t,1) - \mathrm{chi}(1,t,2))\cos(n\pi t)dt = 0$$

$$\int_0^2 (\mathrm{chi}(0,t,1) - \mathrm{chi}(1,t,2))\sin(n\pi t)dt = \frac{2}{n\pi}(1-(-1)^n)$$

The outcome $\sum_{n=1}^{N} \frac{2}{n\pi}(1-(-1)^n)\sin(n\pi t)$ can immediately be equated to the series (1) found above .

All this does not take long and there are no annoying computing errors and the resulting graphs provide instant feedback.
The use of complex numbers is especially important as it provides an opportunity for a theoretical review.

The Fourier-coefficients a_n and b_n can be summarised as:

$$\alpha_n = \tfrac{1}{2}(a_n - ib_n) = \frac{1}{T}.\int_T f(t).e^{-in\omega t}dt$$

The Fourier-series then shows : $\sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega t}$

The concise notation doesn't help to make things any clearer to the average student. However , using Derive, we can rewrite it as :

$n \in \text{integer}(0, \rightarrow)$

$$\sum_{n=-5}^{-1} \frac{i}{n\pi}((-1)^n - 1).e^{in\pi t} + \sum_{n=1}^{5} \frac{i}{n\pi}((-1)^n - 1).e^{in\pi t}$$

$$\frac{2\sin(5\pi t)}{5\pi} + \frac{2\sin(3\pi t)}{3\pi} + \frac{2\sin(\pi t)}{\pi} + i(\frac{2\cos(5\pi t)}{5\pi t} + \frac{2\cos(3\pi t)}{3\pi t} + \frac{2\cos(\pi t)}{\pi t}) +$$

$$\frac{2\sin(5\pi t)}{5\pi} + \frac{2\sin(3\pi t)}{3\pi} + \frac{2\sin(\pi t)}{\pi} - i(\frac{2\cos(5\pi t)}{5\pi t} + \frac{2\cos(3\pi t)}{3\pi t} + \frac{2\cos(\pi t)}{\pi t}) \qquad (2)$$

At a glance it is clear that we used : $e^{in\alpha t} = \cos(n\omega t) + i\sin(n\omega t)$
and that adding the last two lines of expression (2) produces function (1).

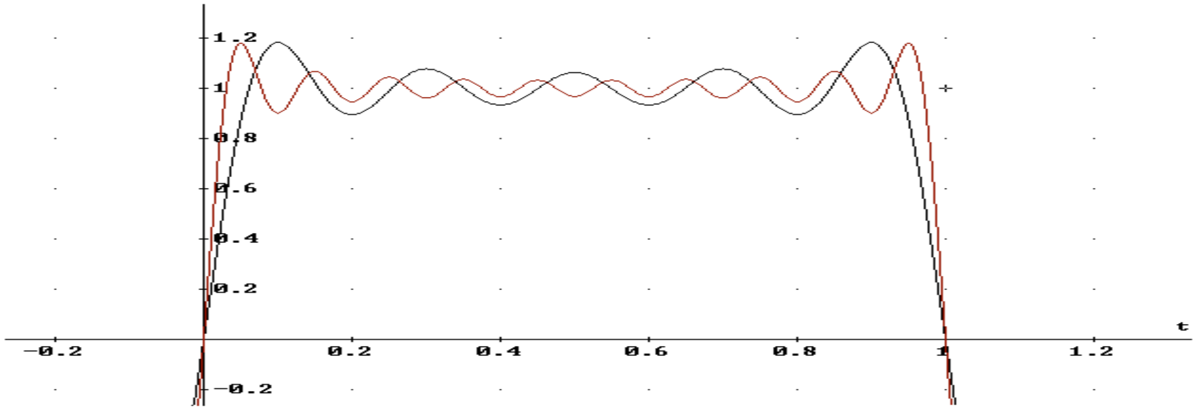
This new approach has led to a substantial change in the syllabus, since doing away with uninspiring computing work has meant that more time can be invested in discussing theory.

The consequences of shifting a function for the amplitude spectrum and the phase spectrum are investigated by using the statements

vector([[n,0],[n,abs(α_n)]],n,a,b)
vector([[n,0],[n,phase(α_n)]],n,a,b)

The outcome can be checked to see if it is consistent with the theoretically predicted result.

Until recently, any investigation of the Gibbs Phenomenon used to be impossible, whereas in the present situation each student can perform calculations in this field.



$$\frac{d}{dt}\ \text{FOURIER(CHI(0, t, 1) - CHI(1, t, 2), t, 0, 2, 10)}=0$$

$$4\cos(9\pi t)+4\cos(7\pi t)+4\cos(5\pi t)+4\cos(3\pi t)+4\cos(\pi t)=0$$

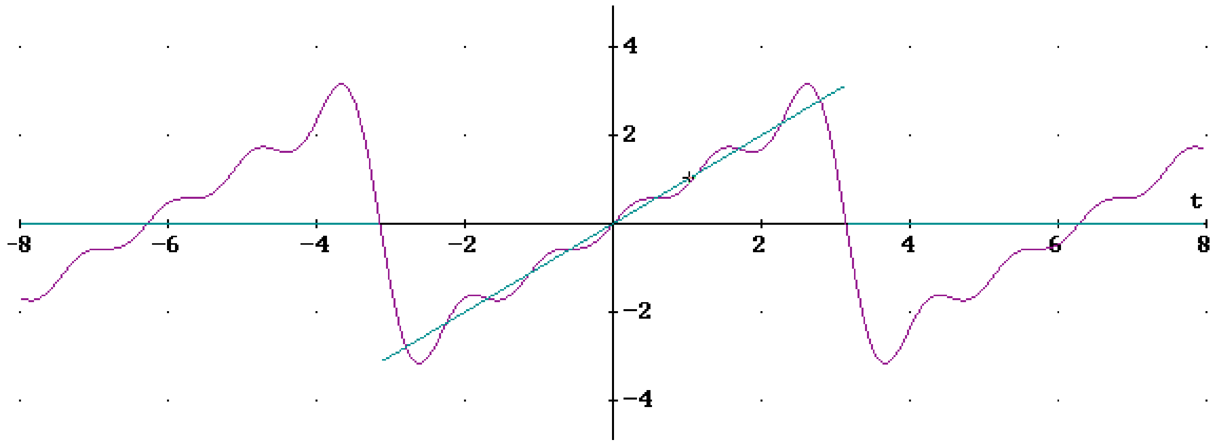
t=0.9

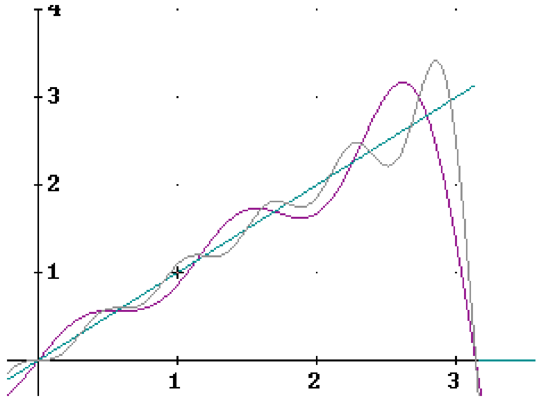
Substituting t=0.9 in (1) gives the maximum 1.18232

Compared with the limit $\frac{2}{\pi}\int_0^{\pi}\frac{\sin(x)}{x}dx=1.17897$ this is a good result.

For 100 terms Derive offers a maximum of 1.179013 which amounts to 8.95 % of the total jump.

The students perform this calculation also on a sawtooth with jumps of 2π.





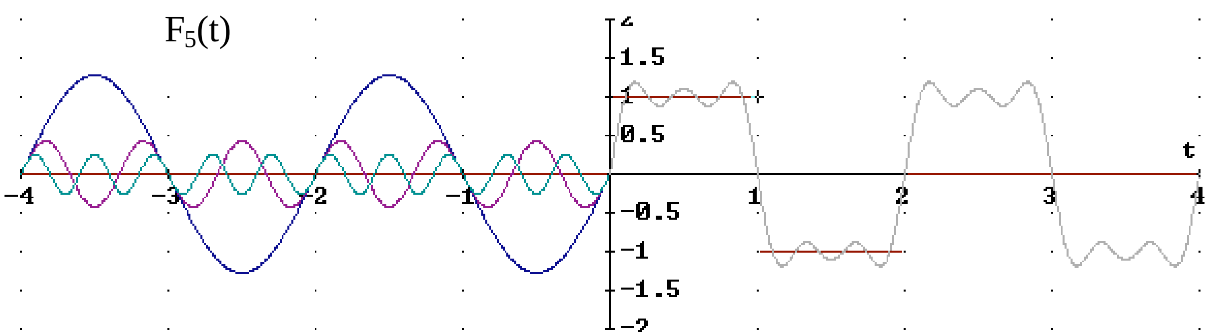
and find that the deviation produces the same results.
This invariably proves to be an incitement to a further study of the Gibbs Phenomenon.

Another major application is using Derive to introduce a first step in the filtering theory.
A low-pass filter can be mathematically described by the convolution-integral :

$$\int_{-\infty}^{\infty} F_N(x).\frac{\sin(\tau(t-x))}{\pi(t-x)}dx \qquad (3)$$

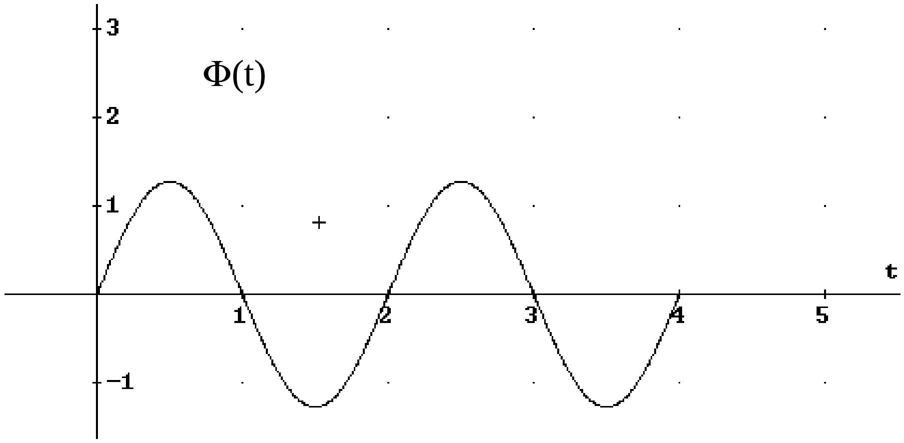
where τ represents the bandwidth.

If we substitute for $F_N(x)$ the series (1) and τ=4, only the 1st harmonic will be passed.
The students are unable to calculate expression (3) but drawing an approximation using Derive is indeed possible, albeit a time-consuming process.



$$\Phi(t) = \int_{-10}^{10} F_5(x).\frac{\sin(4(t-x))}{\pi(t-x)}dx$$ gives the graph shown at the top of the next page.

Contents



Examination methods have been altered as well.
At the end of the course the student's progress is assessed in three different ways:

1) by oral presentation on the outcome of the investigations .
2) a written report on the practical coursework.
3) a written examination covering mainly theoretical knowledge.

One of the major considerations leading to the revised approach was whether the emphasis in coursework could be shifted from acquiring skills to gaining insight. In my view, experience gained over recent years working with Derive tells me we are on the right track.