Contents

Using CAS and the Internet to Communicate Mathematics Effectively

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Introduction

The Professional Standards for Teaching Mathematics (NCTM, 1991) recommended teachers to encourage students to participate in problem solving that extends their capacity to reason and communicate mathematically. Computer technology serves as one avenue for mathematical communication, both in the processing and dissemination of information and knowledge. Computer–based environments can open windows for learning on mathematical structures and problem solving methods, as well as solving specific problems. Noss and Hoyles (1996) have suggested a theory that uses the computer as a window that shows the multiple ways that mathematical meanings are constructed. They coined the term *Webs of meaning,* explaining that "The idea of *webbing* is meant to convey the presence of a structure that learners can draw upon *and reconstruct* for support - in ways that they choose as appropriate for their struggle to construct meaning for some mathematics" (p. 108). One example they presented was the work of Scardamelia and Bereiter (1991) who designed computer learning environments specifically to support cooperative commenting in the context of knowledge-building activities.

Researchers have dealt extensively with the issue of making sense in mathematics, and the contribution of technology in this regard. Schoenfeld (1992) pointed out that learning to think mathematically means developing a mathematical point of view and applying it for understanding mathematical structures. Tall (1991) recommended that students should participate in a full creative cycle of activity in advanced mathematical thinking, including the interaction of different modes of visual and logical thinking. DiSessa (1997) has introduced a new class of software entitled "open toolsets" that involves a highly modifiable, extendable number of units capable of being combined with each other. He has claimed that open toolsets "meet teachers half way" by allowing more substantial control over the technology, thus enabling open toolsets to become a flexible base for learning and instruction that grows and can continue to grow organically.

Here we consider a combination of two types of "all purpose" technologies: (1) CAS and (2) the Internet. Computer Algebra Systems (CAS) require students to use concise symbolic language and to understand the underlying mathematics of a given topic. Interpreting the output of a graphic display involves recognizing equivalent forms of non-graphic representations and judging if the results are reasonable. The Internet, by exploring various Web pages, enables long distance interactive learning of mathematics, and makes possible a dialogue between learner and instructor. Hence, our aim is to explore didactic strategies that make use of these technologies. Our research and development work includes designing computer learning environments in the form of netcourses.

Magic circles

“Magic circles” is a netcourse for high school teachers and students that provides challenging problem-solving situations. The problems presented deal with concepts and skills from the regular high school mathematics curriculum and are associated with abstract constructs (Zehavi & Bruckheimer, 1982). The CAS software *Derive* (Soft Warehouse) was used for investigating and solving these problems. Magic circles involve a rich variety of mathematical structures, concepts and ideas, such as composing linear functions, inverse functions, fixed points, complex numbers, group theory and graphical representation. Figure 1 presents a closed circle of 10 linear functions. We begin at the top of the circle and substitute a number of our choice. The outcome is substituted in the next expression and so on, proceeding clockwise, until we complete the circle, and the final output turns out to be the same number with which we started.

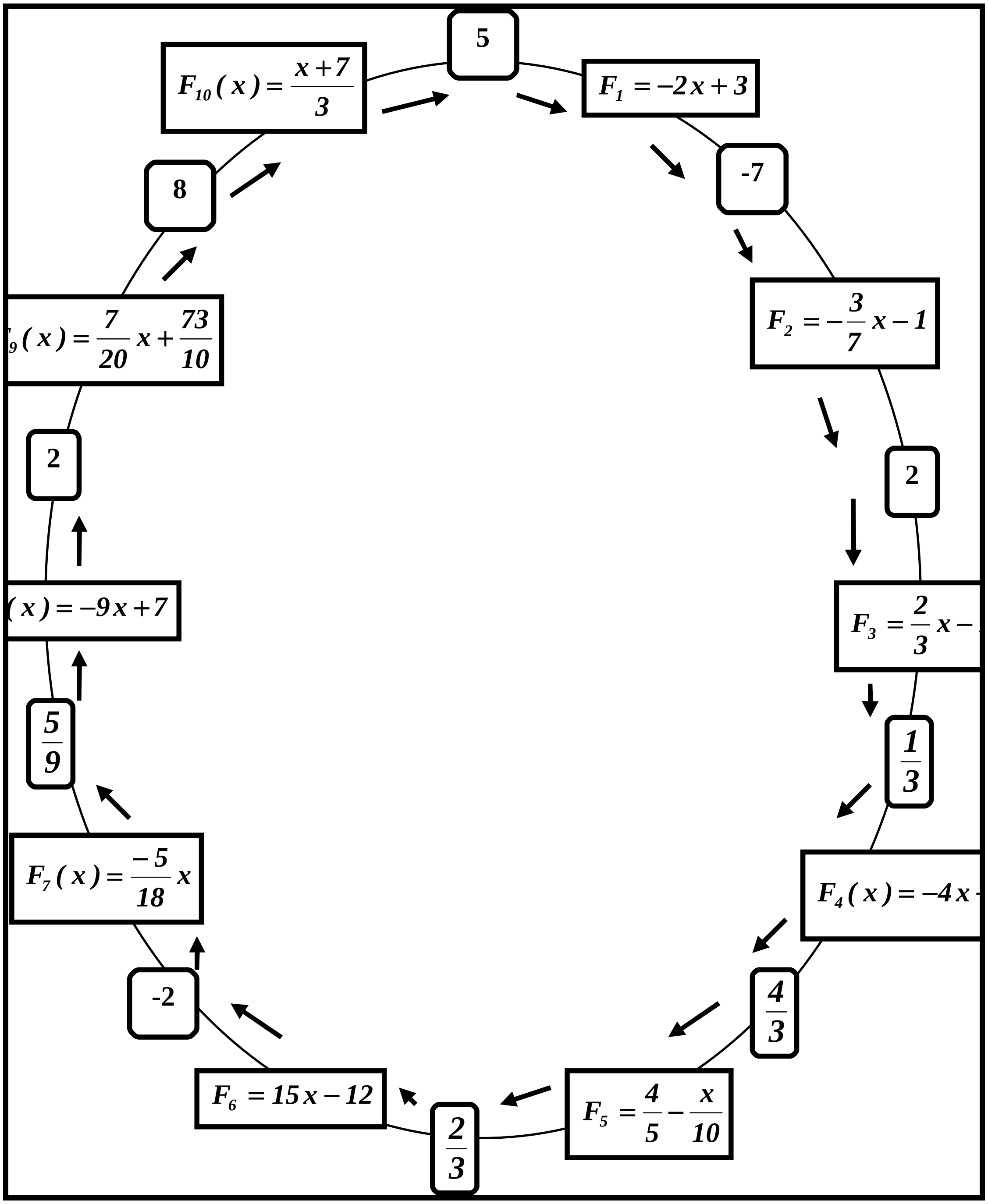

Figure 1. A closed circle

Now we pose several problems:

1. If we start with any number and proceed clockwise, in a given circle, will we always finish with the same number with which we started?
2. If we break the circle at any other point (function), substitute a number, proceed clockwise and complete the circle at the new point of entry, will we always finish with the same number with which we started?
3. Do you know how to construct such circles?

4. If we traverse the circle counter clockwise will we end with the same number with which we started?
5. Is it possible to construct magic circles such that if we traverse them in either direction, the output is the same as the input?

At this stage another circle is presented for the students to explore the magic.

6. Do you know how to construct magic circles such that after only two completed circles the output will be the same as the input?
7. Is it possible to construct magic circles such that after only K completed circles the output will be the same as the input?

There are several ways of solving the problems, each involving meaningful mathematical activities. Perhaps the obvious way to deal with problem 1 is to substitute a variable (for example, a) using *Derive* to perform the tedious manipulation of algebraic expressions:

$$F_{10}(F_9(F_8(F_7(F_6(F_5(F_4(F_3(F_2(F_1(a))))))))))=a$$

Since the computer is doing the technical work under the student's control, students are free to engage in mathematical reasoning related to the composition of linear functions. If they realize that the whole circle reduces to a single linear function, the identity function, they may be able to justify the " breaking of the circle" in problem 2. Because function composition is associative we may write

$$F_{10}\circ F_9\circ F_8\circ F_7\circ F_6\circ F_5\circ F_4\circ F_3\circ F_2\circ F_1=I(x)=x$$

$$(F_{10}\circ F_9\circ F_8\circ F_7)\circ(F_6\circ F_5\circ F_4\circ F_3\circ F_2\circ F_1)=I(x).$$

A linear function $y=ax+b,\quad a\neq 0$ has an inverse. Thus the functions in the two brackets are inverses and we may commute them

$$(F_6\circ F_5\circ F_4\circ F_3\circ F_2\circ F_1)\circ(F_{10}\circ F_9\circ F_8\circ F_7)=I(x).$$

If we remove the parenthesis (applying again associativity) we see that a cyclic permutation of the functions is also the identity function.
How can closed circles be constructed?
Again there are several possible approaches to the problem. Understanding that the linear functions $y=ax+b,\quad a\neq 0$ form a group with composition of functions as an operation helps in determining an efficient method for building circles. Choose any 9 linear functions, find their composition, and then the 10^{th} function is their inverse.
The answer to question 4 is clear since the group of linear functions with composition is non-commutative. If the students are able (or are directed) to find a necessary and sufficient condition needed for the general magic circle, they can build one using the software (see Figure 2).

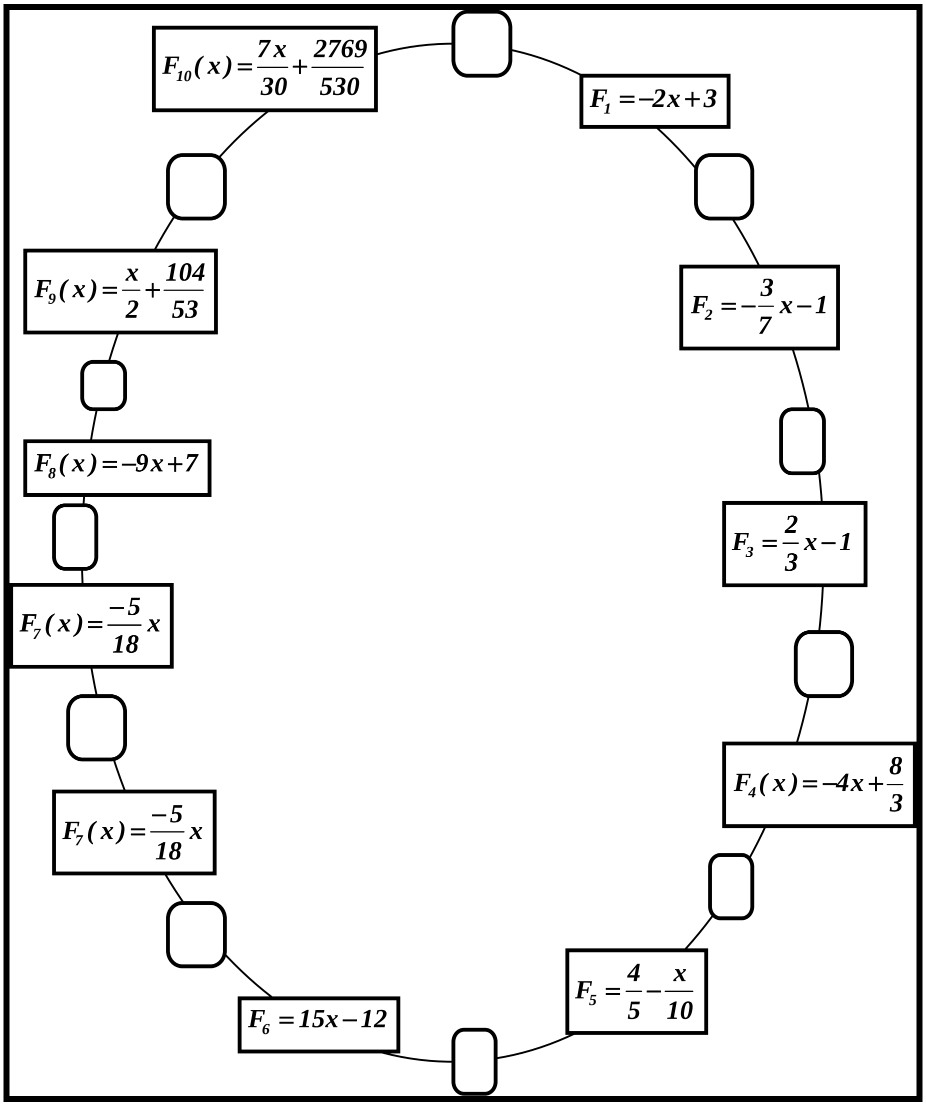

Figure 2. A magic circle

Let us compose the first eight functions in the given circle (shown in Figure 1.)

We define

$$G(x)=F_8\circ F_7\circ F_6\circ F_5\circ F_4\circ F_3\circ F_2\circ F_1$$
$$H(x)=F_1\circ F_2\circ F_3\circ F_4\circ F_5\circ F_6\circ F_7\circ F_8$$
$$F_9(x)=mx+n,\quad m\neq 0.$$

By solving the equation $F_9(G(x))=H(F_9(x))$ we obtain a family of candidate functions: $F_9(x)=mx-\frac{286m}{53}+\frac{39}{53},\quad m\neq 0.$

We choose one of them, $F_9(x)=\frac{x}{2}-\frac{104}{53}$, and proceed as before: find the composition $F(x)=\frac{30x}{7}-\frac{8307}{371}$ (of the first 9 linear functions), and then the 10th function should be the inverse function $F_{10}(x)=\frac{7x}{30}+\frac{2769}{530}$.

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Whereas in a closed circle of n linear functions we have n-1 degrees of freedom, in a magic circle we have n-2 degrees of freedom, which means that we can choose any n-2 linear functions to construct a magic circle. Now what can we expect when we plan to build a magic circle of order K. Let us take an example of a magic circle of order 3. It is a circle of linear functions such that only after 3 completed circles the output will be the same as the input. The search for such circles is very stimulating. Figure 3 shows that we used the same nine first linear functions from our closed circle (see Figure 1). The new webs of meaning that have been opened should be clear to the readers.

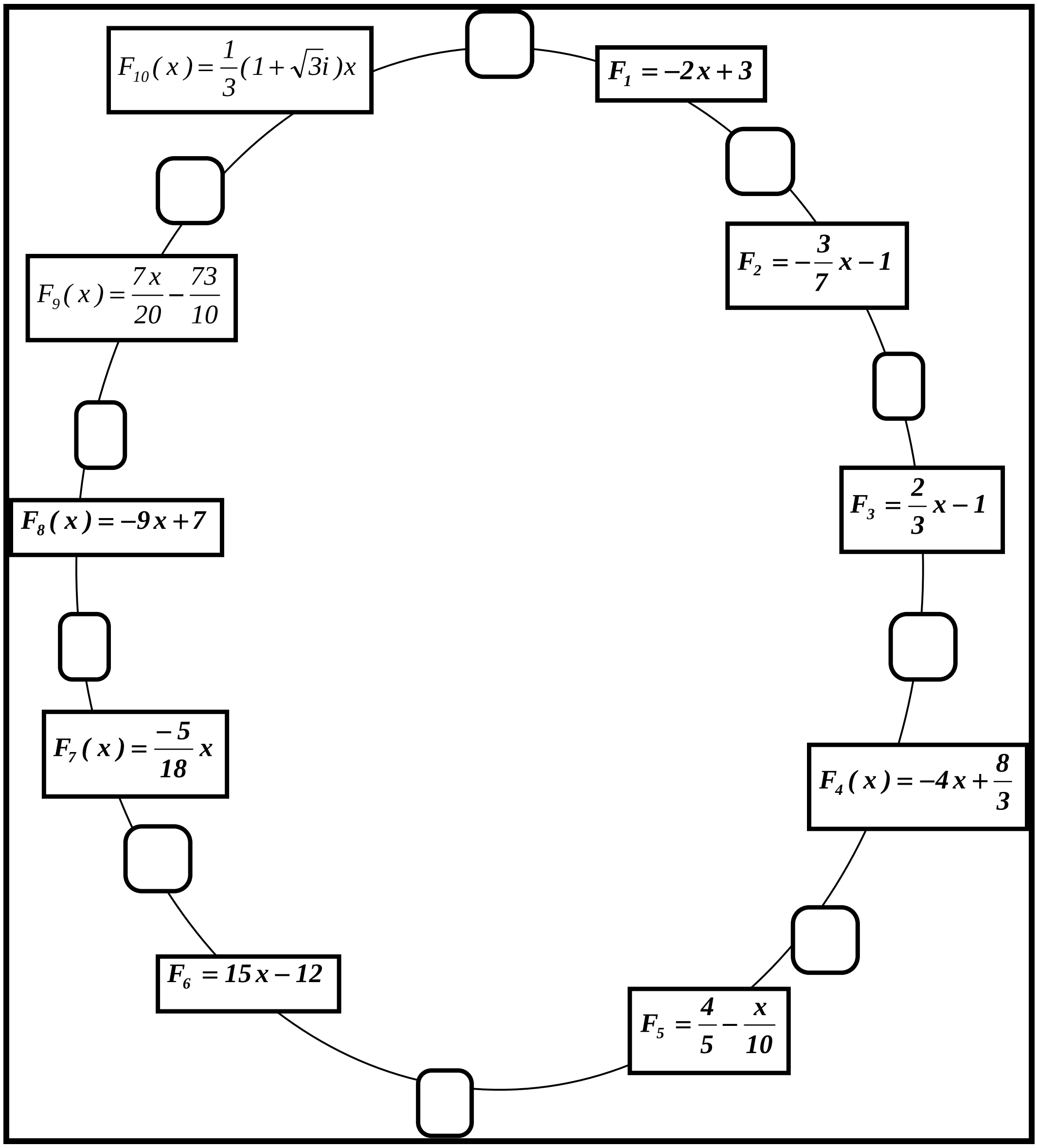

Figure 3. A magic circle of order 3

The pilot study

As part of the formative development of the computerized learning environment we challenged 25 teachers to solve the problems. All the teachers had previously learned to use *Derive* in inservice teacher courses. Most of them gave partial solutions to the problems; only 5 of them managed to complete the whole assignment. We show here two different approaches taken by two teachers.

Teacher A maintained an algebraic technique approach throughout the solving process and looked for mathematical patterns. He did most of his "thinking" by pencil and paper. Here is a sample of his work:

"Given **n** linear functions

$$F_1(x) = a_1 x + b_1, \quad F_2(x) = a_2 x + b_2, \quad \ldots \quad F_n(x) = a_n x + b_n$$

the composition of all the **n** linear functions can be written as

$$F_n(F_{n-1}(F_{n-2} \quad \dots \quad F_3(F_2(F_1(x)))\dots))=$$
$$a_n a_{n-1}\dots a_2 a_1 x + a_n a_{n-1}\dots a_2 b_1 + a_n a_{n-1}\dots a_3 b_2 + a_n a_{n-1} b_{n-2} + a_n b_{n-1} + b_n =$$
$$a_n a_{n-1}\dots a_2 a_1 x + F_n(F_{n-1}(F_{n-2} \quad \dots \quad F_3(F_2(b_1))\dots)).$$

The necessary and sufficient conditions for the general magic circle are

1. $a_n a_{n-1}\dots a_2 a_1 = 1$

2. $a_n a_{n-1}\dots a_2 b_1 + a_n a_{n-1}\dots a_3 b_2 + a_n a_{n-1} b_{n-2} + a_n b_{n-1} + b_n = 0$

3. $a_1 a_2\dots a_{n-1} b_n + a_1 a_2\dots a_{n-2} b_{n-1} + a_1 a_2 b_3 + a_1 b_2 + b_1 = 0.$

Clearly, the two 'last' functions in a magic circle should be constructed according to the solution of these three equations."

Teacher A used *Derive* to verify his findings in certain cases and to create new examples. All the patterns that he found were algebraic. However he didn't associate his findings to the concept of inverse function nor to the concept of the identity function. He justified the patterns by substituting the variable a and after proceeding clockwise he obtained a again. It was difficult for him to communicate mathematically with the other teachers because of the "heavy" algebraic algorithms he had developed. However, he succeeded in completing all the problems and in showing, systematically, how to build the circles.

Teacher B used a graphical approach throughout the solving process. She did most of her "thinking" with *Derive*. She did some algebraic manipulations on the circle's linear functions and then looked at the graphic display trying to interpret the outcome. She succeeded in identifying the visual data obtained by connecting them to mathematical structures such as groups, inverse functions, fixed points and complex numbers. High school teachers are familiar with the graphical representation of the composition of linear functions and of its inverse function. However, teacher B admitted that "it was amazing to realize that the graphical display of the $F_9(x)$ family of functions (in every magic circle) opens another window of 'Web meaning'. Here we present a part of her problem solving process. Figure 4 presents the two parallel lines of *G(x)* and *H(x)* (as presented previously) and a pencil of $F_9(x)$ lines through a point. The intersection point of these lines, $[\frac{286}{53}, \frac{39}{53}]$, can be easily found by solving any two $F_9(x)$ functions

$$m_1 x - \frac{286 m_1}{53} + \frac{39}{53} = m_2 x - \frac{286 m_2}{53} + \frac{39}{53}, \qquad m_1, m_2 \neq 0.$$

Now the following questions arise: Is there any interesting mathematical structure related to this point? Will it always be located outside the parallel lines? Can it fall on one of the parallel lines? (We give here just a little hint, $x = \frac{286}{53}, \frac{39}{53}$ are the fixed points of *G(x)* and *H(x)*, respectively.)

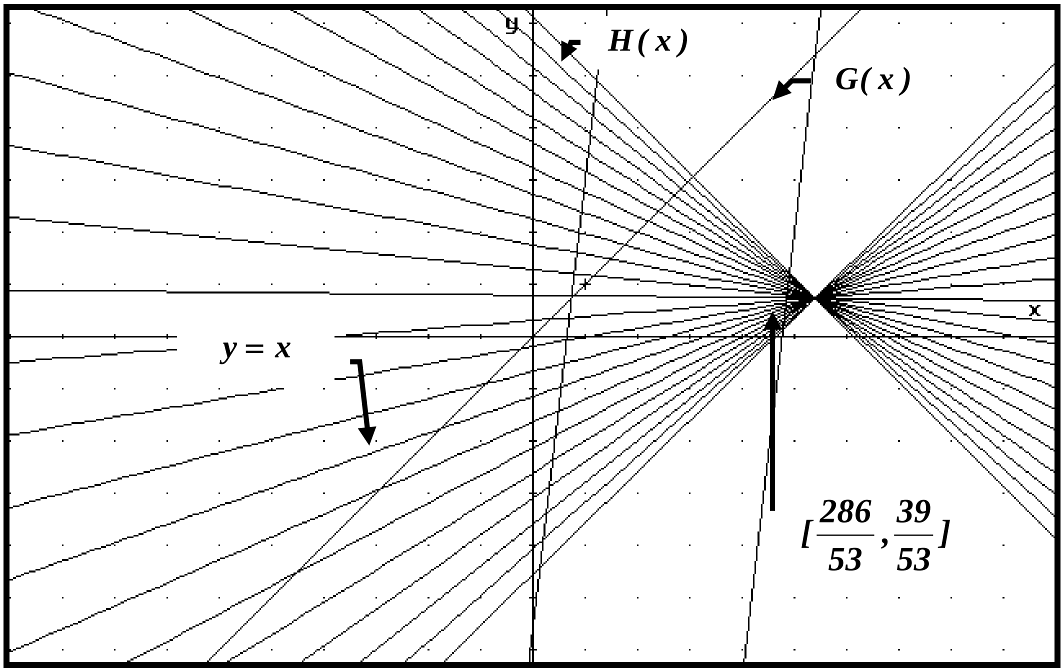

Figure 4. A 'pencil' of $F_9(x)$ functions

Another exciting Web of meaning was opened to the other teachers when teacher B talked about the graphical display she had created while she was trying to solve the magic circle problem of order **K**.

The design of a netcourse

Investigating non routine questions using *Derive* encourages diverse mathematical thinking. With this in mind we have designed special guidance techniques to help students solve such problems in a CAS environment. After analyzing the types of mathematical knowledge that were applied in solving the problems, we identified the relationships and the webs of meanings that were involved in the solution processes. We utilized computer software tools commonly used in the internet, such as html, CGI, Java Script, and e-mail to design an interactive guidance system in the format of Web pages. Three surfing paths are suggested:

'Assignments', *'Mathematical knowledge'* and *'How to use Derive'*.

Assignments - Problems and instructions are presented.

Mathematical knowledge - Mathematical concepts and skills connected to the subject matter are revealed. The students can study the new mathematical topics or just refresh their knowledge.

How to use Derive - Instructions for the use of the software.

The learner can navigate between paths. In every Web page we have installed six different icons:

One) An explanation: definitions of mathematical concepts.

Two) Technical support: guidance in the use of *Derive* for a particular purpose.

Three) Examples: demonstration of specific examples.

Four) Hints: reference to Webs of meanings, i.e., mathematical concepts, skills and ideas or additional explanations and examples.

Five) Solutions: answers to the problems with explanations and proofs.

Six) “At your request”: encourage students to ask or suggest what they want. It also encourages students to participate in conferences among themselves or with the tutors to enhance mathematical communication and construct webs of mathematical meaning.

Learning in the environment of the netcourse is under the control of the learner. The support system suggests possible paths rather than indicating specific way of accomplishing the task.

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