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The Use of *DERIVE* in Assessed Student Work

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Abstract

The issue of assessment in the presence of Computer Algebra Systems (CAS) has, to date, received little attention. The increasing prevalence of affordable computer systems means that more and more students are gaining access to and using CAS. Since it is impossible to ban their use in assignments, teachers have to pay particular attention to the design of questions that test mathematical knowledge and competence rather than the candidate's ability to do mathematics by computer with little or no understanding of what they are doing. This paper explores the use of *DERIVE* as an intrinsic part of assessed student work. The author provides examples showing the use of *DERIVE* in both assignment and examination papers and describes the performance of his students engaged in this type of assessment.

Introduction

As part of the undergraduate modular degree scheme at Anglia Polytechnic University, mathematics modules are taught in basic and advanced analytical techniques normally as part of engineering and technology based programmes. These modules are text book based and one of the stated learning outcomes is that students will be able to "*use appropriate mathematical software for both symbolic and numerical computation*". During the delivery of these modules, a PC coupled to an image projection system is used in the lecture theatre for both teaching and learning purposes. Students have commented favourably on the presence of this facility, claiming that the graphical use of *DERIVE* in particular has increased their understanding of the concepts and applications presented in lectures. In addition, students have an opportunity to indulge in the "What if?" interaction enabled by having the PC as an on line resource. Additional course materials take the form of *DERIVE* handouts which fully integrate with the explanations, problems etc. contained in the text books. As a consequence of regular exposure to *DERIVE* in both lectures and practicals, students are expected to take assessment either permitting or requiring its use. This is a challenge for the instructor who now has to design questions to best test mathematical ability in the presence of a CAS.

Assignments involving the use of *DERIVE*

Presented here, is a selection of problems set as part of assignments for modules in basic and advanced analytical techniques.

Having taught a class how to differentiate a function of a function, it is desirable to consider some general cases whereby the student can determine some "rules" for themselves as shown in this question.

Example 1 "Using *DERIVE*, **author** $f(x) :=$, then **author** $\sin(f(x))$. Use the **calculus, differentiate** commands with variable x and order 1 then **simplify**. This tells us that if you differentiate $\sin(f(x))$ you end up with $f'(x)\cos(f(x))$ irrespective of what $f(x)$ might be. Make a note of this result. Now use *DERIVE* to construct a table of derivatives for the following general cases:

(a) $\cos(f(x))$, (b) $\ln(f(x))$, (c) $e^{f(x)}$, (d) $(f(x))^n$"

The student can now be asked to state the derivatives of a variety of specific functions based on the rules determined above. The question can be further extended to deal with integrals of the form $\int f'(x)\sin(f(x))dx$ etc. This question is generally well answered except for the inevitable error in authoring the exponential "e". One of the aims of this question is to encourage students to think in terms of the general rather than the specific.

Example 2 This question makes use of *DERIVE*'s very useful DIRECTION_FIELD command.

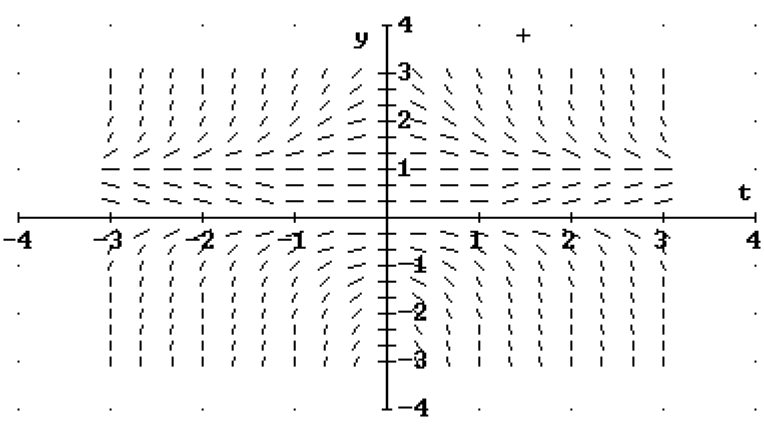
"Using *DERIVE* draw the direction field of the equation $\dfrac{dy}{dt} = y(1-y)t$.

(i) Superimpose, by hand, some of the solution curves suggested by the direction field.

(ii) Find, analytically, the general solution of this equation and plot some of the particular solutions.

(iii) Use an appropriate numerical technique (commands can be found in ODE_APPR) to find say 10 solution points between $t=0$ and $t=1$ lying on the curve which passes through the point with co-ordinates (0,1/2). Comment on your findings."

Comments: The direction field is as follows:



The analytical solution is given by $y = \dfrac{Ae^{t^2}}{1+Ae^{t^2}}$. It is quite a challenge for the student to now generate particular solutions to superimpose onto the direction field and to provide a full treatment for the three cases $A>0$, $A<-1$ and $A \in (-1,0)$. Moreover, determining the location of the variable vertical asymptotes stretches even the more able student. In practice, only a few students were able to obtain the explicit analytical solution and hardly any were able to

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generate superimposed solutions in the band $y \in (0,1)$. It is arguable whether one could set such a question without using a CAS.

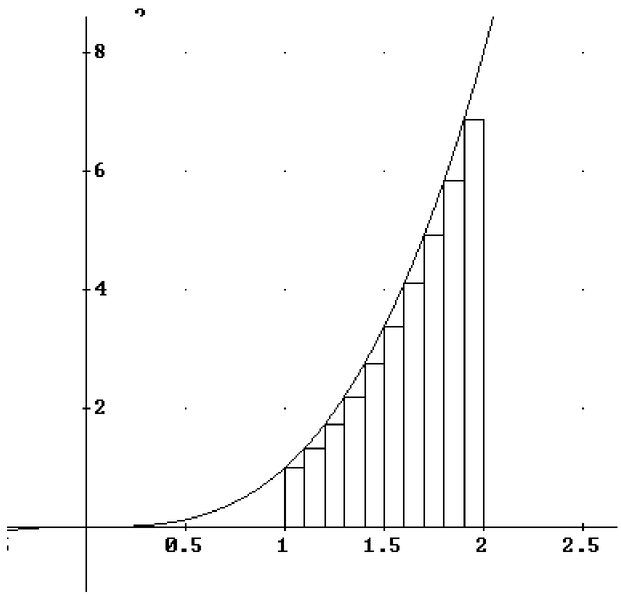
Examinations involving the use of *DERIVE*

Here we consider the design of examination questions based on a module delivery that makes frequent use of *DERIVE* and where direct access to the software during the examination is **not** permitted. Where appropriate, students are provided with relevant utility commands to help in answering questions. Using *DERIVE,* the examiner now has the advantage of being able to generate diagrams to assist in testing a variety of mathematical concepts as shown in the examples which follow.

Example 1 This example deals with the concept of a definite integral as the limit of a sum.

"The extract below shows how *DERIVE* can be used to find estimates of the area enclosed by the curve $y = x^3$, the x-axis and the lines $x = 1$ and $x = 2$.

```
                        3
#1:  LEFT_BOUND(x , x, 1, 2, 10)
                  3
#3:  S_LEFT(x , x, 1, 2, 10)
#4:  "Simplifying the above expression gives:"
#5:  3.4075
                  3
#6:  S_LEFT(x , x, 1, 2, n)
#7:  "Simplifying the above expression gives:"
```



Simplifying and plotting line 1 (line 2 has been omitted) results in the following

diagram

Explain the results shown on lines 5 and 8. By considering the limiting value of the expression shown on line 8 as $n \to \infty$, deduce the exact area enclosed by the curve $y = x^3$, the x-axis and the lines $x = 1$ and $x = 2$. Obtain this area by using direct integration."

Using calculus techniques when analysing rational functions can be heavy going. The next example shows how *DERIVE* can be used in designing a question in connection with the analysis of a rational function which tests interpretive skills rather than the ability to perform stodgy algebraic manipulation.

<u>Example 2</u> "The extract below shows a *DERIVE* session during which analysis is performed on $f(x) = \dfrac{2x^2 - x - 1}{2x - 3}$.

#1: $F(x) := \dfrac{2 \cdot x^2 - x - 1}{2 \cdot x - 3}$

#2: "Find the values of x for which numerator = 0"

#3: $\left[x = 1,\ x = -\dfrac{1}{2}\right]$

#4: "Find the values of x for which denominator = 0"

#5: $\left[x = \dfrac{3}{2}\right]$

#6: $\dfrac{d}{dx}\,\dfrac{2 \cdot x^2 - x - 1}{2 \cdot x - 3} = \dfrac{4 \cdot x^2 - 12 \cdot x + 5}{(2 \cdot x - 3)^2}$

#7: "Find the values of x for which the above expression = 0"

#8: $\left[x = \dfrac{1}{2},\ x = \dfrac{5}{2}\right]$

#9: $F\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$

#10: $F\left(\dfrac{5}{2}\right) = \dfrac{9}{2}$

#11: $\left(\dfrac{d}{dx}\right)^2 \dfrac{2 \cdot x^2 - x - 1}{2 \cdot x - 3} = \dfrac{16}{(2 \cdot x - 3)^3}$

#12: $\dfrac{16}{\left(2 \cdot \dfrac{1}{2} - 3\right)^3} = -2$

#13: $\dfrac{16}{\left(2 \cdot \dfrac{5}{2} - 3\right)^3} = 2$

#14: $F(x) := \dfrac{2}{2 \cdot x - 3} + x + 1$

Use the print out above to answer the following questions:

(i) What are the roots of $f(x)$?

(ii) Write down the equations of any vertical asymptotes

(iii) Write down the co-ordinates of any stationary points

(iv) What can be deduced from line 11 with regard to any points of inflexion?

(v) What do lines 12 and 13 confirm about the nature of any stationary points?

(vi) What can be deduced from line 14 regarding the behaviour of $f(x)$ as $x \to \pm\infty$?

Use the answers you have given to sketch the graph of $f(x)$ with all important points clearly labeled and indicate any asymptotic behaviour."

Comment: This is normally a popular question in an examination where choice is allowed. Parts (ii), (iv) and (vi) are the least well answered with very few candidates deducing the oblique asymptote given by $y = x + 1$.

Students are often deterred from tackling questions involving the generation of Fourier series because the integration required to determine the Fourier coefficients can be tedious. Harnessing the power of a CAS, it is possible to design more interesting

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questions that avoid the need to perform stodgy integration but test properties and applications of Fourier series which could not readily be done otherwise.

Example 3 "A "saw tooth" function $f(t)$ is defined in the following way:

$$f(t) = t \qquad -\pi < t < \pi$$

$$f(t+2\pi) = f(t) \text{ for all } t$$

(a) Sketch $f(t)$ between $t = -3\pi$ and $t = 3\pi$. Explain why the Fourier series representation for the saw tooth function will consist entirely of sine terms only. In the *DERIVE* session shown below, line 1 shows the simplest form of the integral expression for the Fourier coefficients.

#1: $\dfrac{2}{\pi}\cdot\displaystyle\int_0^{\pi} t\cdot \mathrm{SIN}(n\cdot t)\ dt$

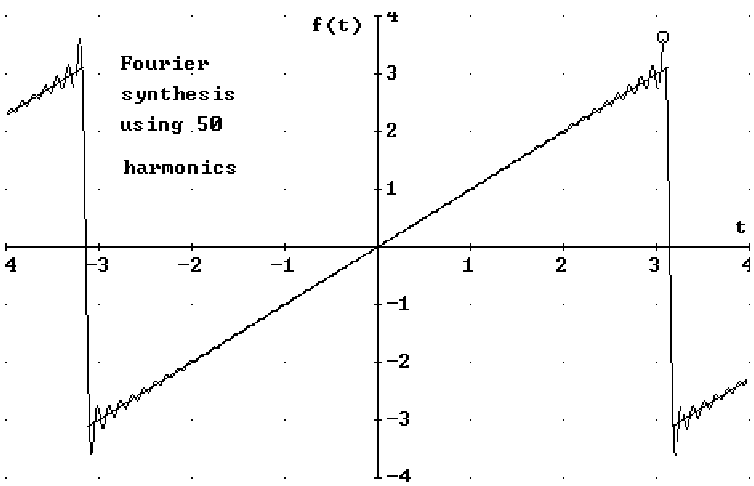
#2: "The above integral simplifies to:"

#3: $\dfrac{2\cdot \mathrm{SIN}(\pi\cdot n)}{\pi\cdot n^2} - \dfrac{2\cdot \mathrm{COS}(\pi\cdot n)}{n}$

By referring to the standard formulae for determining Fourier coefficients, explain briefly how this integral expression was formed. Using line 3, write down the first 5 terms of the Fourier series for the saw tooth function.

(b) By letting $t = \dfrac{\pi}{2}$ in the result obtained in part (a), find a series representation for $\dfrac{\pi}{4}$.

(c) With reference to the graph below showing the synthesised wave form obtained by using the first 50 harmonics, comment briefly on the behaviour around integer multiples of π.



The trace mode is on and shows the peak value just to the left of $t = \pi$ to have co-ordinates (3.0758,3.6389). Use the numerical information provided to estimate the total % overshoot and compare this with that given by the general theory."

Comments: Those candidates who chose to do this question performed quite well in comparison with those performances involving the construction of Fourier series by having to perform tedious integration by hand. Responses relating to the onset of Gibbs' phenomenon were particularly pleasing.

The way that the *DERIVE* software operates can be the source of some searching questions in connection with mathematical concepts such as convergency issues as shown by the following two examples.

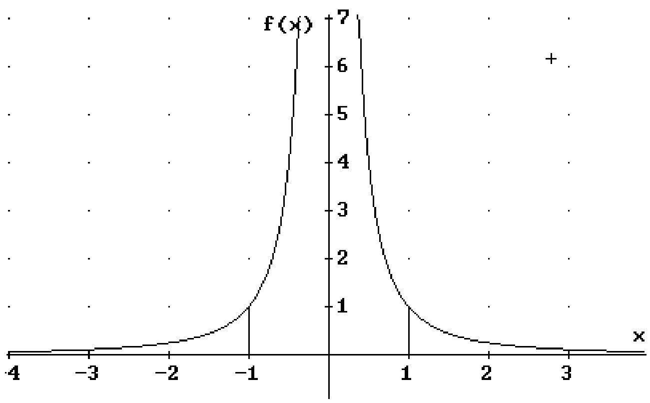
Example 4 "The session below shows how *DERIVE* can be used to find the Laplace transform of e^{3t}.

```
#1:  "Load in the utility file INT_Apps"
                3·t
#2:  LAPLACE(ê     , t, s)
#3:  ?
```

Explain why *DERIVE* returns "?" when we attempt to simplify the expression on line 2. Explain further how you would use *DERIVE* to give the desired result of $\frac{1}{s-3}$."

Example 5 "*DERIVE* is used to evaluate the definite integral $\int_{-1}^{1} \frac{1}{x^2} dx$. The result obtained via *DERIVE* and the area represented by this definite integral are shown below:

$$\#1: \int_{-1}^{1} \frac{1}{x^2}\, dx = -2$$



With reference to the area shown, explain why *DERIVE*'s result for the definite integral must be incorrect. How did *DERIVE* obtain the value –2 and what "mistake" did *DERIVE* make in the process? Find the true value of the given definite integral."

Here we consider the design of examination questions based on a module delivery that makes frequent use of *DERIVE* and where direct access to the software during the examination is permitted. The author has successfully arranged such examinations for small groups of students but several practical considerations need to be made. The configuration of the examination room requires careful attention. For example, there needs to be access to both PCs and writing space in such a way that students are unable to see their neighbours' screens. It is inadvisable to use *DERIVE* on a network for security reasons and to avoid problems caused by a potential network crash during the examination. Where appropriate, students are provided with relevant utility commands to help in answering questions. Part of the examination paper rubric states:

"*The computer algebra package DERIVE is available for use during this examination but, except where stated otherwise, each step of a solution must be shown and explained. In those cases where you may have used a utility command, make a note of the values that you allocated to the parameters in the command. When specified,* *any graphs produced by DERIVE should be copied over to paper by hand in sketch form."*

Two examples of examination questions permitting the use of *DERIVE* now follow.

<u>Example 1</u> "(a) Draw the direction field of the equation $\frac{dx}{dt} = 2x(x-1)$, where t varies from –2 to 2 in 12 steps, and x varies from –2 to 3 in 15 steps. Sketch some of the solution curves suggested by the direction field. Verify that the general solution of the equation is $x = \frac{1}{1-ke^{2t}}$, where k is an arbitrary constant and check that the members of this family resemble the solution curves you have sketched from the direction field. (*12 marks*)

(b) Use the integrating factor method to solve the equation $\frac{dx}{dt} + 3x = 2e^{-t}$, given when $t = 0, x = 2$. By using *DERIVE*'s EULER command with $h = 0.1$ and $n = 8$, obtain an approximation for $x(0.5)$ and compare this value with the exact value. (*13 marks*)."

Comments: The question was answered either quite well (marks of 18/25 or above) or quite poorly (marks of 6/25). Obtaining the direction field was good as was general parameter passing. Aspects of the question that were poorly handled were mainly in connection with separating the variables and with integration. In general, students did not even take advantage of the CALCULUS command to give them a clue as to what the results should be. Also there was no evidence of access to the SEPARABLE _GEN or LINEAR1 commands. None of the students who attempted this question made use of the VECTOR command to generate particular solutions by varying k in part (a). Some general observations and remarks are given at the end of this paper.

<u>Example 2</u> "A repeating waveform of period 2π is described by:

$$\begin{cases} \frac{1}{2}t + \pi & (-\pi \le t \le 0) \\ \frac{1}{2}t - \pi & (0 \le t \le \pi) \end{cases}$$

(i) Define $f(t)$ in terms of the unit step function and hence use *DERIVE* or otherwise to sketch a few cycles of this waveform.

(ii) Find the Fourier series representation of $f(t)$, making use of any properties of the waveform that you can identify before any integration is performed. **Note**: any integrals may be evaluated using *DERIVE*.
By using the first three non-zero terms of the Fourier series, find an approximation for $f(\pi/3)$."

Comments: The above question was worth 25 marks. Those students who attempted this question did well (marks of 19/25). However, as with the previous example, there was no evidence of the use of *DERIVE*'s FOURIER command, but the VECTOR command was used to generate the first few Fourier coefficients from the general form.

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Conclusions

In the author's view, allowing the use of *DERIVE* in student assessed work has resulted in more emphasis being placed on investigation and interpretation instead of mechanistic manipulation. This is not to suggest that manipulative skills are unnecessary, but a balance needs to be sought and we can no longer ignore the increasing prevalence of CAS on ever decreasing smaller devices. With the presence of software packages such as *DERIVE*, new skills can be identified and tested such as parameter passing. We can also test intelligent and efficient use of the software such as using the VECTOR command. It may even be reasonable to test programming skills in *DERIVE* e.g. the use of boolean operators. After several years of using *DERIVE* as an intrinsic part of module delivery for both teaching and learning, the author is of the view that the students who best use the software are those who are more able and confident in Mathematics. It was most interesting to observe the way in which students made use of *DERIVE* in examinations. More often than not, they demonstrated an inability to make the best use of the software and did not even take full advantage of the routine menu commands. Quite often, in lectures, we are inclined to make statements such as "Well, you can do this on *DERIVE*", or "Try this on *DERIVE* later" etc. What the student rarely sees is how to best use the software to work through a problem. It is for this latter reason that having the PC on-line in the lecture room has proved invaluable for both the students and the instructor.

REFERENCES

Graham, E., Berry, J.S. and Watkins, A.J.P. (1997) Mathematical Activities with *DERIVE*, Chartwell-Bratt 67-77 – Kempski, B.L. Concepts of Differential and Integral Calculus.