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Combining Visual and Symbolic Skills in the Teaching and Learning of Mathematics

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Introduction

Given the ease with which computers can generate animated, 3-D colour, pictorial displays of information, it is not surprising that calls are made for educators to use visual displays to try and enhance student understanding and to develop in students the ability to interpret and analyse visual data. The desire to include visualisation skills as a more prominent feature in mathematics education has been discussed for example by Leinbach et. al.[1], and Cochrane [2]. The former consider the impact of a CAS on generic mathematical skills such as generalisation, conjecture, etc. The latter makes the point ' .. we should expect an increasingly pictographic, interactive and virtual world to predicate radical changes in mathematical representation. The ultimate challenge is to achieve faster and higher levels of understanding for an audience with inherently less time available and shorter attention spans. Electronic visualisation is currently the only answer.'. Is the above 'radical change' likely to influence an education process that is (has been?) relatively conservative over the years?

Current computing technology, whether it be hand-held such as the TI-92, computer algebra systems (CAS) such as DERIVE, or bespoke teaching software, should enable more emphasis to be placed on the use of visualisation in teaching to enhance student understanding of mathematical concepts. However, the shift from a more conventional pattern of teaching mathematical processes with perhaps the use of visualisation at the end of the process, to one where the visual aspects of a concept are integrated into the teaching as a whole is not always straightforward. In the UK for example, pioneering work by Tall and his co-workers (e.g. [3],[4]) has led to the production of visualisation software available for the teaching of concepts in 16-19 year A-level mathematics and beyond, but this does not appear to be in widespread use at this time. Informal discussions with teachers at this level suggest that the issue is one of emphasis on the analytic form and its manipulation being dominant in the teaching, and the visualisation aspects being played down in the classroom. This analytical prowess is often perpetuated by the nature of the assessment process which tends to favour the routine use of symbolic processes to solve problems. Such a state of affairs is not new (see for example [5]).

In this paper, the authors are concerned to discover the extent to which the integrated use of an algebra and a graphics window in DERIVE could be used to effect a greater degree of understanding of mathematical concepts. Coupled with this, questions are asked such as 'can students interpret visual information and assimilate it into symbolic notation?' and if so 'how competently can they do this?' and 'does the use of a CAS help?'. In terms of a skills base, the emphasis should be more on interpretation and conjecture rather than one of following a mechanical process. The use of algebraic, numerical and graphical windows and the switching from one representation to another has been referred to as the Window Shuttle method [6]. Here the authors illustrate some teaching and learning advantages of such an approach via examples chosen from the area of functions and graphs. This topic naturally lends itself to a high level of visualisation and much research has been undertaken with this subject domain (see for example [5],[7] and indeed the whole of the text indicated in reference [7]). The visual skills needed to interpret and then conjecture are generic to many branches of mathematics however and the need to combine symbolic and visual skills applies in many mathematical topics.

Visualising Functions

Malabar [8] has been investigating the mathematical skills of 16-19 year old, pre-University, mathematics students in the UK, with a view to determining the extent to which their visual abilities aid their higher order, cognitive skills. The research was based on students who had little or no exposure to DERIVE or other visual software in their teaching or learning and had studied the subject of functions. The sort of questions posed were :

When given a problem such as

$$\int_{-\pi}^{\pi} \sin(x)\ \mathrm{dx}$$

how many students automatically perform the integration and input the limits rather than 'see' that the answer is zero by consideration of the sine graph? Is it just the 'bright' students who see the answer quickly or is it that students are more trustful of an answer obtained by symbolic manipulation? A similar situation arises when having been taught integration by parts, many students will not hesitate to use that method to show that

$$\int_{-1}^{1} x^3 \cos(x)\ \mathrm{dx}$$

is zero also, without any consideration of graphing the integrand.

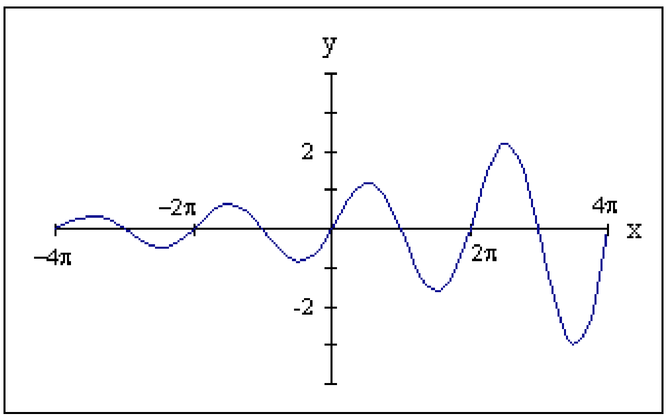
Students are tested on their ability to curve sketch by such questions as:

Sketch the graph of $f(x) = x^4 - 5x^3 + x^2 + 9x + 2$.

How many can then proceed to use this to find the range of values of k such that the equation $x^4 - 5x^3 + x^2 + 9x + k = 0$ has precisely two roots?

Most students can competently substitute values into a given function form to produce a graph sketch (and confirm it using a graph plotting calculator). How readily can they interpret graphical information to derive an appropriate symbolic form of the function? For example:

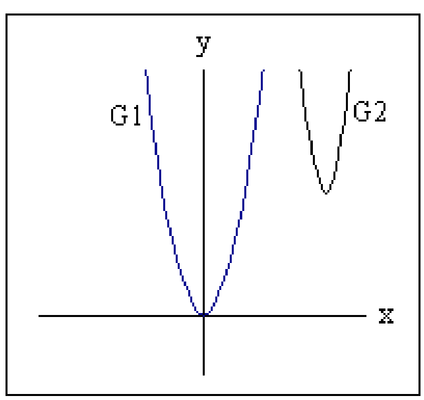
The graph below is made up of the addition or multiplication of some or all of the functions $\sin(x)$, $e^{0.1x}$, $e^{-0.1x}$. Find the symbolic representation of the function.



The final two examples are taken from a paper by Galbraith and Haines [9] who were testing for interpretative skills on their first year University students.

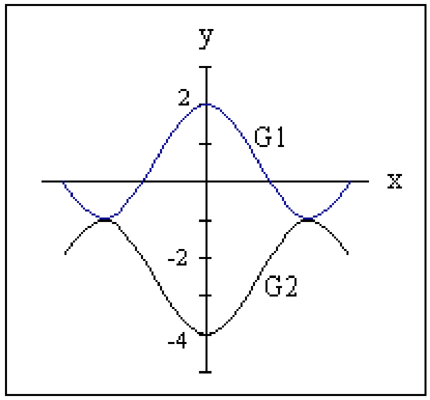
Graph G1 below has the equation $y = 2x^2$. Which of the following might describe the graph of G2?

A: $y = (x - 3)^2 + 2$ **B**: $y = 2(x - 3)^2 + 2$ **C**: $y = 4(x - 3)^2 + 2$
D: $y = (x - 3)^2/4 + 2$ **E**: none of the above.



Graph G1 below has the equation y = f(x) and hence G2 has the equation

A: y = - f(x) - 2 **B**: y = - f(x) **C**: y = f(x) - 6
D: y = f(-x) **E**: none of these.



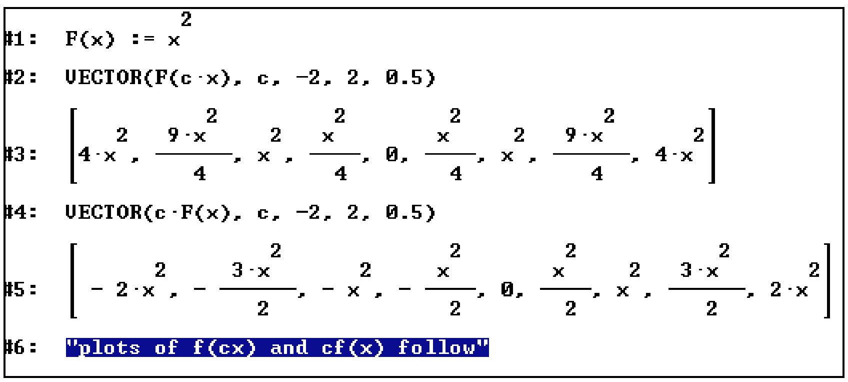
These last two examples were included to test the belief that many students have difficulty in grasping the concepts behind transformations of functions such as translation, rotation, enlargement, etc. The first example gives multiple-choice answers in terms of symbolic representations, but co-ordinate values are excluded from the diagram so that students cannot achieve the correct answer merely by substituting co-ordinates into the functional form. The latter question, without any symbolic form, was generally answered badly by all students taking part in the research.

A more complete description of the testing procedure, the questions posed and the outcomes can be found in [8], but it seems generally the case that these students would invariably try to solve problems initially by symbolic manipulation whenever possible and that the assessment process at this level generally seems to encourage that approach. Although not conclusive, it was observed that students studying statistics at this level generally performed as well at the visual questions as in the algebraic ‘procedural’ type of question compared with those students studying mechanics who tended to favour procedural questions. This may be because statistics students are more naturally inclined to plot data and interpret graphs in the initial stages of their work. It may also be that mechanics students have to manipulate and solve numerous algebraic equations and that this is seen as a more important task than producing and interpreting the correct force/vector diagram.

Teaching Visualisation Skills with DERIVE

Malabar [8] attempted to see if visual skills and, more importantly, an appropriate balance between symbolic manipulation and visual thinking could be achieved by using visual software and a ‘constuctivist’ approach to the teaching of functions and in particular function transformations. A teaching approach using DERIVE would allow interaction between the algebra and graphics windows so that the students could experiment by observing the effect on a given function f(x) when plotting f(x + c), f(cx), cf(x), f(x) + c, etc. (see for example [10]). After these experiments, the students are encouraged to conjecture general results. In a similar way, examples are presented which ask students to experiment by combining simple functions to plot and match a given, more complex function in the graph window. For success, the students must be able to interpret visually and also understand the effect of arithmetic operations on functions, function composition, etc.

These experiments are now described in more detail. A function f(x) whose graph is familiar to students is chosen to act as a reference figure. This is defined in an algebra window at the start and from then on all further investigation is performed with f(x) rather than with the explicit algebraic form. Students are encouraged to plot f(x + c), f(x) + c, f(cx), cf(x) for a range of values of a constant c (including c = 0), in each case producing a family of plots as c varies. For example, the DERIVE commands for the case f(x) = x^2 and for f(cx) and cf(x) together with their respective plots are illustrated below.


#1: $F(x) := x^2$

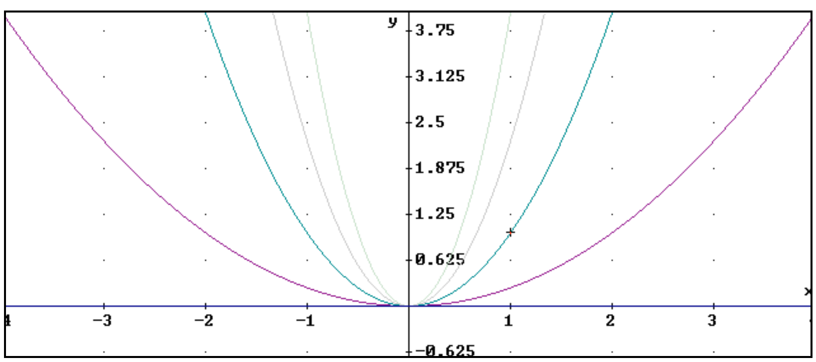
#2: VECTOR(F(c·x), c, −2, 2, 0.5)

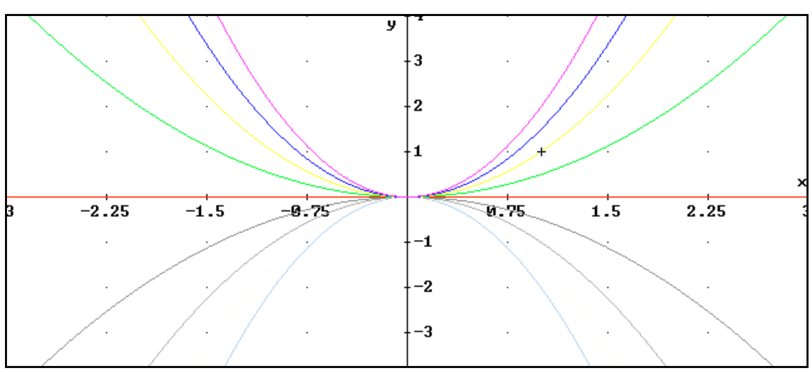
#3: $\left[4\cdot x^2,\ \frac{9\cdot x^2}{4},\ x^2,\ \frac{x^2}{4},\ 0,\ \frac{x^2}{4},\ x^2,\ \frac{9\cdot x^2}{4},\ 4\cdot x^2\right]$

#4: VECTOR(c·F(x), c, −2, 2, 0.5)

#5: $\left[-2\cdot x^2,\ -\frac{3\cdot x^2}{2},\ -x^2,\ -\frac{x^2}{2},\ 0,\ \frac{x^2}{2},\ x^2,\ \frac{3\cdot x^2}{2},\ 2\cdot x^2\right]$

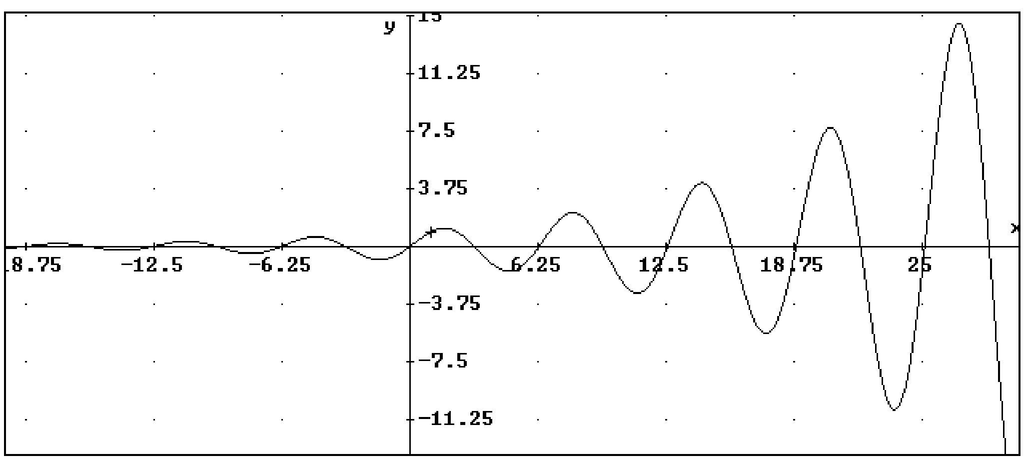
#6: "plots of f(cx) and cf(x) follow"





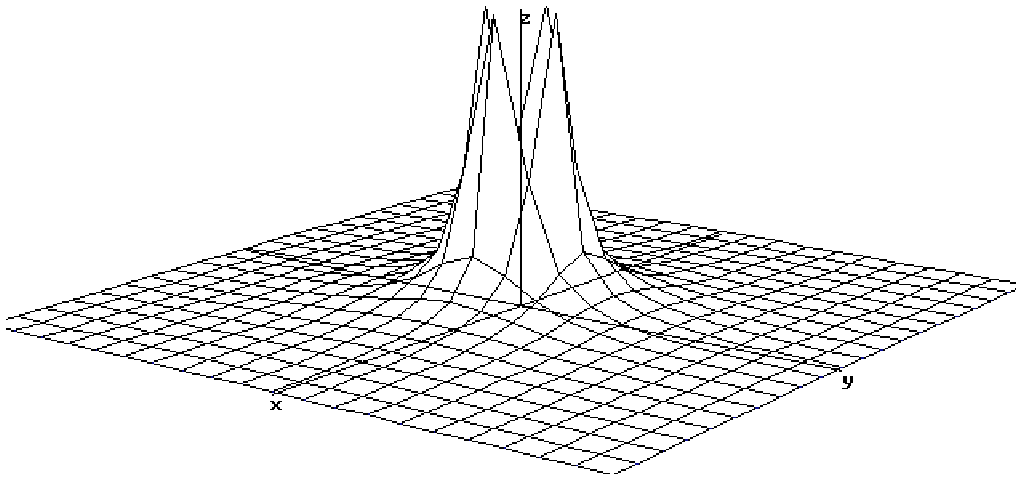
Note that the VECTOR command allows for easy generation of families of curves but needs to be Simplified prior to plotting. Students see the functional form of the curves being plotted and have the opportunity to link together (for example) what f(cx) means both symbolically and visually compared with the original function f(x). Experiments with other basic functional forms can be easily repeated by changing the definition of f(x). Malabar's work suggests that this investigative approach helps students to 'think pictorially' in terms of shifts, scaling, etc., rather than the attempt to algebraically transform the function form and re-plot it and also appears to motivate further investigation work.

Another variant of this investigative approach was to present students with a graph plot in a plot window opposite a blank algebra window, and ask the students to try and produce a functional form which when plotted would match the given plot exactly. For example, given the graph below and the added help that it was composed of some arithmetic combination of sin(x) and $e^{0.1x}$ only, students were encouraged to find this combination. Trial and error approaches soon give way to more serious thought as to what clues the given picture has to offer such as 'what happens as x increases (decreases)?', what happens at x = 0 (y = 0)?' and 'how does this relate to the addition (subtraction, multiplication, division) of sine and exponential functions?'



Similar questions relating to the composition of sine and cosine wave forms of varying amplitude, frequency and phase, problems common in engineering mathematics, can also be considered in this way to reinforce the idea of what changes in amplitude, frequency and phase mean pictorially.

The above was restricted to functions of one variable primarily because of the limits of the syllabus. An extension to $z = f(x , y)$ can easily be achieved via DERIVE and should add to the students' understanding of functions. For example, using DERIVE to investigate whether $g(cx , y)$ is the same as $cg(x , y)$ for a constant c and function $g(x , y)$ is a straightforward task. Trying to match the graph below, for example, with a symbolic form made up of powers of x and/or y, or their inverses and addition, subtraction, multiplication in some way is a more taxing (but more fun?) analytical task for the 'brighter' students.



Visualising with DERIVE in the Future?

Given the relative ease with which complex data can now be represented pictorially, it would seem important for this interaction between visual and symbolic senses to be a feature even more emphasised in future CAS design. For example, in future versions of DERIVE, the above teaching scenarios would be enhanced if the graph window had a tool which could take a given function plot, interact with it by (say) stretching it and the resulting algebraic effect be automatically recorded in the algebra window. Another example could be when teaching differentiation by first principles. It would be handy to have a tool to select two points on a curve so that a chord is drawn between them and the gradient of the chord displayed. In this way the gradient of the tangent as a limiting process can be easily investigated and the analytical approach using the ratio $(f(x+h)-f(x))/h$ could be conjectured by students rather than being fed to them at the start. Clicking on a curve to display the values of the first or higher derivatives at a point might help students understand the concepts of stationary points more easily, especially in three dimensions when discussing saddle points.

All of the above are given as examples where the use of DERIVE's computer algebra capability might encourage more of an integrated approach using both symbolic and visual forms in the teaching and learning of mathematics, particularly amongst pre-University students. Assessment processes are still likely to determine teachers' priorities in the near future, but perhaps some of the above ideas could help DERIVE to be seen by students and teachers as even more of a 'mathematical assistant'.

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