Explorations in Plane Geometry in Cabri and Derive Environment

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Introduction

Information technologies have changed all kinds of human activities. Mathematics is not the exception - it becomes technologically dependent. The most important changes take place in the process of doing mathematics - discovering new facts and their proof. Special mathematical packages offer the user with suitable environment for arranging computer experiments in the problem field with the aim of finding mathematical regularities on the first step of exploration and then support the process of proof with the powerful opportunities of computer algebra. No doubt that the future of mathematics is symbiosis of a human and computer. Using mathematical packages becomes the inalienable component of mathematical culture.

Innovative trends in mathematical education lay in the framework of constructive approach - involving students in the process of constructing their own mathematical system which consists of mathematical knowledge and mathematical beliefs. One of the most effective ways of the realization of constructive approach is the method of learning explorations when students explore open-ended problems on their own. Solving open-ended problems can be regarded as the model of the professional mathematical work. Therefore it is naturally to use information technologies in mathematical education just in the same manner: computer experiments as the source of powerful ideas, computer algebra as a tool of deductive method. Using information technologies in arranging learning explorations and carrying out the proofs can not only do this work more effective but acquaint students with modern technologies of mathematics.

The authors have had the pleasure of playing Cabri-Geometer in one well-known problem [1] and as a result have conjectured its generalization which then they have proved by the help of Derive. They were highly impressed with these computer games and decided to share their experience with their students. Thereby the 2-week computer practice with undergraduate students at the Mathematical Teachers Department of Kharkov state pedagogical university in 1997 was organized in the following form: 1st week - acquainting with packages Cabri-Geometer and Derive, 2nd week - playing with the above mentioned problem with the aim of its generalization. The students have coped with the posed task successfully without any help. The current article outlines activities which were used in the process of generalization of this problem (it is rather common for authors and students).

Problem area

Two quadrates $A_1QC_1D_1$ and $A_2B_2C_2Q$ with the centers O_1 and O_2, and common vertex Q of the identical orientation are given. Let E and F are the midpoints of the segments A_1A_2 and C_1C_2. Then the quadrangle O_1EO_2F is a quadrate as well.

Haw can the above mention problem be generalized?

Computer experiments in the problem solving

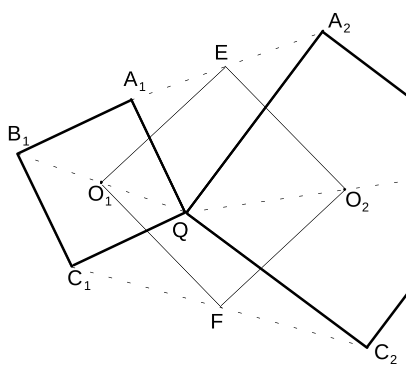
Experiment 1

At first we construct a computer model of a given problem in the Cabri-Geometer environment. With this purpose we construct at first a macro-construction Quadrate which draw a quadrate using two given points as the endpoints of the diagonal.

Construct a computer **Model 1** of a problem:

1. Construct a macro-construction *Quadrate* which draws a quadrate on two given points as the ends of the diagonal.
2. Define three points D_1, Q and B_2 and draw the squares $A_1QC_1D_1$ and $A_2B_2C_2Q$ using pairs of points D_1, Q and Q, B_2 as the endpoints of the diagonals.
3. Draw the segments A_1A_2, QB_2, C_1C_2, D_1Q.
4. Define points E, O_2, F, O_1 as the midpoints of the segments constructed in the previous step 3.

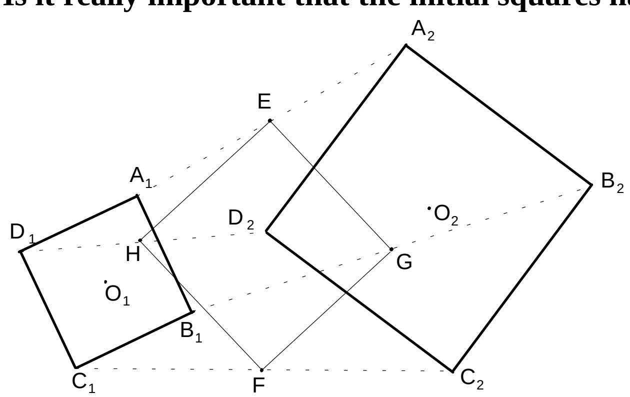
Now the model is ready and we can play with it. Really, our model is dynamic («alive») in the sense that we can dynamically change its parameters. All initial points *(D_1, Q and B_2)* are movable - they can be moved with the mouse. Playing with this model we really see that the third (depending) quadrangle is a square (we can convince in it by view or by measuring the parameters of a figure with the *Cabri-Geometer* instruments - ruler and protractor). Of course our experimental assurance needs the deductive proof, it is only the hypothesis, but this hypothesis is of a great confidentiality. We put aside now the attempts to prove the hypothesis and proceed our computer experiments.



Experiment 2 **(disjointing the vertexes)**

Is it really important that the initial squares have a joint vertexes?

Unfortunately, the previous model cannot be modified for these purposes. Therefore we must repeat the described above algorithm with a little change in the second step - the initial points will be four independent points B_1, D_1, B_2, D_2, the diagonal points of the future quadrates.



By playing with a new **Model 2** we can convince ourselves that in this case the resulting quadrangle is a square as well. As a result of described experiments we can formulate the next hypothesis:

Generalization 1

Two quadrates $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ with the centers O_1 and O_2, of the identical orientation are given. Let *E, F, G, H* be the midpoints of the segments A_1A_2,, B_1B_2, C_1C_2 , D_1D_2 . Then the quadrangle *EFGH* is a quadrate as well.

Experiment 3 (moving the middle points)

Why must the vertexes of the resulting figure be the midpoints?

Maybe the result remains valid in the case of arbitrary points which divide the segments joining the correspondent points in a given proportion?

Modify our previous model. With this purpose a new macro-construction would be needed. This construction will be the macro-construction Divide which divides the segment in a ratio given by the segment and a point on it. Such a macros can be defined with the ideas of the Fales's theorem for example. Then the construction of the **Model 3** we modify with this macros picking an arbitrary point *E* on the segment A_1A_2 and then defining all the other points *F, G, H* with the macros *Divide* as points which divide the correspondent segments in the same ratio.

Now with the mouse we can change the sizes of the squares and their mutual positions and the place of the division points as well. As in previous case we can easily convince ourselves that the resulting figure is a square. Thus we obtain the next generalization.

Generalization 2

Two quadrates $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ with the centers O_1 and O_2, of the identical orientation are given. Let A_2, B_2, C_2 and D_2 be the points of the segments A_1A_2, B_1B_2, C_1C_2 and D_1D_2 respectively which divide them in the same rate. Then the quadrangle *EG**FH*** is a quadrate as well.

Proof in Derive environment

Prove the generalization 2 with *Derive*.

Let $O_1(a, b)$ and $O_2(c, d)$ are the centers of the squares $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$. Denote vectors $O_1A_1 = n_1\,(p , q)$, $O_2A_2 = n_2\,(r , s)$.

Declare vector constants in package *Derive* (using commands *Declare*, *Constant*):

$O_1 := [a, b]$; $O_2 := [c, d]$; $n_1 := [p, q]$; $m_1 := [-q, p]$; $n_2 := [r, s]$; $m_2 := [-s, r]$.

Express coordinates of the vertexes of the squares (using the same commands *Declare*, *Constant*):

$A_1 := O_1 + n_1$; $B_1 := O_1 + m_1$; $C_1 := O_1 - n_1$; $D_1 := O_1 - m_1$; $A_2 := O_2 + n_2$; $B_2 := O_2 + m_2$; $D_2 := O_2 - m_2$; $C_2 := O_2 - n_2$;
Declare the function *Divide*:
$Divide\,(x , y , t) : = (x + ty) / (1 + t).$
Declare coordinates of the points *E, G, F, H*, which divide the segments A_1A_2, B_1B_2, C_1C_2, D_1D_2 in the ratio $t : 1$:
$E := Divide\,(A_1, A_2, t)$; $G := Divide\,(B_1, B_2, t)$;
$F := Divide\,(C_1, C_2, t)$; $H := Divide\,(D_1, D_2, t)$.
Find the difference of the vectors *HE - FG:*
$(E - H) - (G - F)$.

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Simplifying this expression (command *Simplify*) we obtain***:*** *[0, 0]*. Consequently, the quadrangle *EGFH* is a parallelogram.
Evaluate now the scalar product of the vectors *FG* and *GE* (input in author mode the expression *(G - F).(E - G)* and then simplify (command *Simplify*) this expression we obtain zero**.** Thus we have proved that the quadrangle *EGFH* is a rectangle.
Compare the length of the sides *FG* and *GE* of this rectangle. For this purpose declare a new function *SModV* (**S**quare of the **Mod**ule of the **V**ector):
SModV(x): = x.x

Evaluate the difference: *SModV(G - F) - SModV(E - G)*. Simplification of this expression gives zero, consequently this rectangle is the square. Remark that the square *EGFH* can degenerate in a point.

Further generalization

Experiments with arbitrary similar quadrangles allow one to formulate the following generalization:

Generalization 3

Let F_1 and F_2 are two similar figures in the plain of the same orientation (it means that there exists a similarity *f* of the first class, which maps the figure F_1 onto the figure F_2. For each point X_1 of the figure F_1 and a point X_2=f(X_1) of the figure F_2 define a point X, which divide the segment X_1X_2 in a ratio *t* $(X_1X : XX_2 = t)$ Then the figure *F*, which consists of all such points *X*, is similar to two original figures or consists of the single point.

Proof this hypothesis.

The radius-vector of a point *M* will be denoted as $\overline{M}$ in further.

Let X_1 and Y_1 are two arbitrary points of the figure F_1. Let $X_2 = f(X_1)$ and $Y_2 = f(Y_1)$.

Denote through *X* and *Y* the points, dividing the segments X_1X_2 and Y_1Y_2 in a ratio *t*.

Then:

$\overline{X} = (\overline{X}_1 + t\overline{X}_2) / (1 + t)$,
$\overline{Y} = (\overline{Y}_1 + t\overline{Y}_2) / (1 + t)$.
As a consequence we obtain: $\overline{XY} = (\overline{X_1Y_1} + t\,\overline{X_2Y_2}) / (1+t)$.

Therefore we have: $|\overline{XY}|^2 = (\overline{X_1Y_1}^2 + 2t\,\overline{X_1Y_1}.\ \overline{X_2Y_2} + t^2\,\overline{X_2Y_2}^2) / (1+t)^2$.

Denote as φ the angle between the ray X_1Y_1 and its image under the similarity *f*. This angle φ is a constant because of the map *f is of the* first class similarity. With respect to the relation $|X_2Y_2| = k|X_1Y_1|_1$, where *k* is the coefficient of the similarity we finally obtain:

$$|\overline{XY}|^2 = \frac{1+2t\cos\varphi + t^2k^2}{(1+t)^2}|X_1Y_1|^2.$$

Consequently $|XY| = C|X_1Y_1|$, where *C* is a constant.

Thus, the figure *F* is similar to the figure F_1 if $k > 0$, in the case *k=0 the figure F* is a point.

Remark that in the most general case (**Generalization 3**) the proof was done without the computer help and was more simple and natural. It is rather general situation because of from the general point of view the unimportant detail are disappearing and the matter of the fact became more clear and obvious. Nevertheless the computer experiments have played the substantial role in the process of generalization. By the way the discussed above proof could be done on a computer as well but all the analytic calculations were so simple that the computer help was unnecessary.

Last remarks

As it is clearly seen from the given above proof the fact stay valid in the case of an arbitrary dimension (not only in the case of *2D*, but *3D*, *4D* etc. as well).

The successfully proved fact can be used in computer animation for modeling continuous transformations of figures in plane or space.

Other problems

In this paragraph we'll discuss some new results in plane geometry (by our mind), which were discovered through the computer experiments in Cabri and then proved in Derive by authors just at the same mentioned above manner. The details of the correspondent experiments and proofs will not be discussed in view thereby the lack of space (the methodology and technique of them could be shared from the previously discussed problem.

Theorem 1

Let three equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ and $A_3B_3C_3$ of the same orientation are given. Let the points P,Q,R are the middle points of the segments C_1B_2, C_2B_3 and C_3B_1 respectively. Then the triangle PQR is quadrilateral if end only if the triangle $A_1A_2A_3$ is quadrilateral.

Remark 1

The necessary part of the Theorem 1 is well-known (see for example [2], p. 100), the sufficient part is seemed to be new.

Theorem 2

Let two equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ of the same orientation are given. The equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$ and $C_1C_2C_3$ of the same orientation are built on the segments A_1A_2, B_1B_2 and C_1C_2 respectively. Then the triangle $A_3B_3C_3$ is equilateral.

Remark 2

Last two theorems as it could be easily seen are the particular cases of the next theorem which could proved in Derive in general form as well.

Theorem 4

Let n regular m-polygons at the plane $A_1^1A_1^2\ldots A_1^m$, $A_2^1A_2^2\ldots A_2^m$ and $A_n^1A_n^2\ldots A_n^m$ of the same orientation are given. Then if two n-polygons $A_1^1A_2^1\ldots A_n^1$ and $A_1^2A_2^2\ldots A_n^2$ are regular then all the remain n-polygons $A_1^iA_2^i\ldots A_n^i$ $(i=3,4,\ldots,m)$ are regular as well.

Literature

Contents

1. Boltyansky V.G., About one parquet, Mathematics in School, 1984, № 1, p.65-66.

2. Skopets Z.A., Geometric miniatures, Moscwa, «Prosveschenie», 1990, 224p.