DERIVE in ODE : some examples

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Examples of the advantages of using *DERIVE* in the classroom in an Ordinary Differential Equation course (ODE course) are described in this paper. Our students are engineering students who have already taken a one variable calculus course. They still solve problems by hand and *DERIVE* is used to provide graphics support, to check exact solution and to perform calculations in numerical methods. The pedagogical qualities of the software allow teacher to continue to focus on maths.

1- Introduction

We have used *DERIVE* (for DOS and, since january 1997, for Windows) in our Differential equations courses at the École de technologie supérieure (school of engineering) since september 1994. Parts of the utility files **solve.mth**, **dif_apps.mth**, **int_apps.mth**, **ode1.mth**, **ode2.mth** and **ode_appr.mth** were used to create our own ODE file called **kit235.mth**. We added the improved Euler method, a function that automates the sequence of the Picard iteration, a function that gives the components of the Fourier coefficients, functions that perform numerical integration and pictures of Riemann sums ; we gave examples how to create the Dirac impulse « function » and, finally, a function for the convolution of two signals. We found important that students continue to recognize the kind of ODE they are asked to solve. It is a remarquable pedagogical aspect of the **ode1** file with functions like LINEAR1, SEPARABLE, etc ...

In the following paragraphs, we are going to show some examples. For those who might be interested in our complete paper (written in French), just e-mail to us.

2- Piecewise continuous right hand side

We all know that *DERIVE* doesn't have a « Laplace library ». Maybe one reason is the fact that the functions of **ode1** and **ode2** (or, at the end, the INT function) will accept discontinuous functions in the input. With the STEP and CHI functions, it is very easy to do so. Consider the following example.

Example 1 : we want to solve an *RC*-circuit with a constant voltage during a finite interval of time. More precisely, a resistance of 100kΩ = 10^5 Ω and a capacitance of 10μF = 10^{-5} F are connected in series with an emf $E(t)$ such as

$$E(t)=\begin{cases}10000\text{ V} & \text{if } 1<t<3\\ 0 & \text{else}\end{cases}.$$

At $t = 0$, the capacitor voltage is 0. So, we need to solve the first order linear equation

$$\frac{dv_C}{dt}+\frac{v_C}{RC}=\frac{E(t)}{RC}$$

where $RC = 10^5\,10^{-5} = 1$, $E(t)$ = 10000CHI(1, t, 3) and $v_C(0) = 0$. We use *DERIVE* for Windows to compute the capacitor voltage. That kind of problem is normally solved by Laplace transform. The capacity of integrating piecewise continuous function allows *DERIVE* to handle that kind of problems without having to wait too long! When we author

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LINEAR1(1, 10000·CHI(1, t, 3), t, v, 0, 0)
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and then, simplify and plot, we obtain the following graph as an eloquent illustration :

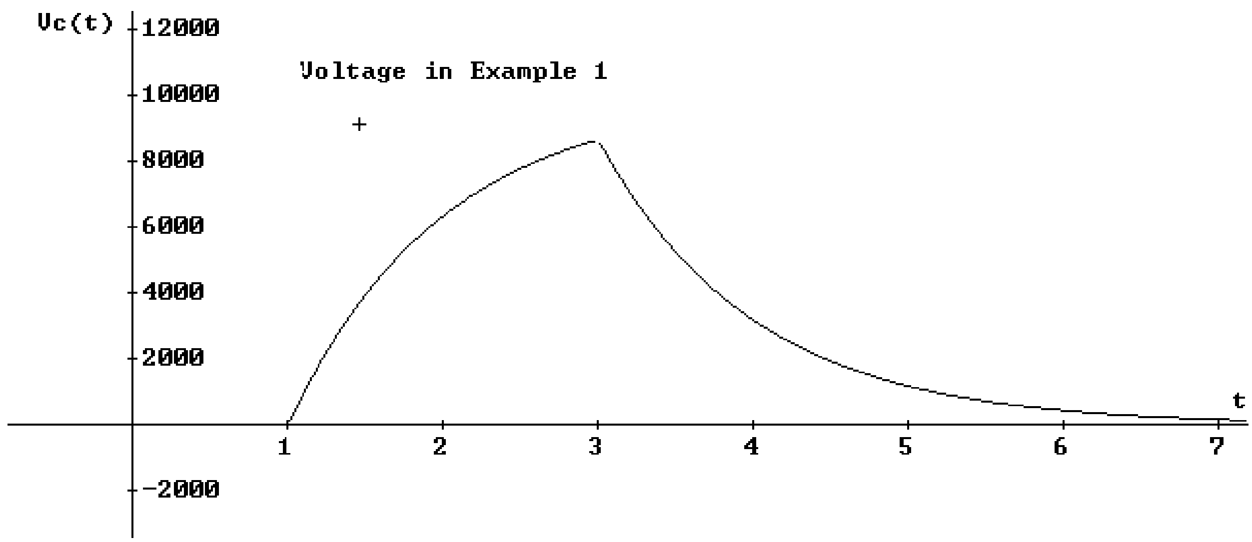

FIGURE 1

We see that the capacitor is beginning to charge at $t = 1$ and, at $t = 3$, the charge begins to decrease to 0. The current at time t will simply be $i(t)=C\dfrac{dv_C}{dt}$. The point $t = 3$ in the above graph will cause a discontinuity in the current. The current is

$$-\frac{ê^{3-t}\cdot\mathrm{SIGN}(t-3)}{20}+\frac{ê^{1-t}\cdot\mathrm{SIGN}(t-1)}{20}+\frac{ê^{1-t}\cdot(1-ê^{2})}{20}$$

and figure 2 shows its graph. We note the jump discontinuity at $t = 3$:

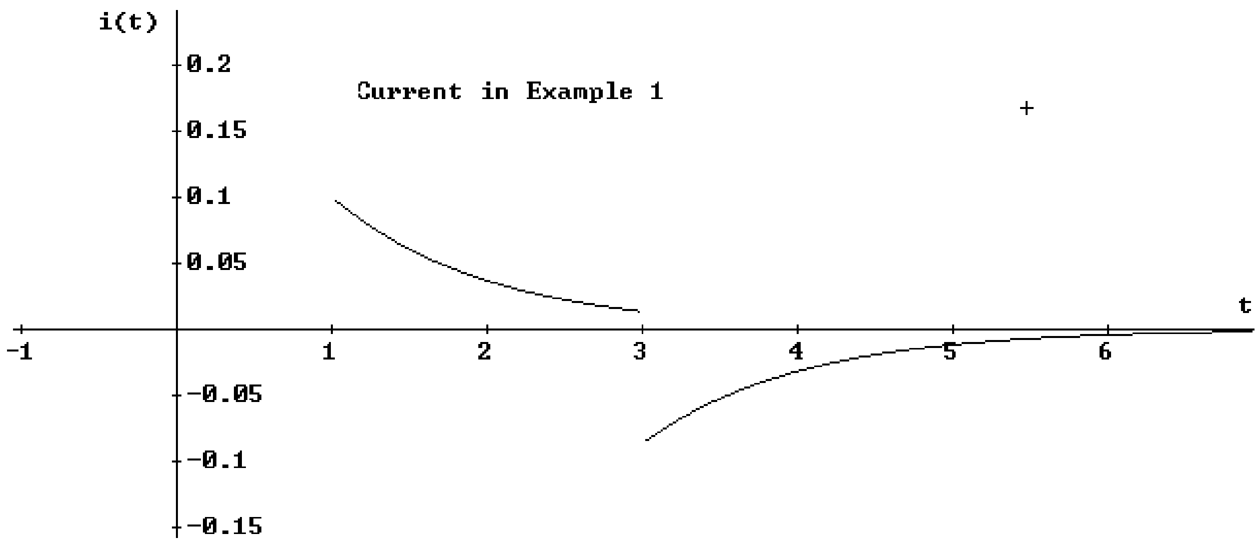

FIGURE 2

Moreover, the jump in the graph for this emf is always the ratio of the emf and the resistance (1/10 here). Let *DERIVE* work with appropriate limits (from right and from left) :

$$i := -\frac{\hat{e}^{3-t}\cdot \mathrm{SIGN}(t-3)}{20} + \frac{\hat{e}^{1-t}\cdot \mathrm{SIGN}(t-1)}{20} + \frac{\hat{e}^{1-t}\cdot (1-\hat{e}^{2})}{20}$$

$$\left(\lim_{t\to 3-} i\right) - \lim_{t\to 3+} i = \frac{1}{10}$$

3- No Dirac delta « function » in *DERIVE*? So what!

In most classical ODE courses, the Dirac delta « function » is presented as the limit of indicator functions. We can use functions of the form $\frac{1}{\varepsilon}\mathrm{CHI}(0,t,\varepsilon)$ to approximate $\delta(t)$. After, we just need to perform a limit as $\varepsilon \to 0^{+}$. We think it's a good way to understand the meaning of that « special function ». In vibrational motion, what does an external force like $2\delta(t - 8)$ mean? To answer that question, let's consider an example.

Example 2 : suppose we have an object of mass 1 kg suspended from a spring with spring constant of 4N/m. The object is initially at rest. At $t = 0$, an external force $f(t)$ is applied to the system. What will be the motion of the object?

Letting $y = y(t)$ denoting the motion, the differential equation is

$$\frac{d^2y}{dt^2} + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

We will first consider $f(t) = f_1(t)$ where $f_1(t) = \begin{cases} 3t/5 & \text{if } 0 < t < 5 \\ 3 & \text{if } 5 < t < 7 \end{cases}$

$$\mathrm{f1} := \frac{3 \cdot t}{5} \cdot \mathrm{CHI}(0,\ t,\ 5) + 3 \cdot \mathrm{CHI}(5,\ t,\ 7)$$

and we author

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DSOLVE2_IV(0, 4, f1, t, 0, 0, 0)
```

The simplification yields the following graph :

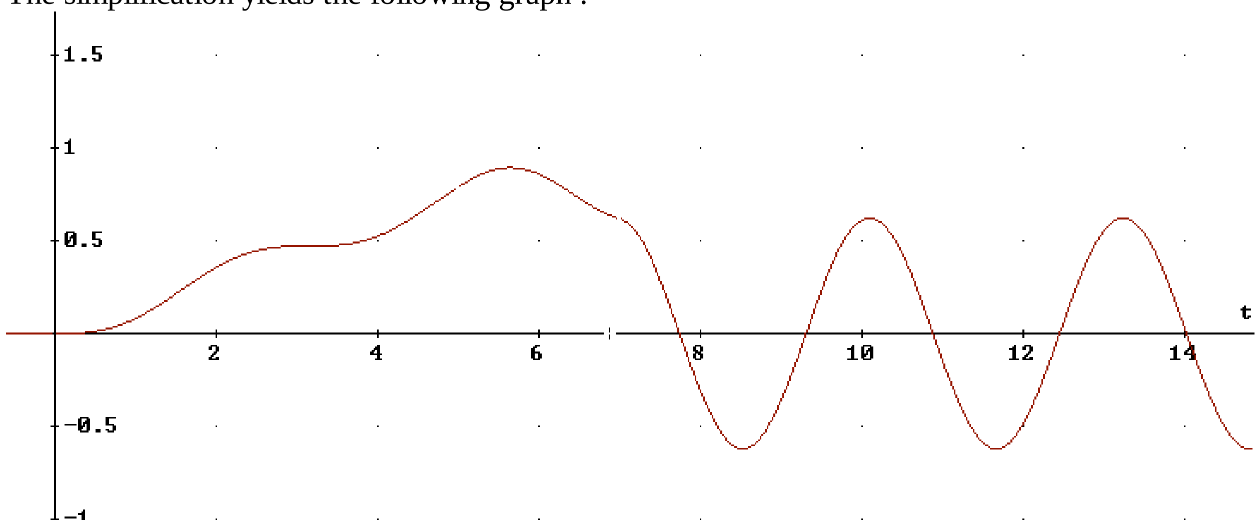

FIGURE 3

If we declare the variable t to be greater than 7 and if we use trigonometry collect, we obtain the movement after $t = 7$:

```
t :∈ Real (7, ∞)
Trigonometry := Collect
0.6217764441·SIN(2·t + 0.2326493804)
```

So the amplitude of the movement is about 0.62. What will be the effect on the movement if we add to the external force an impulse of 2 Ns at $t = 8$ ($2\delta(t-8)$) ? In order to answer this question, we define a function that will help. First, we don't forget to go back to original « t » :

```
t :=
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$$\mathrm{f2} := \frac{2}{\varepsilon} \cdot \mathrm{CHI}(8,\ t,\ 8 + \varepsilon)$$

The above function $f_2(t)$ is not $2\delta(t-8)$ but so is $\lim_{\varepsilon \to 0^+} f_2$. Simplifying and plotting the result of

$$\lim_{\varepsilon \to 0+} \mathrm{DSOLVE2\_IV}(0,\ 4,\ \mathrm{f1} + \mathrm{f2},\ t,\ 0,\ 0,\ 0)$$

w get the same curve as in figure 3 for $t \le 8$ but the movement changes when $t \ge 8$. Figure 4 compares the two curves :

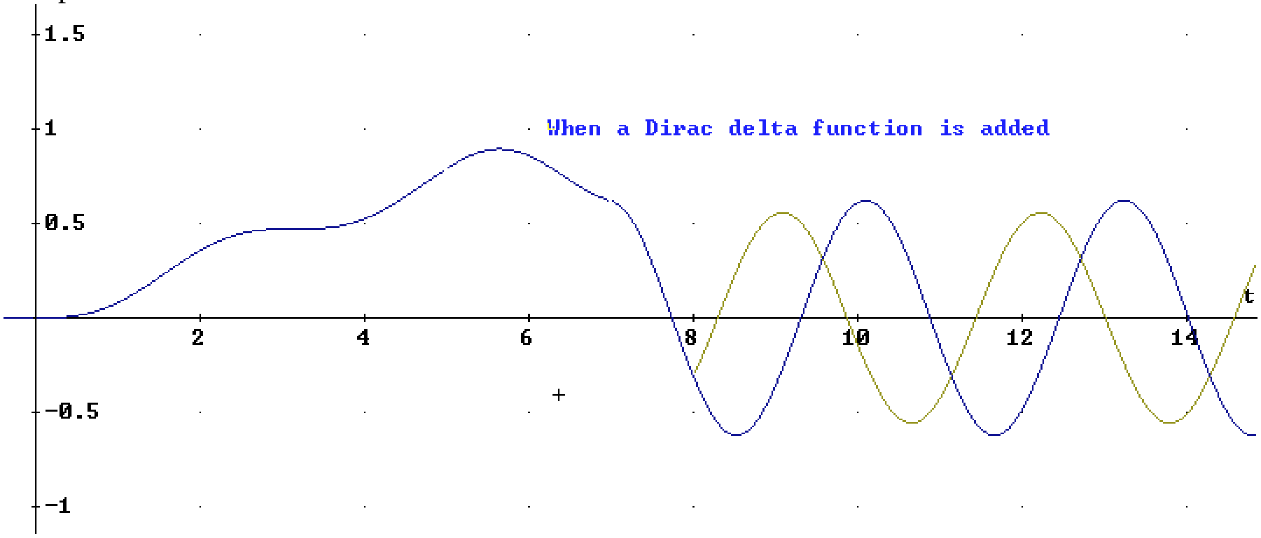

FIGURE 4

Amplitude has decreased too :

`t :∈ Real (8, ∞)`

`0.5570765398·SIN(2·t − 10.31014066)`

Example 3 : in signals courses, convolution is defined on the whole line by the integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

It is an important concept. If $x(t)$ represents the input in a linear time-invariant system with response impulse $h(t)$, then $y(t)$ is the output. The Dirac delta « function » is the neutral element of convolution and, because of the shifting proprerty, we can easily see that $y(t) * \delta(t-a) = y(t-a)$. Before continuing, we define the convolution in *DERIVE* , using the word CONVOL :

$$\text{CONVOL}(x,\ h,\ t) := \int_{-\infty}^{\infty} \left(\lim_{t\to\tau} x\right) \cdot \lim_{t\to t-\tau} h \, d\tau$$

Computing the convolution of two signals by hand is quite long and we have to check all cases for the variable t. For example, let $x(t)$ be a rectangular pulse and $h(t)$ a ramp :

`[x := − 2·CHI(0, t, 2), h := (2·t + 2)·CHI(−2, t, 1)]`

Properties of convolution tell us that the resulting convolution will last between –3 and 3 (if two signals $x(t)$ and $h(t)$ are right-sided and also left-sided, they have finite duration and the signal $y(t) = x(t)*h(t)$ has finite duration, the sum of the two durations). Here is the convolution :

`y := CONVOL(x, h, t)`

`y = − (t + 1)·|t − 3| − t·|t + 2| + (t + 3)·|t − 1| + (t − 2)·|t|`

It is interesting to note the absolute values, indicating which values of *t* have to be checked when performing the convolution by hand. Graph of the above result is shown in the following figure :

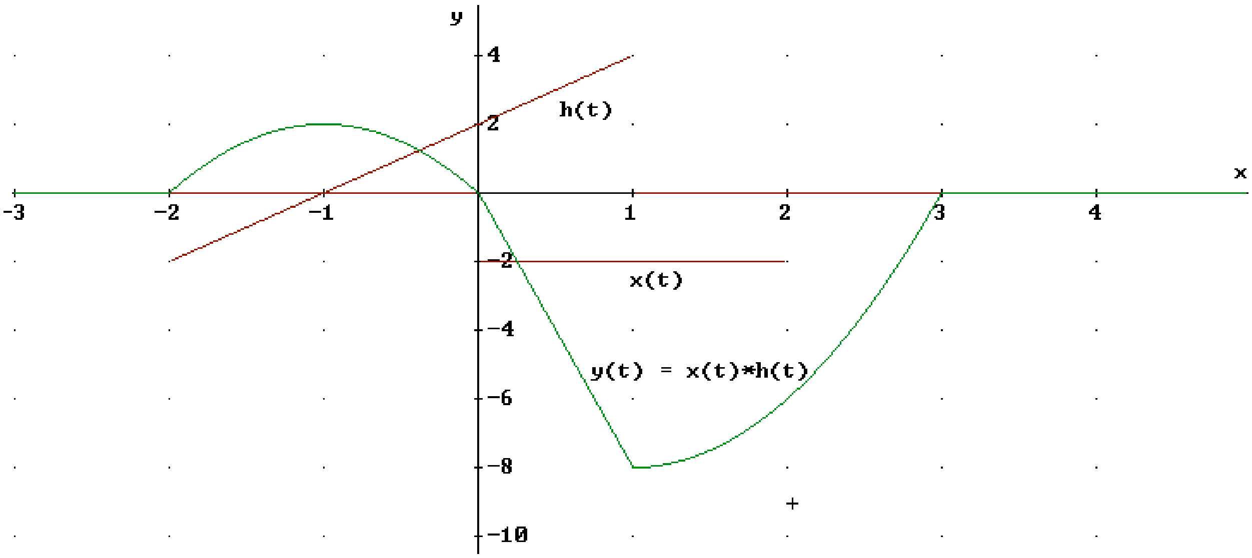

FIGURE 5

Finally, let us see that $y(t)*\delta(t-a) = y(t-a)$ for $a = 1$. We only have to simplify and plot

$$\lim_{\epsilon \to 0+} \text{CONVOL}\left(y,\ \frac{1}{\epsilon}\cdot\text{CHI}(1,\ t,\ 1+\epsilon),\ t\right)$$

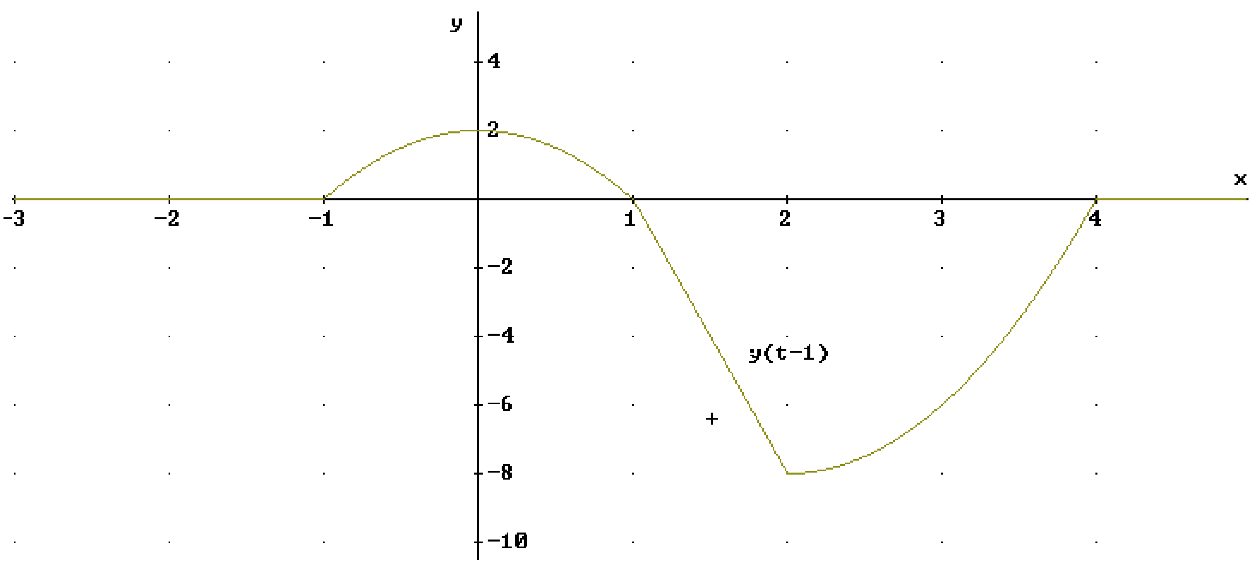

FIGURE 6

As we can see in the above figure, this is exactly the graph of $y(t-1)$.

4- Observing the Gibbs phenomenon

Fourier series are very interesting if we can plot partial sums of the periodic function! And, as shown by Jos Verhoosel in Plymouth (July 1994), the MOD function in *DERIVE* will allow us to plot a periodic function on the whole line. It is difficult with students that are not mathematics students to prove the Gibb's phenomenon. We illustrate with the square wave in our next example.

Example 4 : let $f(x) = \begin{cases} -\dfrac{\pi}{2} & \text{if} \quad -\pi < x < 0 \\ \dfrac{\pi}{2} & \text{if} \quad 0 < x < \pi \end{cases}$ and $f(x + 2\pi) = f(x)$.

This is an odd function and the Fourier series is $2\sum_{n=1}^{\infty} \dfrac{\sin(2n-1)\pi x}{2n-1}$ as confirmed by *DERIVE* for $n = 3$ for example :

$$\#1: \quad F(x) := -\frac{\pi}{2}\cdot \mathrm{CHI}(-\pi, x, 0) + \frac{\pi}{2}\cdot \mathrm{CHI}(0, x, \pi)$$

$$\#2: \quad \mathrm{FOURIER}(F(x), x, -\pi, \pi, 5) = \frac{2\cdot \mathrm{SIN}(5\cdot x)}{5} + \frac{2\cdot \mathrm{SIN}(3\cdot x)}{3} + 2\cdot \mathrm{SIN}(x)$$

Writing $s_n(x) = 2\sum_{k=1}^{n} \dfrac{\sin(2k-1)x}{2k-1}$ and using the Dirichlet conditions, we can write that $\lim_{n\to\infty} s_n(x) = \dfrac{\pi}{2} \cong 1.57$ if $0 < x < \pi$. Increasing n is a good way to be convinced of this : here's a graph of f, s_3, s_{10} and s_{50} :

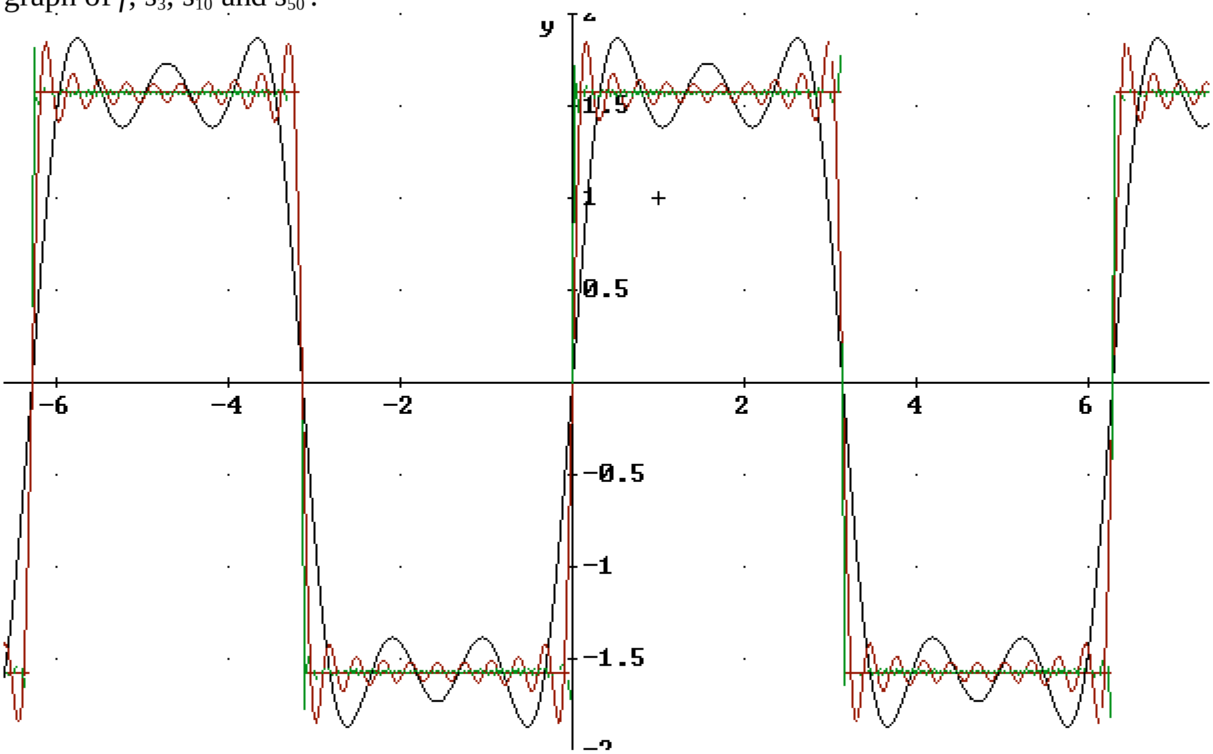

FIGURE 7

But, if we look carefully, the first maximum on the right of zero doesn't seem to go to 1.57 but to 1.85. This is the Gibbs phenomenon : there exists a number σ such that

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{2n}\right) = \sigma = \int_0^{\pi} \frac{\sin y}{y}\, dy \cong 1.85$$

(the overshoot don't disappear !). We will prove this using *DERIVE* as a mathematical assistant.

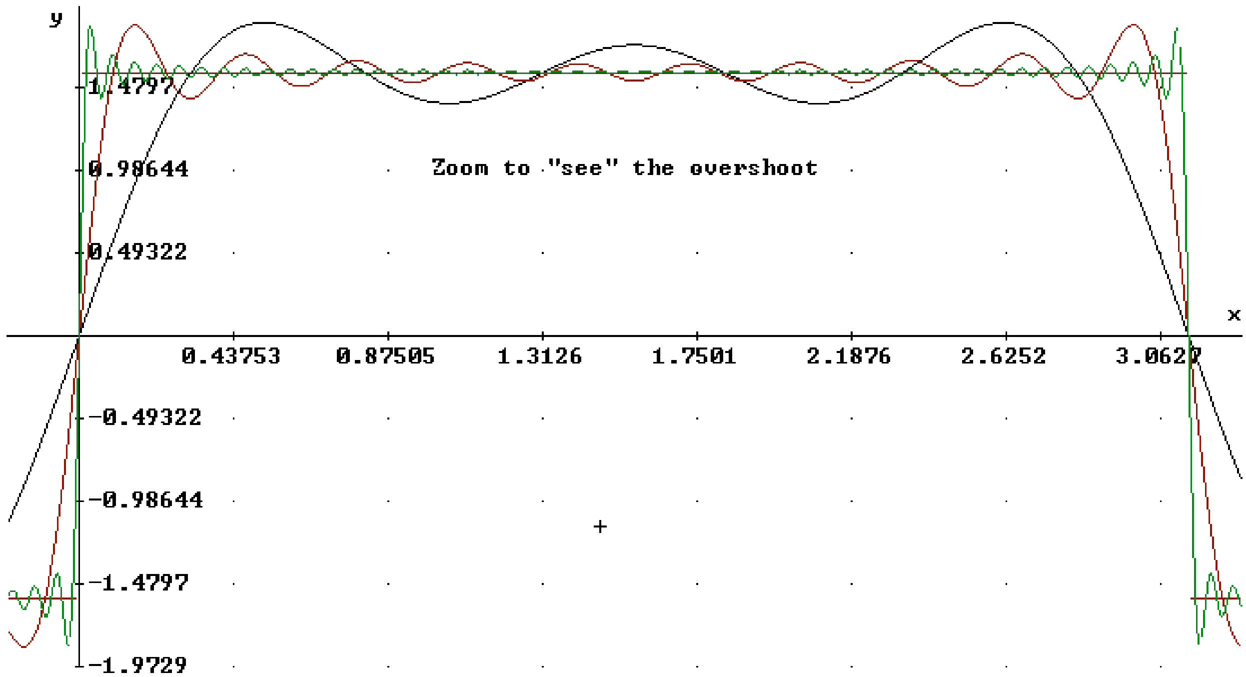

FIGURE 8

First, we show that the first critical point located on the right of zero is a maximum (clear from the graphs !) whose x-coordinate is $\pi/(2n)$:

#3: $S(x, n) := \sum_{k=1}^{n} \frac{2\cdot SIN((2\cdot k - 1)\cdot x)}{2\cdot k - 1}$

#4: $\frac{d}{dx} S(x, n) = \frac{SIN(2\cdot n\cdot x)}{SIN(x)}$

#5: $SOLVE\left(\frac{SIN(2\cdot n\cdot x)}{SIN(x)} = 0, x\right) = \left[x = \frac{\pi}{2\cdot n}, x = -\frac{\pi}{2\cdot n}\right]$

#6: $S''\left(\frac{\pi}{2\cdot n}, n\right) = -\frac{2\cdot n}{SIN\left(\frac{\pi}{2\cdot n}\right)}$

In line #6, second derivative is negative because $\pi/(2n)$ is between 0 and π. This shows that $x = \pi/(2n)$ is the first maximum located just after 0. As n goes to infinity, this point gets closer to 0, but its y-coordinate stays at about 1.85. To prove this, let us point out that

$$\#7: \quad S\left(\frac{\pi}{2\cdot n}, n\right) = 2\cdot \sum_{k=1}^{n} \frac{\mathrm{SIN}\left(\frac{\pi\cdot k}{n} - \frac{\pi}{2\cdot n}\right)}{2\cdot k - 1}$$

We want to perform the limit, as n goes to infinity, of the right hand side of #7. *DERIVE* won't find it. But this right hand side is a Riemann sum : the midsum of the function sin(y)/y (thanks to Benny Evans at www.math.okstate.edu for those files in numerical integration).

$$\#8: \quad \mathrm{MID}\left(\frac{\mathrm{SIN}(y)}{y}, y, 0, \pi, n\right) = 2\cdot \sum_{i=1}^{n} \frac{\mathrm{SIN}\left(\frac{\pi\cdot i}{n} - \frac{\pi}{2\cdot n}\right)}{2\cdot i - 1}$$

Here is a picture of the midsum with n = 20 :

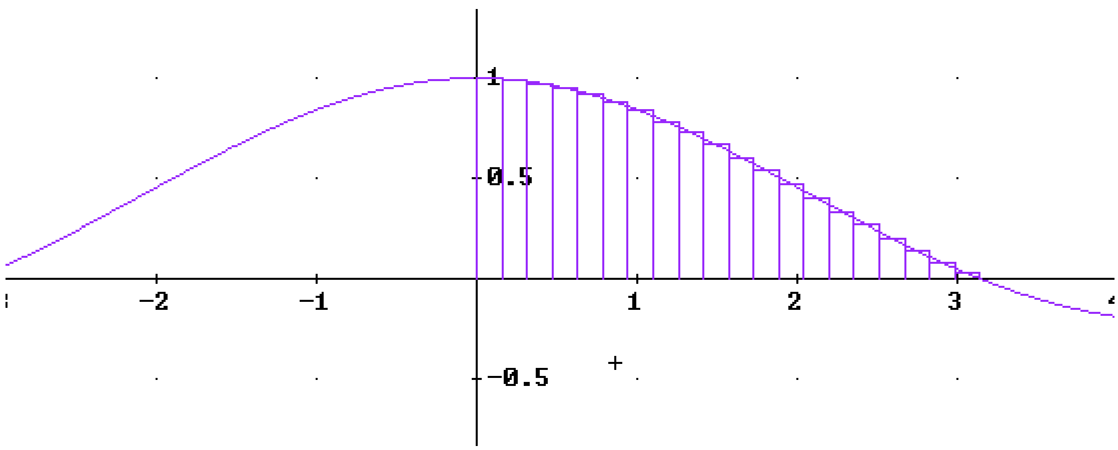

FIGURE 9

Consequently, the limit is just the value of $\int_0^{\pi} \frac{\sin y}{y}\, dy$ and *DERIVE* will approximate it :

$$\#9: \quad \int_0^{\pi} \frac{\mathrm{SIN}(y)}{y}\, dy$$

$$\#10: \quad \mathrm{APPROX}\left(\int_0^{\pi} \frac{\mathrm{SIN}(y)}{y}\, dy, 10\right) = 1.851937051$$

5- Conclusion

Other examples using direction fields, Euler and Runge-Kutta methods, Picard iteration, Taylor series methods are interesting and quite easy to solve with *DERIVE*. They are now <u>entire parts</u> of our ODE courses. Our students go 3 or 4 times to the lab during the semester. They get 2 or 3 homeworks.

Special thanks to Anthony P. Ferzola who gave us some ideas for using *DERIVE* in a ODE courses during a workshop in Plymouth, U.K. in July 1994 and to Benny Evans for his remarquable article in the IDJ (Volume 2 No. 3) and for the workshop in Bonn, Germany, July 1996.

I have to say that Albert Rich and David Stoutemyer—the fathers of *DERIVE*— created a sofware that allows teachers like myself to continue to focus on maths *while using* the software : this is why I like *DERIVE* so much. Finally, Theresa Shelby made a superb job for the Windows version. Mahalo to all of them!

Finally, we started to use the TI-92 in class with some students (there is no computer in class for students, only in labs) and they were astonished to see how easy it is to define our own functions that will solve first-order differential equation or apply the method of variation of parameters and so on!

We will appreciate ideas and comments from anyone interested in ODE and *DERIVE*.

6- References

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