Contents

Exploring Population Models on the TI-92 and TI-92 Plus Using Slope-Fields, Exponential, and Logistic Regression

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Adapted with permission from Hughes-Hallett, Deborah and Andrew M. Gleason, et al, **Calculus**, *John Wiley & Sons, 1994, pp. 526-534.*

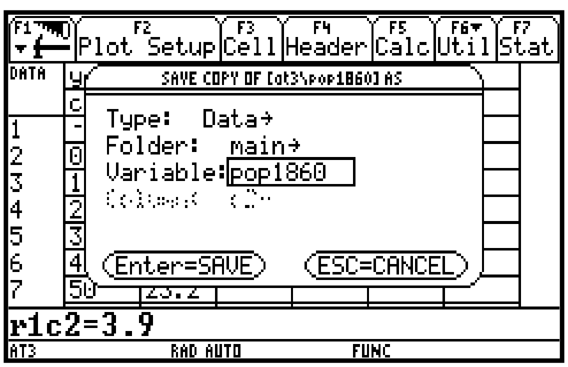
US Population, Exponential Approximation

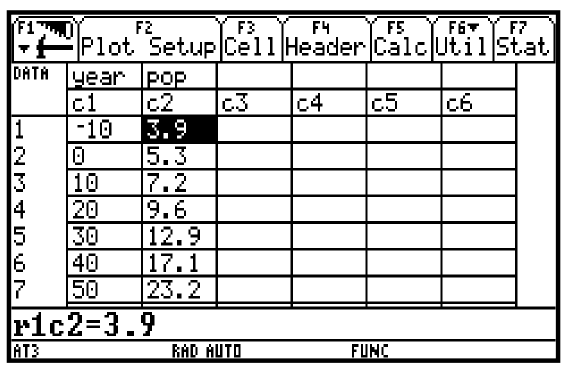
We frequently approximate population data by an exponential model. By using US census data, we see that an exponential model is appropriate until the Civil War, but data for more years fits a logistic model (which also fits the pre-Civil War data).

Year	1790	1800	1810	1820	1830	1840	1850	1860
Pop	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4

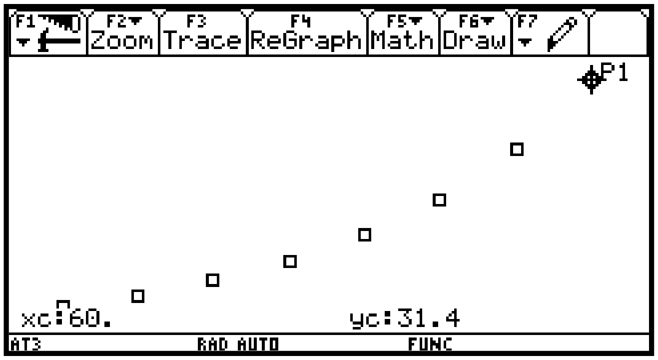
US Population census data in millions

In entering the above US census data (in millions) from 1790 to 1860, make it easy to determine the year by letting t = 0 in 1800, thus t = -10 in 1790. Press **APPS** **6:Data/Matrix Editor** **3:New** Type: **Data** Folder: **Main** Variable**:** enter **pop1860** **ENTER** Type -10, 0, 10, 20, . . . , 60 in column 1 for the years, right arrow and up arrow to the top of column 2, enter the population data above.



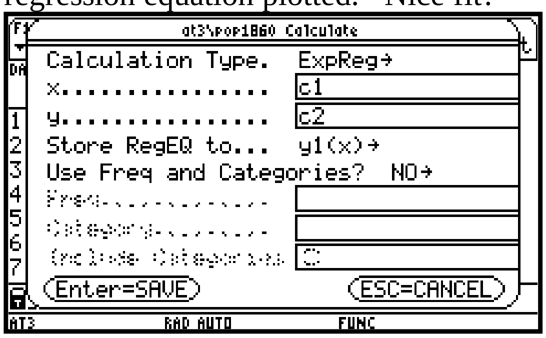


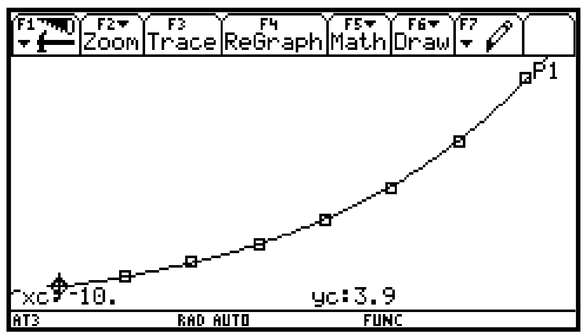
To set-up a scatterplot of this data, press **F2 Plot Setup** **F1 Define** **Plot Type--Scatter** **Mark--Box** in **x** type **c1** in **y** type **c2** **ENTER**
To see the scatterplot, press **Y =** **F5 All** **3: Functions Off** (to turn all the functions off) **F2: Zoom** **9:ZoomData** Voila!



US Population Census Data in Millions, 1790 to 1860

To use the regression capabilities of the TI-92 to fit an exponential regression, press **APPS 6:Data/Matrix Editor 1:Current F5 Calc** for Calculation Type, right arrow to see menu and press **4:ExpReg ENTER** in **x** type **c1 ENTER** in **y** type **c2 ENTER** for Store RegEQ to right arrow, down arrow to **y1(x)** press to see the regression coefficients (the equation has been entered in **y1** which rounds to **y1=5.3 (1.03)**^{x}) press **ENTER GRAPH** to see the scatter plot and regression equation plotted. Nice fit?



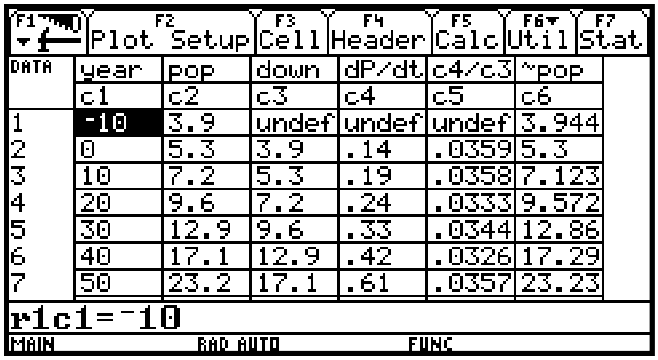


We can verify that this data fits an exponential model and find the equation mathematically. If the relative growth rate, $\frac{dP}{dt} / P$, is relatively constant, that is if

$\frac{dP}{dt} / P \approx k$, then

$P = P_0 e^{kt}$, and the population is exponential. To calculate the relative growth rate for each year above, we use one-sided approximations with 10-year intervals. In the table below, **c3 = shift (c2)** [a vertical shift down of population to complete the following computations; up arrow to c3, press **ENTER** and type; press **ENTER** again to see the results]. $c4 = \frac{c2 - c3}{10} = \frac{dP}{dt}$, $c5 = \frac{c4}{c3} = \frac{dP}{dt} / P$. Since the average ten-year relative growth rate is .03 for the years 1790 to 1860, the population of the United States can be described by the differential equation $\frac{dP}{dt} = .03\,P$. Its solution is $P = 5.3\, e^{.03t}$

or **$P = 5.3 \cdot 1.03^t$**. We use the latter equation to predict the population, **c6 = 5.3 * 1.03^c1** and compare it to **c2** to see that the approximation is good. [To see all six columns in your table, press ◆**F**, right and up arrow to a **cell width** of **5**, **ENTER.** For ~ in **~pop**, press **CHARacter** [**2nd** +] **3:Punctuation** **D:** ~]


F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA	year	pop	down	dP/dt	c4/c3	~pop
	c1	c2	c3	c4	c5	c6
1	-10	3.9	undef	undef	undef	3.944
2	0	5.3	3.9	.14	.0359	5.3
3	10	7.2	5.3	.19	.0358	7.123
4	20	9.6	7.2	.24	.0333	9.572
5	30	12.9	9.6	.33	.0344	12.86
6	40	17.1	12.9	.42	.0326	17.29
7	50	23.2	17.1	.61	.0357	23.23

r1c1=⁻10

MAIN RAD AUTO FUNC

An exponential model fits well until the civil war when the birth rate changed significantly.

Lincoln , in his **1862 Annual Message to Congress**, recommended that slavery be abolished in each slave-holding state by the year 1900 and that each state be reimbursed for each slave owned as of the eighth census (1860). He used an exponential model to predict that by 1900 there would 100,000,000 people to share this burden [y1(100)] by saying that since the “average decennial increase of 34.60% [±2%] in population through the seventy years from our first to our last census yet taken”, this growth is “inflexible” and will continue ”if we do not relinquish the chance by the folly and evils of disunion or by long and exhausting war spring from the only great element of national discord among us.” This quote, from **The Writings of Abraham Lincoln,** The Modern Library, Random House, 1940, pp. 741-2, was brought to my attention by Rob Parsons, Bakersfield College.

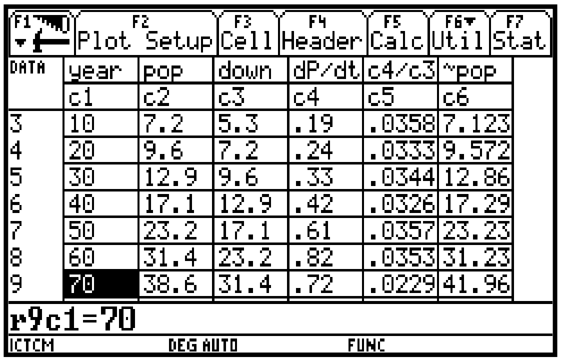
Fortunately this model did not continue to the present time which would give us a population in the United States of about 1.5 billion!

US Population, Logistic Approximation

If we extend the US population data through the year 1940 (prior to World War II), we clearly see that the relative growth rate [column 5] does not remain around .03 and the population does not continue to grow exponentially [compare columns 2 and 6].

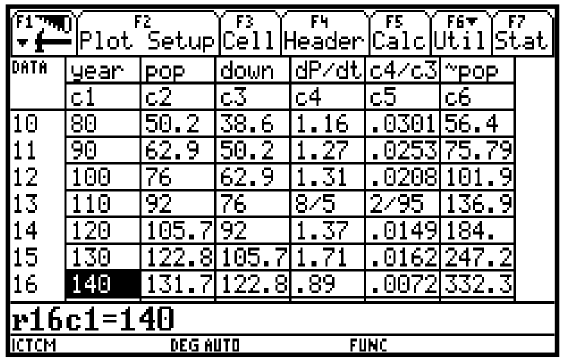
Year	1790	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890
Pop	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4	38.6	50.2	62.9

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990
Pop	76.0	92.0	105.7	122.8	131.7	150.7	179.0	205.0	226.5	248.7



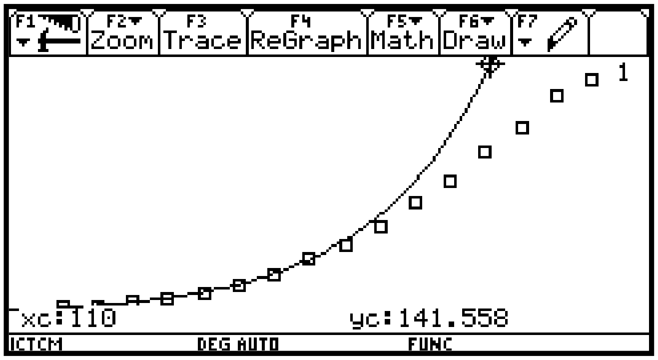
DATA	year	pop	down	dP/dt	c4/c3	~pop
	c1	c2	c3	c4	c5	c6
3	10	7.2	5.3	.19	.0358	7.123
4	20	9.6	7.2	.24	.0333	9.572
5	30	12.9	9.6	.33	.0344	12.86
6	40	17.1	12.9	.42	.0326	17.29
7	50	23.2	17.1	.61	.0357	23.23
8	60	31.4	23.2	.82	.0353	31.23
9	70	38.6	31.4	.72	.0229	41.96

r9c1=70



DATA	year	pop	down	dP/dt	c4/c3	~pop
	c1	c2	c3	c4	c5	c6
10	80	50.2	38.6	1.16	.0301	56.4
11	90	62.9	50.2	1.27	.0253	75.79
12	100	76	62.9	1.31	.0208	101.9
13	110	92	76	8/5	2/95	136.9
14	120	105.7	92	1.37	.0149	184.
15	130	122.8	105.7	1.71	.0162	247.2
16	140	131.7	122.8	.89	.0072	332.3

r16c1=140



US Population in Millions, 1790 to 1940, with Exponential Fit

However, if the relative growth rate, $\frac{\mathbf{dP}}{\mathbf{dt}}$ **/ P**, is linear, that is if $\frac{\mathbf{dP}}{\mathbf{dt}}$ **/ P** $\approx$ **k - a P**, then the population is logistic. To get a good fit, we will use the symmetric rate of change: $\frac{\mathbf{dP}}{\mathbf{dt}} \approx \frac{\mathbf{P(t+10)-P(t-10)}}{\mathbf{20}}$. To do this, use column 7 (not shown in screens below) and let **c7 = shift (c3)**; calculate $\frac{\mathbf{dP}}{\mathbf{dt}}$ in column 4 by **c4 =(c2-c7) / 20** which gives the average growth rate for the population in column 3 or the year **t - 10**, and calculate the relative growth rate $\frac{\mathbf{dP}}{\mathbf{dt}}$ **/ P** in column 5 by **c5 =c4 / c3.** The predicted population in column 6 has been calculated using the solution to the differential equation found below.

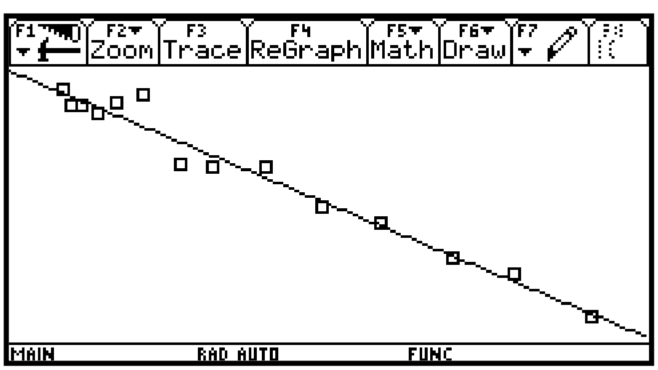
DATA	year	pop	down	dP/dt	c4/P	~pop
	c1	c2	c3	c4	c5	c6
3	10	7.2	5.3	.165	.0311	7.215
4	20	9.6	7.2	.215	.0299	9.793
5	30	12.9	9.6	.285	.0297	13.22
6	40	17.1	12.9	.375	.0291	17.74
7	50	23.2	17.1	.515	.0301	23.58
8	60	31.4	23.2	.715	.0308	31.01
9	70	38.6	31.4	.77	.0245	40.2

r9c5=.024522292993631

DATA	year	pop	down	dP/dt	c4/P	~pop
	c1	c2	c3	c4	c5	c6
10	80	50.2	38.6	.94	.0244	51.22
11	90	62.9	50.2	1.215	.0242	63.95
12	100	76.	62.9	1.29	.0205	78.05
13	110	92.	76.	1.455	.0191	92.93
14	120	105.7	92.	1.485	.0161	107.9
15	130	122.8	105.7	1.54	.0146	122.1
16	140	131.7	122.8	1.3	.0106	135.1

r16c5=.010586319218241

The graph below shows $\frac{\mathbf{dP}}{\mathbf{dt}}$ **/ P** as a function of **P** (**c5** graphed as a function of **c3)** for the data through 1940 (prior to World War II). To find the linear regression for this data, press **F5 Calc** right arrow **5:Lin Reg** down arrow **c3** down arrow **c5** down and right arrow to **y1(x)** to store the linear regression in y1(x) **ENTER**. We see that the equation is $\frac{\mathbf{dP}}{\mathbf{dt}}$ **/ P = .032 - 1.70 *10**$^{-4}$ **P** with a correlation of **-.98**. To see the graph, press **Y =** **F2 Zoom** **9:ZoomData**.



Linear Regression: $\frac{dP}{dt}$ / P as a function of t for 1790-1940

In the logistic model, $\frac{dP}{dt} / P \approx k - a\,P$, as **P** increases the relative growth rate approaches 0; thus $P \to \frac{k}{a}$ and the largest supportable population is $L = \frac{k}{a}$.

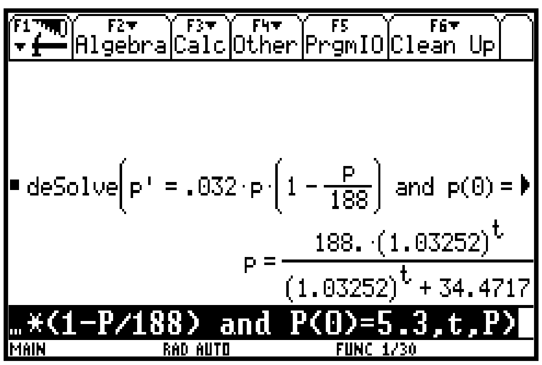
Then $\frac{dP}{dt} \bullet k\,P - a\,P^2 = k\,P\,(1 - \frac{a}{k}P) = k\,P\left(1 - \frac{P}{L}\right)$.

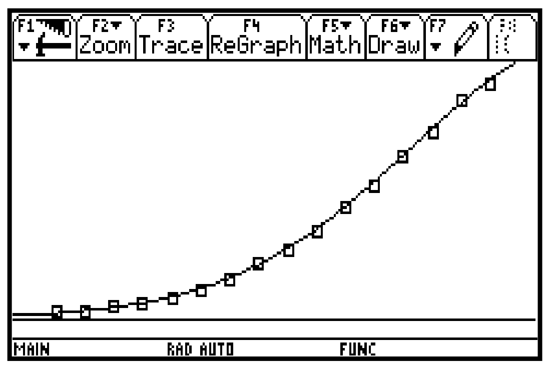
For 1790 to 1940, $\frac{dP}{dt} / P = .032 - 1.70 * 10^{-4}\,P$ and $L = \frac{k}{a} = 188$;

thus $\frac{dP}{dt} = .032\ P\ (1 - .0053\ P) = .032\,P\left(1 - \frac{P}{188}\right)$.

This **Advanced Mathematics Software** on the new **TI-92 Plus Module** contains a new command in the **F3 Calc** menu **C:deSolve(** for solving differential equations. To use it, enter the differential equation **and** the initial condition followed by the independent, then dependent variable, separated by commas.

[To type the "prime" in **p**', press **2nd b**.]





Thus, $P = \frac{188\,(1.0325)^t}{1.0325^t + 34.5} = \frac{188}{1 + 34.5\,(1.0325)^{-t}}$.

Entering this equation in **y1** and graphing it with the scatterplot Plot 1, we see how well this solution fits the data. The predicted population (column 6) in the tables on the previous page were calculated using this equation. Good fit!!!.

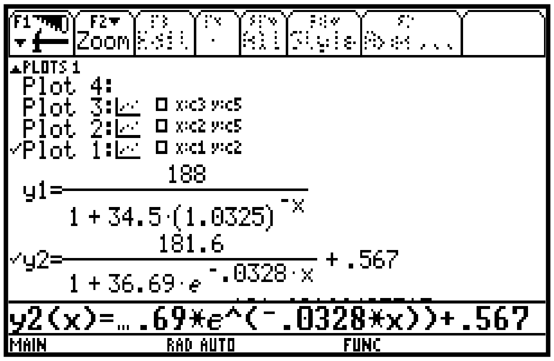
The **TI-92 Plus Module** also contains a logistic regression equation of the form

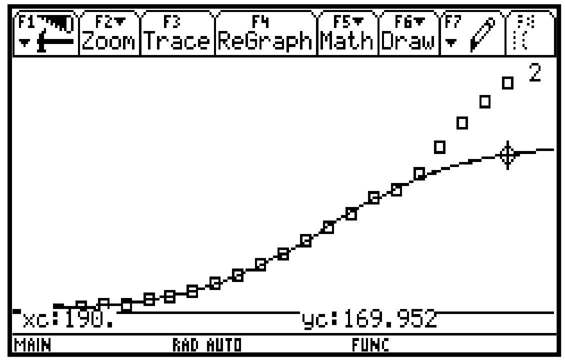
$y = \frac{a}{1 + b\,e^{cx}} + d$. To find this logistic regression, press **APPS** **6:Data/Matrix**

Contents

Editor **1:Current** **F5 Calc** right arrow **C:Logistic** down arrow **c1** down arrow **c2** down and right arrow to **y3(x)** to store the logistic regression in y3(x) **ENTER**.

Rounding, we see that the equation is $\mathbf{y_2(x) = \dfrac{181.6}{1+36.69\,e^{-.0328\,x}} + .567}$.

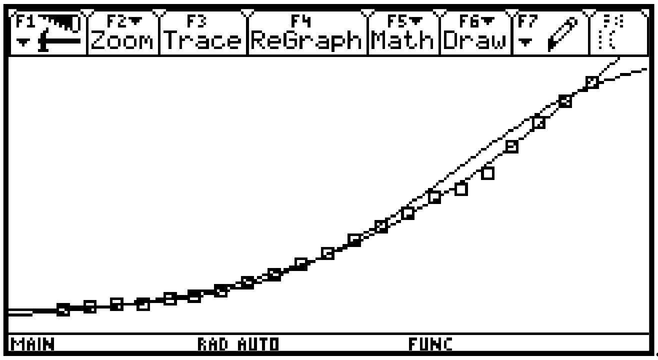




Now we shall develop two different logistic equations using the data from 1790 to 1990 with very different predicted maximum populations. Add the population data for the years 1950 to 1990 to the table. Drawing the scatterplot for all this data, we see above that the logistic equation fitted through 1940 does not predict the remaining data well. Using the TI-92 Plus' logistic regression on this full set of data, we find that

$\mathbf{P = \dfrac{550}{1+30.77\,e^{-.0176\ t}} - 13.94}$ which predicts a population of 536 million (550 - 14) by about the year 2300. However, if we find the linear regression for $\mathbf{\dfrac{dP}{dt} / P}$ as a function of **P** (**c5** graphed as a function of **c3**), we obtain $\mathbf{\dfrac{dP}{dt} / P = .029 - .0001}$ or

$\mathbf{P = .029P\left(1-\dfrac{P}{290}\right)}$. This differential equation has the solution $\mathbf{P = \dfrac{290}{1+53.7(1.03^{-x})}}$ and a limiting population of 290 million by about the year 2100 which we see below is already beginning to level off by 1990. Which do we hope is the better predictor? How do we decide which is more accurate? How much do we trust technology?



US Population in Millions, 1790 to 1990, with Two Logistic Fits