New perspectives on conic sections

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Abstract

For a given hyperbola, what are the points in the plane through which pass tangents to the hyperbola? Can we identify regions from which the hyperbola is viewed in an acute angle? What are the roles of the asymptotes and the director circle of the hyperbola? What is the contribution of methods in dynamic geometry to our exploration with computer algebra software?

In a Resource E-book for Teaching Analytic Geometry with CAS (developed at the Weizmann Institute of Science), the director circle is used as an additional unifying point of view for conic sections. At first glance, the director circle of a hyperbola seems to play a role similar to the role of the director circle of an ellipse; but, in fact, there are a couple of differences. The case of the parabola can be seen as a limiting case for ellipses whose eccentricity approaches 1, that is, the length of the major axis tends to infinity. In this case, the directrix can be viewed as a circle with an infinite radius.

1. New perspectives on the parabola

Take a parabola with canonical equation \( y^2 = 2px \). We wish to find whether there exist tangents to the parabola through a given point with coordinates \((X,Y)\) in the plane. For that purpose we explore the set of solutions of the following system of equations:

\[
\begin{align*}
    y^2 &= 2px \\
    y &= mx - mX + Y.
\end{align*}
\]

We find the following classification (see Figure 1):

- For a point out of the parabola (i.e. for which the inequality \( y^2 > 2px \) holds), there exist two tangents;
- Through a point on the parabola, there exist a single tangent;
- There exist no tangent to the parabola through an interior point (i.e. a point for which the inequality \( y^2 < 2px \) holds).
It is well known that the directrix is the geometric locus of the points from which the parabola is seen in a right angle. We check graphically (see Figure 2), and justify geometrically that the directrix divides the exterior of the parabola into two regions. One of them is the locus of points from which the parabola is seen under an acute angle, the other one is the locus of points from which the parabola is seen under an obtuse angle.

(Figure 2 was created by defining a dynamic procedure which enabled us to click on the mouse at points \([X:=h\text{Cross}, Y:=v\text{Cross}]\) to generate a viewing angle from that point.)
The last result intrigued the authors and the in-service teachers who attended a professional development course in Analytic Geometry. In order to obtain more details, we explored the locus of points from which the parabola is seen under a given angle. To our surprise, for specific angles we found branches of hyperbolas (see Figure 3): an unfamiliar relationship between parabolas and hyperbolas.

![Figure 3](image)

2. A one-parameter equation for a family of conic sections

For a unified exploration of the director circle of a conic section we use a one-parameter equation. Consider a general conic section, whose focus is the origin of the coordinate system and the directrix is a line whose equation is \(x + p = 0\). An equation for this curve is \(x^2 + y^2 = e^2(x + p)^2\) where the eccentricity is \(e\). If we set \(p = \frac{1}{e}\) we have a one-parameter equation of a family of conic sections, that is \(x^2 + y^2 = e^2(x + \frac{1}{e})^2\). A simplified form of this equation \(x^2 + y^2 = (ex+1)^2\) looks better. Each conic section of the two parameter family is similar to a conic section of the single parameter family.

Examples are displayed on Figure 4 for \(e = 0, 0.7, 1, 1.7\). They have been obtained using a slider bar to demonstrate the continuous deformation from ellipses to hyperbolas, resulting from increasing the eccentricity. Each conic section divides the plane into a convex and a concave domain. The concave region, which is always connected, comprises the points through which pass tangents to the conic.
As before (Zehavi & Mann, 2005; Mann et al, to appear), we look for the loci of intersection points of pairs of perpendicular tangents to the various curves. By solving the system of equations

$$
\begin{align*}
xy + e &= x + y \\
\frac{x}{m_1} + \frac{y}{m_2} &= XY
\end{align*}
$$

we obtain two expressions for the slopes $m_1, m_2$ of the tangents; we substitute values for $e$ and explore the equation $m_1 \cdot m_2 = -1$. 

*Figure 4*
Figure 5 shows the cases of an ellipse and of a hyperbola.

![Figure 5](image)

For this family, we obtained a "director" circle \((e^2 - 1)X^2 + (e^2 - 1)Y^2 + 2eX + 2 = 0\) in the following cases: \(0 \leq e < 1\) (any ellipse) and \(1 < e < \sqrt{2}\) (some hyperbolas). Are the director circles of ellipses and hyperbolas of the same nature?

3. Changing the angle of view on an ellipse

We define \(\text{ellipse}(e) := x^2 + y^2 = (ex + 1)^2, \ 0 \leq e < 1\). Next we define \(\text{DMZ}(e, \alpha)\)\(^1\) to be the locus of points from which \(\text{ellipse}(e)\) is viewed under an angle \(\alpha\):

\[
\text{DMZ}(e, \alpha) := \text{If } \alpha = \pi/2, \text{ then } m_1\cdot m_2 = -1 \quad \text{and} \quad (m_2 - m_1)/(1 + m_1\cdot m_2) = \tan(\alpha)
\]

Consider (in Figure 6) \(\text{DMZ}(0.9, \pi/2), \text{DMZ}(0.9, \pi/3), \text{DMZ}(0.9, \pi/6)\). \(\text{DMZ}(0.9, \pi/2)\) is the director circle of the ellipse? \(\text{DMZ}(0.9, \pi/3)\) and \(\text{DMZ}(0.9, \pi/6)\) are located outside the director circle. These DMZ curves seem to have a clear pattern, which motivates further exploration. The natural question now is how \(\text{DMZ}(e, \alpha), e > 1\) will behave?

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\(^1\) A hint regarding the name DMZ is given by the list of authors.
4. DMZ-curve for a hyperbola

Figure 7 illustrates DMZ curves (as defined above) for various hyperbolas, where the viewing angles are, as above, $\pi/2, \pi/3, \pi/6$. The figures raise concerns such as: (a) Where is the director circle for $e = 1.45$? (b) Are all viewing angles on $DMZ(1.3, \pi/6)$ really $\pi/6$? (c) What is the number of tangents, to the hyperbola, through points in the exterior of the hyperbola?
The answers to the above questions reveal a complex picture (Figure 8)

- The **outside of the** (canonic) hyperbola is partitioned into four loci:
  - **locus0** (of points through which no tangents pass) which is in fact the set \{(0, 0)\};
  - **locus1** (of points through which passes a single tangent) which is the symmetric difference of the two asymptotes;
  - **locus21** (of points through which pass **two tangents to one branch** of the hyperbola) which is the area between the hyperbola and its asymptotes;
  - **locus22** (of points through which pass **two tangents to two branches** of the hyperbola) which is the area outside the asymptotes;

- The viewing angle in locus21 must be bigger then the angle between the asymptotes; the viewing angle in locus22 must be smaller then the angle between the asymptotes;

- DMZ(e, α) may be empty; may look like a circle (α=π/2); may look like the union of two closed curves, or like one closed curve. The words "look like" are important, because what looks like a connected curve is really the union of four open arcs two of them subsets of locus21, and the other two subsets of locus22.

![Figure 8](image-url)
Using Derive 6 we show that the outside of every hyperbola is divided into two disjoint regions by the asymptotes and the director circle: one region of vertices of acute viewing angles; the other of vertices of obtuse viewing angles. The angle between the asymptotes determines if there is actually a director circle, and, hence, if each of the two regions is composed of 4 disjoint sub-regions or only two. Using slider bars enabled us to explore dynamically the unfamiliar results (Figure 9).

![Figure 9](image)

5. **Changes that technology bring to the teaching and learning of analytic Geometry**

A CAS enables us to explore unfamiliar curves (loci of angles of view) by plotting implicit equations with which we did not bother to deal in the past because they were inhibiting. The possibility of drawing curves point by point may be very problematic, because of all kinds of distorted impressions we might get. This calls for extensive use of reflective thinking. More explicitly - in such a way we have no possibility to know whether the drawing is really a "preview" of the actual curve, because of a lack of information about vertices, singular points (without a well defined tangent), points of self intersection, etc. All these topics necessitate the usage of more advanced theorems, such as the Implicit Function Theorem to decide how to divide the curve into arcs which can be studied by Calculus methods (Walker, 1978).

Using a CAS as a dynamic geometry software, we can (a) bypass the lack of theoretical knowledge of the students at a certain extent (Dana-Picard 2005), and (b)
give an introduction to more advanced Mathematics. Another product of such a work is the rediscovery of "ancient notions" which disappeared from the syllabi a long while ago (see the complaints of Thom (1962) on this). Integration of the analytic and graphic powers of CAS enables us to sum a complex situation in a single "picture".

References


